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REAL HYPERSURFACES IN A COMPLEX SPACE FORM WITH RECURRENT RICCI TENSOR

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Abstract. In this paper we show that there are no real hypersurfaces in a non-flat complex space form with recurrent Ricci tensor.

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1. Introduction. Let $M_n(c)$ be an n-dimensional non-flat complex space form with constant holomorphic sectional curvature $4c$. It is known that a complete and simply connected non-flat complex space form is either a complex projective space ($c > 0$) or a complex hyperbolic space ($c < 0$).

It is well known that there are no real hypersurfaces M in $M_n(c)$ with parallel Ricci tensor S, i.e., $\nabla S = 0$ (cf. [**3**]), where ∇ denotes the Levi-Civita connection on M. Therefore, it is interesting to study real hypersurfaces M in $M_n(c)$ under certain conditions that are weaker than the Ricci-parallel condition. Many results have been obtained along this direction (cf. [**2**], [**4**], [**6**], [**7**], [**8**], [**10**], [**11**]). In this paper, we investigate the condition that the Ricci tensor is *recurrent*, i.e., there exists a 1-form ψ in M such that

$$\nabla_X S = \psi(X)S$$

for any vector field X tangent to M. We prove the following:

THEOREM. *There are no real hypersurfaces M in $M_n(c)$, $N \geq 3$, with recurrent Ricci tensor.*

REMARK. A similar result has been obtained by Hamda [**2**] for $c > 0$ under the assumption the vector field $\xi = -JN$ is principal, where N is a unit normal vector field on M.

2. Preliminaries. Let M be an orientable connected real hypersurface of $M_n(c)$, $c \neq 0$, and let N be a unit normal vector field on M. Denote by $\bar{\nabla}$ the Levi-Civita connection on $M_n(c)$ and ∇ the connection induced on M. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N$$
$$\bar{\nabla}_X N = -AX,$$

for any vector fields X and Y tangent to M, where $\langle , \rangle$ denotes the Riemannian metric of M induced from the Riemannian metric of $M_n(c)$ and A is the second fundamental tensor of M in $M_n(c)$. Now, we define a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then it is seen that $\langle \xi, X \rangle = \eta(X)$. Furthermore, the set of tensors $(\phi, \xi, \eta, \langle , \rangle)$ is an almost contact metric structure on M, i.e., they satisfy the following

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1. \tag{1}$$

Let R be the curvature tensor of M. Then the equation of Gauss is given by

$$\begin{aligned} R(X, Y)Z = c\{&\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle \phi Y, Z\rangle \phi X - \langle \phi X, Z\rangle \phi Y - 2\langle \phi X, Z\rangle \phi Y\} \\ &+ \langle AY, Z\rangle AX - \langle AX, Z\rangle AY. \end{aligned}$$

From (1) and the Gauss equation that

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X$$

where $h = traceA$ and S is the Ricci tensor of type (1,1) on M. The real hypersurfaces M is said to be *Ryan* if the Ricci tensor S satisfies

$$(R(X, Y)S)Z = 0$$

for any vector field X, Y and Z tangent to M.

Finally we state some known results for later use.

THEOREM A. [**10**]. *There are no real hypersurfaces M in $M_n(c)$, $n \geq 3$, satisfying the Ryan condition.*

THEOREM B. [**1, 5, 9**]. *There are no Einstein real hypersurfaces M in $M_n(c)$, $n \geq 3$.*

3. Proof of Theorem. Suppose that the Ricci tensor is recurrent. Then

$$(\nabla_Y S)Z = \psi(Y)SZ \tag{2}$$

for any vector fields Y and Z tangent to M. Since M is non-Einsteinian (by Theorem B), S admits at least one nonzero eigenvalue σ, for otherwise, we must have $S = 0$, which contradicts M being non-Einsteinian. Let Z be a unit eigenvector of S corresponding to the eigenvalue $\sigma \neq 0$. By using the relationship (2), we get

$$\begin{aligned} Y\sigma &= \langle (\nabla_Y S)Z, Z\rangle + \langle S\nabla_Y Z, Z\rangle + \langle SZ, \nabla_Y Z\rangle \\ &= \psi(Y)\langle SZ, Z\rangle + \sigma(\nabla_Y Z, Z\rangle + \sigma\langle Z, \nabla_Y Z\rangle \\ &= \sigma\psi(Y). \end{aligned}$$

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This means that

$$d\sigma = \sigma\psi.$$

Therefore

$$0 = d^2\sigma = d\sigma \wedge \psi + \sigma d\psi = \sigma\psi \wedge \psi + \sigma d\psi = \sigma d\psi.$$

Now we look at the open set $\mathbf{W}$ of all points x such that $\sigma(x) \neq 0$. Then we have $d\psi = 0$ or

$$(\nabla_X \psi)Y = (\nabla_Y \psi)X \tag{3}$$

for any X and $Y \in T_xM$ and $x \in \mathbf{W}$.

Next, for any X, Y and $Z \in T_xM$ and $x \in \mathbf{W}$, by differentiating (2) covariantly with respect to X, we obtain

$$\begin{aligned}(\nabla_X \nabla_Y S)Z &= \nabla_X(\nabla_Y S)Z - (\nabla_{\nabla_X Y} S)Z - (\nabla_Y S)\nabla_X Z\\ &= \nabla_X\{\psi(Y)SZ\} - \psi(\nabla_X Y)SZ - \psi(Y)S\nabla_X Z\\ &= \{\nabla_X[\psi(Y)]\}SZ + \psi(Y)\nabla_X SZ - \psi(\nabla_X Y)SZ - \psi(Y)S\nabla_X Z\\ &= \{(\nabla_X \psi)Y\}SZ + \psi(Y)(\nabla_X S)Z\\ &= \{(\nabla_X \psi)Y\}SZ + \psi(Y)\psi(X)SZ.\end{aligned}$$

By exchanging X and Y in this equation, we have

$$(\nabla_Y \nabla_X S)Z = \{(\nabla_Y \psi)X\}SZ + \psi(X)\psi(Y)SZ.$$

From these equations, together with the Ricci identity, we have

$$(R(X, Y)S)Z = \{(\nabla_Y \psi)X - (\nabla_Y \psi)X\}SZ.$$

Together with (3), we find that

$$(R(X, Y)S)Z = 0$$

From Theorem A, this is impossible. Hence the open set $\mathbf{W}$ must be empty and so $\sigma = 0$. This is a contradiction and so we conclude that S cannot be recurrent.

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