J. Austral. Math. Soc. Ser. B 37(1995), 253–266

SIMILARITY SOLUTIONS FOR A QUASILINEAR PARABOLIC EQUATION

JONG-SHENG GUO[1]

(Received 28 June 1993; revised 6 December 1993)

Abstract

In this paper, we use an ordinary differential equation approach to study the existence of similarity solutions for the equation $u_t = \Delta(u^\alpha) + \theta u^{-\beta}$ in $\mathbf{R}^n \times (0, \infty)$, where $\alpha > 0$, $\beta > 0$, $\theta \in \{0, 1\}$, and $n \geq 1$. This includes the slow diffusion equation when $\alpha > 1$, the standard heat equation when $\alpha = 1$, and the fast diffusion equation when $0 < \alpha < 1$. We prove that there are forward self-similar solutions for this equation with initial data of the form $c|x|^p$, where $p = 2/(\alpha + \beta)$ if $\theta = 1$; $p \geq 0$ and $2 + (1 - \alpha)p > 0$ if $\theta = 0$, for some positive constant c.

1. Introduction

We are interested in the Cauchy problem for the quasilinear parabolic equation,

$$u_t = \Delta(u^\alpha) + \theta u^{-\beta} \qquad \text{in } \mathbf{R}^n \times (0, \infty), \tag{1.1}$$

where $\alpha > 0, \beta > 0, \theta \in \mathbf{R}, n \geq 1$, and Δ is the standard Laplacian operator, that is,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

This equation arises in many applications, such as the heat flow in materials with a temperature dependent conductivity with or without reaction. This includes the slow diffusion equation when $\alpha > 1$, the standard heat equation when $\alpha = 1$, and the fast diffusion equation when $0 < \alpha < 1$. For earlier work on this type of equation, we refer to the nice survey paper of Kalashnikov [8]. This equation (for $\alpha = 1$ and $\theta = 1$) is related to a parabolic system arising in film development (see for example [1] and [10]).

[1]Institute of Applied Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.

In this paper we use an ordinary differential equation approach to find some special solutions, namely similarity solutions, of (1.1). We want to find the forward self-similar solution of the form

$$u(x,t) = t^{p\delta} w(|x|/t^{\delta}), \qquad |x| = \sqrt{x_1^2 + \cdots + x_n^2} \tag{1.2}$$

for (1.1). Here the similarity exponents are necessarily given by

$$p = \frac{2}{(\alpha+\beta)} \quad \text{for } \theta \neq 0; \qquad p \geq 0 \text{ and } 2 + (1-\alpha)p > 0 \quad \text{for } \theta = 0. \tag{1.3a}$$

$$\delta = \frac{1}{2 + (1-\alpha)p}. \tag{1.3b}$$

Notice that for $\theta = 0$ the exponent p can be arbitrary if $\alpha \leq 1$, and $p < 2/(\alpha - 1)$ if $\alpha > 1$.

Let $r = |x|/t^{\delta}$. Then $w \in C^2([0,\infty))$ and satisfies

$$(w^{\alpha})'' + \frac{n-1}{r}(w^{\alpha})' + \delta r w' - p\delta w + \theta w^{-\beta} = 0, \qquad r > 0, \tag{1.4a}$$

$$w'(0) = 0, \qquad w > 0. \tag{1.4b}$$

We remark that this reduction can be obtained if we consider only the radial solutions of (1.1) by using the operator

$$\Delta = \frac{\partial^2}{\partial \sigma^2} + \frac{n-1}{\sigma}\frac{\partial}{\partial \sigma}, \qquad \sigma = |x|$$

and a solution ansatz as

$$u(\sigma, t) = t^{p\delta} w(\sigma/t^{\delta}).$$

In this way, we have simplified the problem from $n+1$ dimensions to $1+1$ dimensions.

Since (1.1) for $\theta \in \mathbf{R} \setminus \{0\}$ can be transformed to an equation in the same form as (1.1) with $\theta \in \{-1, 1\}$ via the simple transformations $t \to t/|\theta|$, $x \to x/\sqrt{|\theta|}$, we shall henceforth assume that $\theta \in \{-1, 0, 1\}$.

First, we consider the case $\theta \in \{0, 1\}$.

Let $\gamma = 1/(\beta + 1)$ and $\kappa = 0$ if $\theta = 0$; $= (\beta+1)^{\gamma}$ if $\theta = 1$.

Note that for $p = 0$ (and $\theta = 0$), any positive constant is a solution of (1.4), and κ is the only constant solution of (1.4) for any $p > 0$. We shall hence assume that $p > 0$ and p satisfies (1.3a). We say that w is a solution of (1.4) if $w \in C^2([0,\infty))$, $w \not\equiv \kappa$, and w satisfies (1.4).

The main result of this paper is as follows.

THEOREM A. *A solution of (1.4) exists if and only if $w(0) > \kappa$. Moreover, for $w(0) > \kappa$, $\lim_{r\to\infty}[r^{-p}w(r)]$ exists and is positive.*

Let $w(r)$ be a solution of (1.4) with $w(0) > \kappa$. For any $x \neq 0$, we rewrite (1.2) as

$$u(x,t) = |x|^p r^{-p} w(r). \tag{1.5}$$

Letting $t \to 0$ in (1.5), it follows from Theorem A that there exists a positive constant c such that

$$u(x,0) = c|x|^p, \qquad x \neq 0. \tag{1.6}$$

On the other hand, we have $u(0,0) = 0$. This shows that there is a solution of the Cauchy problem for (1.1) with initial data $c|x|^p$. Notice that for $w \equiv \kappa$ the function $u(x,t) = \kappa t^\gamma$ is a solution of (1.1) with $\theta = 1$ and with initial data $u_0 \equiv 0$.

It is well known that in many cases the large time behaviors or the finite time singular behaviors (for example, blowup and quenching behaviors) of solutions for the evolution equation are described by its similarity solutions. See, for example, [2, 3] for blowup behavior and [4] for quenching behavior. Also, the large time behaviors of solutions of the Cauchy problem of (1.1) with $\theta = 1$ are described by the above similarity solutions (see [6] and references therein).

Let u be the solution of (1.1) with $\theta = 1$ and with the initial data $u_0(x) \geq 0$, where $u_0(x)$ is continuous such that

$$\lim_{|x|\to\infty} |x|^{-p} u_0(x) = c$$

for some nonnegative constant c and $p \geq 0$ with $2 + (1-\alpha)p > 0$. Then we have the following theorem.

THEOREM B. *Let δ be given by (1.3b). If $c > 0$ and $p \geq 2/(\alpha+\beta)$, then*

$$|t^{-p\delta}u(x,t) - w(|x|/t^\delta)| \to 0 \qquad \text{as } t \to \infty$$

uniformly on sets $\{(x,t) : |x| \leq Ct^\delta\}$ for any $C > 0$, where $w(r)$ is a solution of (1.4) with $\theta = 1$ if $p = 2/(\alpha+\beta)$; and $\theta = 0$ if $p > 2/(\alpha+\beta)$ such that

$$\lim_{r\to\infty} [r^{-p}w(r)] = c. \tag{1.7}$$

If $c = 0$ and $p = 2/(\alpha+\beta)$, then we have

$$|t^{-\gamma}u(x,t) - \kappa| \to 0 \qquad \text{as } t \to \infty$$

uniformly on sets $\{(x,t) \mid |x| \leq Ct^\delta\}$ for any $C > 0$.

Notice that the case $p < 2/(\alpha + \beta)$ is contained in the second case of Theorem B. Theorem B is proved in [9] for $\alpha \geq 1$ and in [6] for $0 < \alpha < 1$. In these papers, they first obtained the existence and uniqueness for the global solutions of the Cauchy problem for (1.1) with $\theta = 1$ for general initial data. Hence by the scaling invariance property of the problem we obtain the existence of a solution of (1.4) and (1.7).

In [5], the author studied the existence of similarity solutions for (1.1) with $\alpha = 1$ and $\theta = 1$, that is, the case for the standard heat equation with source by a totally different approach. In that paper, Theorem A was proved for $\alpha = 1$ using an ordinary differential equation approach.

Extending the method used in [5] (see also the references cited there), we shall show that a similar argument can also be applied to the cases of the fast and slow diffusion equations. The method we used is fairly elementary and it provides an easy way to obtain the asymptotic behaviors at infinity of solutions for certain ordinary differential equations.

We remark that the existence of self-similar solutions for the equation

$$u_t = \Delta(u^\alpha) - u^\beta \qquad \text{in } \mathbf{R}^n \times (0, \infty)$$

for $1 < \beta < \alpha$ and $n \geq 1$ was studied by McLeod, Peletier, and Vazquez in [11] using the shooting-matching method. For other interesting types of similarity solutions using a totally different approach, we refer to the paper [7] and the references therein.

This paper is organized as follows. In Section 2, we give some preliminary results for solutions of (1.4) and introduce some notation. We then derive the asymptotic behaviors of solutions of (1.4) at infinity and give the proof of Theorem A for the fast diffusion case in Section 3 and for the slow diffusion case in Section 4.

Finally, for the case $\theta = -1$ and $0 < \alpha \leq 1$, the above method can also be applied with some minor modifications. This is done in Section 5.

2. Preliminaries

In this section we shall give some preliminary results for (1.4) for any $\alpha > 0$ and introduce some notation.

Let $\phi(r) = w(r)^\alpha$ and let $g(s) = p\delta s^{1/\alpha} - \theta s^{-\beta/\alpha}$. Then w satisfies (1.4) if and only if ϕ satisfies

$$\phi'' + \left[\frac{n-1}{r} + \frac{\delta}{\alpha} r \phi^{1/\alpha - 1}\right]\phi' - g(\phi) = 0, \qquad r > 0, \tag{2.1a}$$

$$\phi'(0) = 0, \qquad \phi > 0. \tag{2.1b}$$

From now on we shall study the problem (2.1) instead of (1.4). For a given $\phi(r)$, we introduce

$$\rho(r) = \exp\left\{\int_0^r \frac{\delta}{\alpha} s\phi(s)^{1/\alpha-1} ds\right\}, \tag{2.2}$$

$$\sigma(r) = r^{n-1}\rho(r). \tag{2.3}$$

Note that $\sigma(r)$ is defined in terms of $\phi(s)$ for $s \in [0, r]$ for $\alpha \neq 1$. Also, we have

$$\rho'(r) = \frac{\delta}{\alpha} r\phi(r)^{1/\alpha-1}\rho(r), \tag{2.4}$$

$$\sigma'(r) = \left[\frac{n-1}{r} + \frac{\delta}{\alpha} r\phi(r)^{1/\alpha-1}\right]\sigma(r). \tag{2.5}$$

Let $\psi = \phi'$. Then (2.1a) can be rewritten as the following system:

$$\phi' = \psi, \tag{2.6a}$$

$$\psi' = -\left[\frac{n-1}{r} + \frac{\delta}{\alpha} r\phi^{1/\alpha-1}\right]\psi + g(\phi). \tag{2.6b}$$

Solving (2.1) with $\phi(0) = \eta^\alpha > 0$ is equivalent to solving the integral system:

$$\phi(r) = \eta^\alpha + \int_0^r \psi(s) ds, \tag{2.7a}$$

$$\psi(r) = \sigma(r)^{-1}\int_0^r \sigma(s) g(\phi(s)) ds. \tag{2.7b}$$

Since $g(s) > 0$ for $s > \kappa^\alpha$ and $g(s) < 0$ for $s < \kappa^\alpha$, it follows from (2.7b) that $\phi'(r) > 0, \forall r > 0$ if $\eta > \kappa$; and $\phi'(r) < 0, \forall r > 0$ if $\eta < \kappa$. Hence every solution of (2.1) with $\theta = 0$ must be strictly increasing. Using (2.7), the local existence and uniqueness of the solution of (2.1) with $\phi(0) = \eta^\alpha$ follows from the standard fixed point argument. Note that

$$\lim_{r\to 0} \psi(r)/r = g(\eta^\alpha)/n = \lim_{r\to 0} \psi'(r).$$

We shall denote the solution of (2.1) with $\phi(0) = \eta^\alpha$ by $\phi(r; \eta)$. For global existence of solutions of (2.1), we have:

PROPOSITION 2.1. *The local solution $\phi(r; \eta)$ of (2.1) with $\eta \in (0, \kappa)$ cannot be extended to all $r > 0$. If a global solution $\phi(r; \eta)$ for (2.1) with $\eta > \kappa$ exists, then $\phi(r; \eta) \to \infty$.*

PROOF. Suppose that the solution $\phi(r;\eta)$ of (2.1) with $\eta > 0$ and $\eta \neq \kappa$ exists for all $r > 0$. Since ϕ is necessarily strictly monotone, the limit $l = \lim_{r\to\infty}\phi(r)$ exists. Suppose that $l < \infty$. Then

$$\int_0^\infty \phi'(r)dr = l - \phi(0)$$

which is bounded. Since ϕ' has a constant sign, there is a sequence $r_m \to \infty$ such that $\phi'(r_m) \to 0$ as $m \to \infty$. Dividing (2.1a) by r and integrating from 1 to $r_m > 1$, we obtain that the integral

$$\int_1^{r_m} \frac{g(\phi)}{r}dr$$

is bounded as $m \to \infty$, since the integrals

$$\int_1^{r_m} \frac{\phi''(r)}{r}dr = \frac{\phi'(r_m)}{r_m} - \phi'(1) + \int_1^{r_m} \frac{\phi'(r)}{r^2}dr,$$
$$\int_1^{r_m} \frac{n-1}{r^2}\phi'(r)dr \qquad \text{and} \qquad \delta\int_1^{r_m} (\phi(r)^{1/\alpha})'dr$$

are uniformly bounded for all m. On the other hand, we have

$$\left|\int_1^{r_m} \frac{g(\phi)}{r}dr\right| \geq |g(\eta^\alpha)|\ln(r_m)$$

which tends to infinity as $m \to \infty$, a contradiction. Therefore the proposition follows.

From now on we shall assume that $\eta > \kappa$ and we shall let $\phi(r) = \phi(r;\eta)$. Using (2.3), (2.1a) can be rewritten as

$$(\sigma\phi')' = \sigma g(\phi). \tag{2.8}$$

For $\eta > \kappa$, we have the following global existence result.

PROPOSITION 2.2. *The local solution $\phi(r)$ of (2.1) can be continued to all $r > 0$.*

PROOF. Suppose that $\phi(r)$ exists in $[0,\epsilon]$. Fix $r_0 \in (0,\epsilon)$. From (2.8) it follows that

$$\phi'(r) = \frac{\sigma(r_0)}{\sigma(r)}\phi'(r_0) + \int_{r_0}^{r} \frac{\sigma(s)}{\sigma(r)}g(\phi(s))ds. \tag{2.9}$$

Recall $\phi'(r) > 0$ for all $r > 0$.

Suppose first that $n \geq 2$. For $r \geq r_0$, we compute

$$\begin{aligned}
\phi'(r) &\leq \phi'(r_0) + p\delta \int_{r_0}^{r} \left(\frac{s}{r}\right)^{n-2} \left(\frac{1}{r\rho(r)}\right) s\rho(s)\phi(s)^{1/\alpha-1}\phi(s)ds \\
&\leq \phi'(r_0) + p\alpha \frac{\phi(r)}{r\rho(r)} \int_{r_0}^{r} \rho'(s)ds \qquad (2.10)\\
&\leq \phi'(r_0) + p\alpha \frac{\phi(r)}{r}.
\end{aligned}$$

Hence from (2.10) it follows that

$$\phi(r) \leq C_1 r^{p\alpha} + C_2 r + C_3, \quad r \geq r_0, \tag{2.11}$$

for some positive constants C_1, C_2 and C_3 depending only on α, p, and r_0. Hence $\phi(r)$ and $\phi'(r)$ are bounded for r finite. Notice that $\phi(r) \geq \eta^\alpha$ for all $r \geq 0$. Therefore, the result follows by the standard continuation theorem.

For the case $n = 1$, $\sigma(r) = \rho(r)$. By writing $\rho(s) = s^{-1}[s\rho(s)]$ in (2.9) and noting that $1/s \leq 1/r_0$ for $r \geq r_0$, we have

$$\phi'(r) \leq \phi'(r_0) + p\alpha \frac{\phi(r)}{r_0}. \tag{2.12}$$

Then the result follows by the same argument as above and the proof is completed.

Let $z(r) = \phi'(r)/\phi(r)$. Then z satisfies the equation

$$z' + \left[\frac{n-1}{r} + \frac{\delta}{\alpha} r\phi(r)^{1/\alpha-1}\right] z = p\delta\phi^{1/\alpha-1} - \theta\phi^{-1-\beta/\alpha} - z^2. \tag{2.13}$$

It follows from (2.13) that

$$z(r) = \sigma(r)^{-1} \int_0^r \sigma(s)[p\delta\phi(s)^{1/\alpha-1} - \theta\phi(s)^{-1-\beta/\alpha} - z(s)^2]ds. \tag{2.14}$$

3. The fast diffusion equation

In this section we shall study the asymptotic behaviors of solutions $\phi(r)$ of (2.1) at $r = \infty$ for $0 < \alpha < 1$. First, we have the following lemma. For convenience, we let $d = \delta/\alpha$.

LEMMA 3.1. *We have* $\lim_{r\to\infty}[r\phi'(r)] = \infty$.

PROOF. Since $0 < \alpha < 1$ and $\phi(r) \to \infty$ as $r \to \infty$, $\sigma(r) \to \infty$ exponentially as $r \to \infty$. Using (2.9) with $r_0 = 0$ and applying l'Hôpital's rule, we compute

$$\begin{aligned}\lim_{r\to\infty}[r\phi'(r)] &= \lim_{r\to\infty}\frac{1}{r^{-1}\sigma(r)}\int_0^r \sigma(s)g(\phi(s))ds\\ &= \lim_{r\to\infty}\frac{p\delta\phi(r)^{1/\alpha} - \theta\phi(r)^{-\beta/\alpha}}{(n-2)r^{-2} + d\phi(r)^{1/\alpha-1}}\\ &= \lim_{r\to\infty}\frac{\phi(r)[p\delta - \theta\phi(r)^{-(\beta+1)/\alpha}]}{(n-2)r^{-2}\phi(r)^{1-1/\alpha} + d}\\ &= \infty,\end{aligned}$$

since $\phi(r)^{1-1/\alpha} \to 0$ as $r \to \infty$. The lemma is proved.

The following lemma shows that $\phi(r)$ can only grow to infinity polynomially.

LEMMA 3.2. *We have* $\lim_{r\to\infty}[\phi'(r)/\phi(r)] = 0$.

PROOF. Recall (2.14). Then we have

$$0 \le \frac{\phi'(r)}{\phi(r)} \le \frac{1}{\sigma(r)}p\delta\int_0^r \sigma(s)\phi(s)^{1/\alpha-1}ds \equiv K(r). \tag{3.1}$$

Applying l'Hôpital's rule, we compute

$$\begin{aligned}\lim_{r\to\infty}K(r) &= \lim_{r\to\infty}\frac{p\delta\phi(r)^{1/\alpha-1}}{(n-1)r^{-1} + dr\phi(r)^{1/\alpha-1}}\\ &= \lim_{r\to\infty}\frac{p\delta/r}{(n-1)r^{-2}\phi(r)^{1-1/\alpha} + d}\\ &= 0,\end{aligned}$$

and the lemma follows.

The following lemma gives the degree of the polynomial growth for $\phi(r)$.

LEMMA 3.3. *We have*

$$\lim_{r\to\infty}\frac{r\phi'(r)}{\phi(r)} = p\alpha. \tag{3.2}$$

PROOF. Rewrite (2.14) as

$$z(r) = \sigma(r)^{-1}\int_0^r \sigma(s)[p\delta\phi(s)^{1/\alpha-1} + a(s)]ds,$$

https://doi.org/10.1017/S0334270000007694 Published online by Cambridge University Press

where $a(s) \to 0$ as $s \to \infty$. Then, using l'Hôpital's rule again, we obtain

$$\begin{aligned}\lim_{r\to\infty}[rz(r)] &= \lim_{r\to\infty} \frac{p\delta\phi(r)^{1/\alpha-1} + a(r)}{(n-2)r^{-2} + d\phi(r)^{1/\alpha-1}} \\ &= \lim_{r\to\infty} \frac{p\delta + a(r)\phi(r)^{1-1/\alpha}}{(n-2)r^{-2}\phi(r)^{1-1/\alpha} + d} \\ &= p\alpha,\end{aligned}$$

since $1 < 1/\alpha$ and $\phi(r) \to \infty$.

LEMMA 3.4. *For any* $\epsilon > 0$ *there are* $K = K(\epsilon) > 0$ *and* $R = R(\epsilon) > 0$ *such that*

$$\phi(r) \geq Kr^{p\alpha-\epsilon}, \quad \forall r > R. \tag{3.3}$$

PROOF. Given $\epsilon > 0$ it follows from (3.2) that there is a number $R > 0$ such that

$$\frac{\phi'(r)}{\phi(r)} \geq \frac{1}{r}(p\alpha - \epsilon)$$

for all $r \geq R$. An integration gives (3.3).

Finally, we state the main result of this section as follows.

THEOREM 3.5. *The limit,* $\lim_{r\to\infty}[r^{-p\alpha}\phi(r)]$, *exists and is positive.*

PROOF. Take any positive constant $\lambda < 2$. Using (2.14), we rewrite

$$\begin{aligned} r^{\lambda}[rz(r) - p\alpha] = \frac{1}{r^{-\lambda-1}\sigma(r)} \bigg(& p\delta\int_0^r \sigma(s)\phi(s)^{1/\alpha-1}ds - p\alpha r^{-1}\sigma(r) \\ & -\theta\int_0^r \sigma(s)\phi(s)^{-1-\beta/\alpha}ds - \int_0^r \sigma(s)z(s)^2 ds\bigg). \end{aligned} \tag{3.4}$$

Applying l'Hôpital's rule to (3.4), we obtain

$$\begin{aligned} &\lim_{r\to\infty} r^{\lambda}[rz(r) - p\alpha] \\ &\quad= \lim_{r\to\infty} \frac{(2-n)p\alpha r^{-2} - \theta\phi(r)^{-1-\beta/\alpha} - z(r)^2}{(n-\lambda-2)r^{-\lambda-2} + dr^{-\lambda}\phi(r)^{1/\alpha-1}} \\ &\quad= \lim_{r\to\infty} \frac{(2-n)p\alpha r^{\lambda-2}\phi(r)^{1-1/\alpha} - \theta r^{\lambda}\phi(r)^{-(1+\beta)/\alpha} - z(r)^2 r^{\lambda}\phi(r)^{1-1/\alpha}}{(n-\lambda-2)r^{-2}\phi(r)^{1-1/\alpha} + d}, \end{aligned} \tag{3.5}$$

if the last limit on the right hand side of (3.5) exists.

We claim that the limit on the right-hand side of (3.5) is zero. For $\theta = 1$, we have $p = 2/(\alpha + \beta)$. Since $\lambda < 2$ and $\alpha < 1$, there is an $\epsilon > 0$ such that $\lambda < (p\alpha - \epsilon)(1 + \beta)/\alpha$. It follows from Lemma 3.4 that

$$0 \le r^{\lambda}\phi(r)^{-(1+\beta)/\alpha} \le Cr^{\lambda-(p\alpha-\epsilon)(1+\beta)/\alpha}.$$

Therefore, we obtain $\lim_{r\to\infty} r^{\lambda}\phi(r)^{-(1+\beta)/\alpha} = 0$. Next, writing $z(r)^2 r^{\lambda}\phi(r)^{1-1/\alpha} = [rz(r)]^2 r^{\lambda-2}\phi(r)^{1-1/\alpha}$ and using (3.2), it follows that $\lim_{r\to\infty}[z(r)^2 r^{\lambda}\phi(r)^{1-1/\alpha}] = 0$. Hence the limit on the right-hand side of (3.5) is zero and we conclude that

$$\lim_{r\to\infty} r^{\lambda}[rz(r) - p\alpha] = 0. \tag{3.6}$$

From (3.6) and by an integration, it follows that there is a positive constant c such that $\phi(r) = cr^{p\alpha}[1 + o(r^{-\lambda})]$ as $r \to \infty$. Hence the theorem is proved.

Then Theorem A for $0 < \alpha < 1$ follows from Propositions 2.1 and 2.2, and Theorem 3.5.

4. The slow diffusion equation

In this section we shall study the asymptotic behaviors of solutions $\phi(r)$ of (2.1) at $r = \infty$ for $\alpha > 1$. We shall distinguish two cases.

First, we deal with the case $n \ge 2$. Recall (2.11) that there are positive constants C_1, C_2 and C_3 such that

$$\phi(r) \le C_1 r^{p\alpha} + C_2 r + C_3. \tag{4.1}$$

For $n = 1$, we need a better estimate than (2.12). From (2.9) it follows that $\phi'(r) \le p\delta r\phi(r)^{1/\alpha}$, where the fact that $g(\phi)$ and $\phi(s)$ are increasing is used. Then an integration gives

$$\phi(r) \le [C_1 + C_2 r^2]^{\alpha/(\alpha-1)}, \qquad \forall r$$

for some constants C_1 and C_2. In fact, for any $q > p\alpha$ we have

$$\lim_{r\to\infty}[r^{-q}\phi(r)] = 0. \tag{4.2}$$

Otherwise, we have $\limsup_{r\to\infty}[r^{-q}\phi(r)] = A > 0$. Hence we can find a sequence $\{r_m\}$ which tends to infinity as $m \to \infty$ such that

$$\phi(r_m) \ge \frac{A}{2} r_m^q, \qquad \forall m. \tag{4.3}$$

Dividing (2.1a) by $r\phi(r)^{1/\alpha}$ and integrating from 1 to $r_m > 1$, we obtain

$$\begin{aligned}
I_m &\geq \int_1^{r_m} r^{-1}\phi(r)^{-1/\alpha}\phi''(r)dr \\
&= r_m^{-1}\phi(r_m)^{-1/\alpha}\phi'(r_m) - \phi(1)^{-1/\alpha}\phi'(1) \\
&\quad + \int_1^{r_m}\left[r^{-2}\phi(r)^{-1/\alpha} + \frac{1}{\alpha}r^{-1}\phi(r)^{-1/\alpha-1}\phi'(r)\right]\phi'(r)dr \\
&\geq -\phi(1)^{-1/\alpha}\phi'(1),
\end{aligned}$$

for all m, where

$$I_m = \int_1^{r_m}\left[\frac{p\delta}{r} - d\frac{\phi'(r)}{\phi(r)}\right]dr.$$

But, by (4.3) we have

$$I_m = \ln\left[r_m^{p\delta}\phi(r_m)^{-d}\phi(1)^d\right] \leq \ln\left[\phi(1)^d 2^d A^{-d} r_m^{p\delta-dq}\right],$$

which tends to $-\infty$ as $m \to \infty$, a contradiction. Therefore, (4.2) holds.

We observe from (4.1) and (4.2) that $\sigma(r) \to \infty$ exponentially as $r \to \infty$ and that

$$\lim_{r\to\infty} r^{-2}\phi(r)^{1-1/\alpha} = 0, \tag{4.4}$$

In parallel to Lemmas 3.1 and 3.2, we have the following two lemmas whose proofs are the same as the proofs of Lemmas 3.1 and 3.2.

LEMMA 4.1. *We have* $\lim_{r\to\infty}[r\phi'(r)] = \infty$

LEMMA 4.2. *We have* $\lim_{r\to\infty}[\phi'(r)/\phi(r)] = 0$.

In order to obtain the degree of the polynomial growth for $\phi(r)$, we need some extra work. We claim that

$$\lim_{r\to\infty} \phi'(r)\phi(r)^{-(\alpha+1)/(2\alpha)} = 0. \tag{4.5}$$

Let $q = \max(p\alpha, 1)$ for $n \geq 2$ and $q \in (p\alpha, 2\alpha/(\alpha-1))$ for $n = 1$. From (3.1), (4.1) and (4.2), we have

$$0 \leq \phi'(r)\phi(r)^{-(\alpha+1)/(2\alpha)} \leq C\frac{1}{r^{q(1-\alpha)/(2\alpha)}\sigma(r)}\int_0^r \sigma(s)\phi(s)^{1/\alpha-1}ds \equiv CK(r),$$

for some positive constant C. Using (4.4) and l'Hôpital's rule, we compute

$$\begin{aligned}
\lim_{r\to\infty} K(r) &= \lim_{r\to\infty}\frac{\phi(r)^{1/\alpha-1}}{[q(1-\alpha)/(2\alpha)+n-1]r^{q(1-\alpha)/(2\alpha)-1} + dr^{q(1-\alpha)/(2\alpha)+1}\phi(r)^{1/\alpha-1}} \\
&= \lim_{r\to\infty}\frac{r^{-q(1-\alpha)/(2\alpha)-1}}{[q(1-\alpha)/(2\alpha)+n-1]r^{-2}\phi(r)^{1-1/\alpha} + d} \\
&= 0,
\end{aligned}$$

since $q < 2\alpha/(\alpha - 1)$. Hence (4.5) follows.

We are now ready to prove the following lemma.

LEMMA 4.3. *We have*

$$\lim_{r\to\infty} \frac{r\phi'(r)}{\phi(r)} = p\alpha. \tag{4.6}$$

PROOF. Rewrite (2.14) as

$$z(r) = \sigma(r)^{-1} \int_0^r \sigma(s)a(s)ds,$$

where

$$a(s) = p\delta\phi(s)^{1/\alpha-1} - \theta\phi(s)^{-1-\beta/\alpha} - z(s)^2$$

which tends to zero as $s \to \infty$. Then, using l'Hôpital's rule again, we obtain

$$\begin{aligned}\lim_{r\to\infty}[rz(r)] &= \lim_{r\to\infty} \frac{p\delta\phi(r)^{1/\alpha-1} - \theta\phi(r)^{-1-\beta/\alpha} - z(r)^2}{(n-2)r^{-2} + d\phi(r)^{1/\alpha-1}} \\ &= \lim_{r\to\infty} \frac{p\delta - \theta\phi(r)^{-(\beta+1)/\alpha} - (\phi'(r))^2\phi(r)^{-1-1/\alpha}}{(n-2)r^{-2}\phi(r)^{1-1/\alpha} + d} \\ &= p\alpha,\end{aligned}$$

by using (4.4) and (4.5). The proof is complete.

The proof of the following lemma is the same as that of Lemma 3.4 and we omit it.

LEMMA 4.4. *For any $\epsilon > 0$, there are $K = K(\epsilon) > 0$ and $R = R(\epsilon) > 0$ such that*

$$\phi(r) \geq Kr^{p\alpha-\epsilon}, \qquad \forall r > R. \tag{4.7}$$

Finally, we state the main result of this section as follows.

THEOREM 4.5. *The limit, $\lim_{r\to\infty}[r^{-p\alpha}\phi(r)]$, exists and is positive.*

PROOF. Take any positive constant $\lambda < 2(1+\beta)/(\alpha+\beta)$. Note that $\lambda < 2$. Using Lemma 4.4 and (4.6), the theorem can be proved in the same way as Theorem 3.5.

Theorem A for $\alpha > 1$ follows from Propositions 2.1 and 2.2 and Theorem 4.5.

5. Final remarks

The above method can be applied to (1.1) with $\theta = -1$ and $0 < \alpha \leq 1$. More precisely, we can deduce the following theorem. Recall that $p = 2/(\alpha + \beta)$, $\gamma = 1/(\beta + 1)$, and $p\delta = \gamma$.

THEOREM 5.1. *A solution of (1.4) exists if and only if* $w(0) > 0$. *Moreover, for* $w(0) > 0$ *the limit,* $\lim_{r\to\infty}[r^{-p}w(r)]$, *exists and is positive.*

As before, let $g(s) = \gamma s^{1/\alpha} + s^{-\beta/\alpha}$. Hence every solution of (2.1) must be strictly increasing and tends to ∞ as $r \to \infty$. For $s > \eta^\alpha$, since $g(s) \leq K\gamma s^{1/\alpha}$ for some positive constant K, the inequalities (2.10) and (2.12) become

$$\phi'(r) \leq \phi'(r_0) + Kp\alpha\phi(r)/r, \tag{5.1}$$

$$\phi'(r) \leq \phi'(r_0) + Kp\alpha\phi(r)/r_0, \tag{5.2}$$

respectively. Hence Proposition 2.2 holds for any $\eta > 0$.

The results of Section 3 remain the same. Therefore, Theorem 5.1 follows easily in the same way as before.

Acknowledgement

The author thanks the referee and Professor J. M. Hill for some valuable comments. This work was partially supported by National Science Council of the Republic of China under the contract NSC 82-0208-M-007-062.

References

[1] A. Friedman, *Mathematics in industrial problems, Part II* (Springer-Verlag, Heidelberg, 1989).

[2] Y. Giga and R. V. Kohn, "Asymptotically self-similar blow-up of semilinear heat equation", *Comm. Pure Appl. Math.* **38** (1985) 297–319.

[3] Y. Giga and R. V. Kohn, "Characterizing blowup using similarity variables", *Indiana Univ. Math. J.* **36** (1987) 1–40.

[4] J.-S. Guo, "On the quenching behavior of the solution of a semilinear parabolic equation", *J. Math. Anal. Appl.* **151** (1990) 58–79.

[5] J.-S. Guo, "Forward self-similar solutions of a semilinear heat equation", *Bull. Inst. Math. Acad. Sinica* **20** (1992) 119–128.

[6] J.-S. Guo, "Large time behavior of solutions of a fast diffusion equation with source", *Nonlinear Anal.* (to appear) .

[7] J. M. Hill, A. J. Avagliano and M. P. Edwards, "Some exact results for nonlinear diffusion with absorption", *IMA J. Appl. Math.* **48** (1992) 283–304.

[8] A. S. Kalashnikov, "Some problems of the qualitative theory of nonlinear degenerate second-order parabolic equations", *Russian Math. Surveys* **42** (1987) 169–222.
[9] S. Kamin and W. Liu, "Large time behavior of a nonlinear diffusion equation with a source", *IMA Preprint Series # 833* (June 1991).
[10] W. Liu, "A parabolic system arising from film development", *Nonlinear Anal.* **16** (1991) 263–281.
[11] J. B. McLeod, L. A. Peletier and J. L. Vazquez, "Solutions of a nonlinear ODE appearing in the theory of diffusion with absorption", *Differential and Integral Equations* **4** (1991) 1–14.