Discussion on Tangency in Elementary Geometry.

(*Glasgow Meeting, 4th December 1925.*)

The following communication gives a summary of the opening paper by Mr G. Lawson and of the remarks by various members who afterwards took part in the discussion. The question at issue was how to treat definitions and properties of the tangent to a plane curve in teaching beginners.

It is perhaps well to point out that three distinct methods of considering tangents were held in view: (1) the method of *limits,* (2) the method of *coincidence,* (3) the method of *Euclid.*

(1) If P, Q are points on a curve, and the secant PQ is the line PQ produced indefinitely either way, the tangent at P is the limiting position of the secant PQ as Q tends to P.

(2) By this is meant the process of substituting x_1 for x_2 and y_1 for y_2 in the algebraic equation for the secant through points (x_1, y_1) and (x_2, y_2), in any case when the result of the substitution is not identically zero.

(3) This applies primarily to a circle. A straight line is said to be a tangent to a circle, when it meets the circle, and being produced indefinitely either way does not cut it.

Summary of Opening Paper by Mr G. Lawson, Waid Academy, Anstruther.

1. Mr Lawson pointed out that in the Proceedings of the Society, Vol. 22 (1904) Professor Gibson had exposed the radical fault of the proofs of tangency by limits, which were then (1904) appearing in text-books—the fault of regarding a limit as a particular case—and had written "there is absolutely no cogency in the reasoning that is often based on the conception of coincident points and lines." These two pronouncements in the same paper of 1904 may have, in the opinion of the opener, helped to create a tendency which the Mathematical Association Report of 1923 finds prevalent, the tendency to identify the method of limits and the method of coincidence in tangency; and he submitted the question: "Are the two methods distinct, though liable to confusion as $F(a)$ and the limit of $F(x)$ as x tends to a are distinct, and as liable to confusion."

2. Assuming the methods to be distinct, he examined the paper of 1904 to find the pitfalls of the method of coincidence. The pitfall is the assumption that a theorem proved on the express understanding that the points are separate continues to be true when coincidence violates that condition. To find ways of avoiding the pitfall he turns to Gibson and Pinkerton's *Analytical Geometry* (1911), where the tangent is defined by coincidence without a word about limits. There the usual equation of the join of (x_1, y_1) and (x_2, y_2) on the circle $x^2 + y^2 = r^2$ is turned in the usual way into

$$x(x_1 + x_2) + y(y_1 + y_2) = r^2 + x_1 x_2 + y_1 y_2.$$

But before proceeding to coincidence the book inserts this argument:—"It is easy to verify that this equation represents the line through (x_1, y_1) and (x_2, y_2), provided these points lie on the circle; apart therefore from the particular process by which the equation has been reached we know that it represents the required secant." The question was asked whether the saving clause was inserted in 1911 in order to guard against the 1904 pitfall.

3. The opener agreed with the view recorded in the 1904 paper, that the method of limits is a method of reasoning unsuited to beginners, and also with the I.A.A.M. Report, that it is a method unsuited for use by teachers untrained in limits. On the other hand, the method of coincidence is suitable and attractive to both; diagrams like those of the 1911 text-book make clear to them the tangent position of the moving line at coincidence. The pitfall of the method was exposed in 1904; can it be avoided in pure geometry by a circuit corresponding to the 1911 circuit in analytical geometry?

(*a*) If we say—"the line from the centre of a circle perpendicular to the rotating secant passes between P and Q when they are separate, and therefore shifts from one side of P to the other when and only when Q so shifts, and therefore passes through P when Q coincides with P"—have we fallen into the 1904 pit?

(*b*) Twist the argument (in the 1911 way). Through P on a circle let a line be drawn; and let OM be the perpendicular upon it from the centre O; make MQ equal to MP. Then OQ equals OP and Q is on the circle. Is it to fall into the 1904 pit or into another to ask in what position of the line,

Q coincides with P, and to answer, "when OP is the perpendicular"?

4. To define the tangent in analytical geometry by coincidence and to define it in the calculus as a limit, is this not to commit the error of confusing the value $F(a)$ with the limit of $F(x)$? Ought not the second definition to be derived as a theorem from the first?

The following theorem and proof were submitted for criticism. To prove by coincidence that the gradient of the tangent at point $(a, F(a))$ on the curve $y = F(x)$ is $F'(a)$. The theorem of mean value

$$F(a+h) = F(a) + hF'(a+\theta h),$$

reduces the line joining points $(a, F(a))$ and $(a+h, F(a+h))$ to the form $y - F(a) = (x-a)F'(a+\theta h)$. This equation is open to the usual objection that it is established on the understanding that the points are separate. But apart from the particular process of derivation we know by substitution that the equation represents a line meeting the curve at points $(a, F(a))$ and $(a+h, F(a+h))$. At coincidence h is 0 and the gradient is $F'(a)$.

5. The method of coincidence without a word about limits is therefore easy for the beginner and sufficient for the student of the Calculus.

Summary of Remarks by Professor Gibson.

Professor Gibson pointed out, with reference to Mr Lawson's remarks on the article in Vol. 22 of the *Proceedings,* that in it he was dealing solely with certain proofs that professed to be based on the definition of a tangent as a limit while in the textbook by Dr Pinkerton and himself a different definition was used—a definition based on the method of "coincident points" and quite independent of the method of limits. In the former case the distinction between the limit of a function for x tending to a and the value of the function for x equal to a was vital; in the latter the cogency of the proof depended solely on "value" and not at all on "limit." The fact that different definitions of a tangent are employed in different circumstances should hardly be assumed as an admission that either of them must be abandoned in favour of the other.

Whatever method of treating the tangent be adopted it is essential to define or to make clear the property of the line and then to make use of that property in establishing theorems. The criticism in the article in Vol. 22 was directed at the faulty application of the defining property and the opinion was expressed that the method of limits was unsuitable for beginners in geometry. On the other hand, in analytical geometry, the method of coincident points, if adequately explained and illustrated, seems to be both simple and fruitful and to form a useful introduction to work on higher plane curves. At later stages the two methods may be usefully combined; there is no need whatever for adhering rigidly to one definition when the pupil has made considerable progress in his studies. In the first approach to geometry, Euclid's definition seems to be the most suitable; the method of coincident points is not open, when properly expounded, to the criticism directed against certain expositions of the method of limits, but it seems to be more difficult of vigorous treatment than that of Euclid.

Summary of Remarks by Dr Pinkerton, High School, Glasgow.

Dr Pinkerton referred to a passage in the recently published "Life of Lord Rayleigh." The author was looking at a very elementary introduction to the differential calculus, and said that he thought a certain demonstration inconclusive. Lord Rayleigh's reply was: "I daresay it is quite conclusive enough." These two points of view seemed to represent the two attitudes towards any method of treatment of tangents in elementary books. It seemed that intuition as well as logic had a rôle in mathematics, and yet this opinion was probably not at all popular. Dr Pinkerton cited an experience of his own where an able boy could not see his way to believe that it was possible for two lines to be incommensurable even if a proof were given. Chrystal had said that every mathematical book worth anything must be read "backwards and forwards"; his advice to a student in difficulty was this: "Go on but often return to strengthen your faith." This advice should be borne in mind in all mathematical teaching. Teachers naturally liked a method that led straight onwards and never needed revision for a fuller understanding, and learners liked it too. That is one reason why Euclid was supreme for so long in the field of geometry. But methods that were looked on as illegitimate in ancient geometry are now accepted by mathematicians as legitimate. Beginners find a difficulty with some of these modern methods, especially if the logic of the method is insisted on beyond the capacity of the method to stand the strain.

As an example, reference was made to the mode of deriving the property of a tangent from the property of the secant. If P is a point on a curve, say, $y = x^2$, and Q the point whose abscissa is $x_p + h$, the gradient of the secant PQ is $2x_p + h$. A difficulty is often felt when h is put equal to zero, and the gradient of the tangent at P is deduced to be $2x_p$. Now a "family" of secants through P is given by assigning various values to h in the expression $2x_p + h$ (zero being barred). Every member of the "family" is a secant. What about the line through P whose gradient is $2x_p$? It is not a secant. No, because it is the tangent, there being no other value available for the gradient of the tangent. It was claimed that this argument was "quite conclusive enough."

While it was certainly a teacher's business to communicate to his pupils an organised body of definite, clear and accurate knowledge, it was also his business to see that this was done in such a way as would continually stimulate the pupil to work for himself and find a pleasure in understanding and even discovering things for himself. It was one of the advantages of introducing modern ideas into the elementary study of mathematics that students were furnished with a variety of principles that needed more personal exertion to understand and illustrate, that excited the inventive faculties, and intensified the student's pleasure and profit by making him educate himself.

Summary of Remarks by Dr Dougall.

For the beginner, Euclid's definition is still the best: *a tangent to a circle or conic is a straight line which meets the curve in one point, and one point only.* There is no need to go beyond this, either in pure or in co-ordinate geometry, so long as we confine ourselves to curves of the second degree. All the same, the alternative definition using the idea of a limit ought to be introduced at an early stage. I do not think the

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tangent can logically be regarded as a *particular case* of a chord. And the definition of a tangent as a chord through two coincident points must be used with caution when there are double points about. The only safe way is to hold *one point fixed* while the other moves along the curve to coincidence with it.

The method of deducing the equation of the tangent at (x_1, y_1) from the equation of the chord through $(x_1 y_1)$, (x_2, y_2) seems to me to involve a *process*, as distinguished from an *act*, of bringing two points from separation into coincidence, and so to imply the limit definition. I do not see that from the equation of the chord of a circle, say, in Burnside's form

$$x^2 + y^2 - 1 = (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$$

we *prove* any more about the line

$$x^2 + y^2 - 1 = (x - x_1)^2 + (y - y_1)^2$$

than that it passes through (x_1, y_1)—unless we bring in the notion of the limit.

Summary of Remarks by Mr T. P. Black, Leith Academy.

In answer to a request that teachers might give some account of their practice and experience, the speaker outlined the following scheme :—

1. Teach the tangent truths, starting from the definition of a tangent as a straight line perpendicular to the radius. (Euclid's definition follows as an immediate consequence.)

The advantage of this is that pupils are dealing with a perfectly definite straight line.

The limit definition is objectionable here (*a*) since it presents two simultaneous difficulties—new facts by new method; (*b*) since the pupil will almost certainly consider the limit as a particular case.

2. Pass to the idea of Continuity. The aim here is not to use the principle of continuity as a new instrument for the discovery of new truths. The aim is essentially to introduce continuity as a unifying principle.

(*a*) Draw the attention of the pupils to the apparent relation of tangent truths to truths regarding chords and angles, noting that the proof of a chord or angle truth ceases to hold for the corresponding tangent truth.

(*b*) Excite curiosity as to whether we can possibly argue from the one set of truths to the other.

(*c*) Introduce the limit definition of a tangent, being careful to show the approach from both sides.

(*d*) Use the method of limits to connect truths already known, taking only such cases as admit of approach from both sides.

Postscript by Professor Turnbull.

The above summaries of the discussion clearly indicate general agreement that the method of Euclid gives the best introduction to tangency for the beginner. The opener, however, asked several questions, not all of which have received explicit answers in these published notes. Accordingly something should be said here to supplement the omission.

First as to the assumption in § 2 of the Opener's remarks, the general conclusion of the meeting was that the method of *coincidence* is distinct from the method of limits, if the treatment is purely analytical, although Dr Dougall and perhaps others would not entirely agree with this. Everything turns on how the equation of the secant is established. Now unless a geometrical theorem is used to establish it, there appear to be only two possible ways of doing this, one of which involves the removal of a factor which vanishes identically when the point Q is the point P. So this way is obviously excluded, else the method of limits is creeping in. The other way is illustrated by Burnside's equation for the secant already quoted

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = x^2 + y^2 - 1.$$

It is hardly enough to take a pragmatic view and say to the beginner, "Make this your starting point. This is the equation of the secant, as you may verify." For the intelligent student will immediately ask, where does the equation come from? To be sure, its origin is unexceptionable: no vanishing factor has been removed in forming it. But it requires relatively artificial algebra to make it. Also it is difficult to answer the critic that the equation properly belongs to the theory of coaxal circles, the secant in fact being the axis of the circles whose limiting points are P and Q : the equation represents a straight line which is the limiting form of the system of circles through P and Q.

For a straightforward elementary method, free from all hint of the theory of limits, the method of Descartes* seems to be the simplest. The equation of the tangent to a circle or conic can, in each case, be determined as

$$y = mx + f(m)$$

where $f(m)$ is immediately given by the condition that this straight line should have one point only in common with the curve.

It seems worth while following up the remark of Dr Dougall by further suggesting that the method of coincidence is of most value, and is of practical teaching value, when it is regarded as the second step in the operation of finding a limit, quite apart from how the first step has been surmounted. The analagous steps in the process, say, of differentiating x^2 are first the step leading to $2x + h$, and secondly the step leading from $2x + h$ to the limit $2x$. The student is comfortably placed in position for this second step, and then finds no difficulty in taking it; in fact he finds it very attractive, for he is not pre-occupied with difficulties as to the correctness of the first step.

As to the further questions of the Proposer, we venture to say that the answers to § 3 (*a*) and (*b*) are No, but the method is that of limits.

As to the theorem of § 4, this example seems to be of value if used in the spirit suggested by the words of Professor Chrystal quoted above. Regarded as geometry it is quite a satisfactory illustration, provided the mean value theorem is assumed. But how is this last to be proved? For a beginner an analytical proof is difficult, whereas a geometrical proof involves the conclusions it seeks to secure.

**i.e.* The method using the condition that a quadratic equation should have equal roots. Descartes first used this method in geometry.