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UNIQUELY LINE COLORABLE GRAPHS

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1. **Introduction.** A line-coloring of a graph G is an assignment of colors to the lines of G so that adjacent lines are colored differently; an *n-line coloring* uses n colors. The *line-chromatic number* $\chi'(G)$ is the smallest n for which G admits an n-line coloring. Vizing [6] has shown that $\chi'(G)$ is either $\Delta(G)$ or $1+\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of the points of G. Each $\chi'(G)$-line coloring of G partitions the line set of G into $\chi'(G)$ subsets, called *line-color classes*, two lines belonging to the same subset if and only if they are colored the same. If $\chi'(G)=n$ and every two n-line colorings induce the same partition, then G is said to be *uniquely n-line colorable*. For example, the complete bipartite graph $K_{1,n}$ is uniquely n-line colorable. The analogous concept for point coloring was introduced by Cartwright and Harary [2]. Uniquely point colorable graphs were also investigated in [3], [4]. The main object of this note is to prove that every uniquely n-line colorable graph G has $\Delta(G)=n$ unless G is K_3, the complete graph on three points.

2. **Uniquely n-line colorable graphs.** In this section we develop some of the basic properties of uniquely n-line colorable graphs.

THEOREM 1. *If G is uniquely n-line colorable and C is some line-color class, then $G-C$ is uniquely $(n-1)$-line colorable.*

Proof. The graph $G-C$ must have only one $(n-1)$-line coloring, since every $(n-1)$-line coloring of $G-C$ can be extended to an n-line coloring of G by coloring the lines in C with the nth color.

COROLLARY 1.1. *If G is uniquely n-line colorable and $C_1, C_2, \ldots, C_n$ are its line-color classes, then the subgraph induced by $\bigcup_{i=1}^{k} C_i$, $k\leq n$, is uniquely k-line colorable.*

Proof. By Theorem 1, $G-C_n$ is uniquely $(n-1)$-line colorable, and furthermore it is clear that C_{n-1} is a line-color class of $G\dot{-}C_n$. Applying Theorem 1 again we have $[G-C_n]-C_{n-1}$ is uniquely $(n-2)$-line colorable. In general, by applying Theorem 1 $(n-k)$-times we see that $G-\bigcup_{i=k+1}^{n}$ is uniquely k-line colorable.

The next corollary is the analogue to Theorem 4 of [2].

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COROLLARY 1.2. *If G is uniquely n-line colorable, then the subgraph induced by the union of any two line-color classes is connected.*

Proof. The induced subgraph S formed by the union of any two line-color classes is uniquely 2-line colorable by Corollary 1.1. Therefore S must be connected. In fact, since no point of S can have degree larger than two, S must either be a path or a cycle of even length.

In [3], it was shown that every uniquely n-point colorable graph is $(n-1)$-connected. In order to state a corresponding result for uniquely n-line colorable graphs, we need to place a restriction on the minimum degree $\delta(G)$ of the points of the graph.

THEOREM 2. *If $\delta(G)\geq n-1$ and G is uniquely n-line colorable, then G is $(n-1)$-line connected.*

Proof. We observe first that G is uniquely n-line colorable if and only if its line graph $L(G)$ is uniquely n-point colorable. If the removal of fewer than $n-1$ lines disconnects G and $\delta(G)\geq n-1$, then the components of this disconnected graph each contain at least one line. Therefore the removal of the corresponding points in $L(G)$ must disconnect $L(G)$. This, however, contradicts the fact that $L(G)$ is $(n-1)$-connected.

COROLLARY 2.1. *If G is uniquely n-line colorable, $\delta(G)\geq n-1$, and C is a line-color class, then $G-C$ is $(n-2)$-line connected.*

Proof. Since $\delta(G)\geq n-1$ and C is a line-color class $\delta(G-C)\geq n-2$. The result now follows directly from Theorems 1 and 2.

One of the chief results of [4] (see also [5]) is that for all $n\geq 3$ there exists a uniquely n-point colorable graph which contains no subgraph isomorphic to K_n. This result suggests the conjecture that for all $n\geq 3$, there exists a uniquely n-line colorable graph G with $\Delta(G)=n-1$. However, the final and main theorem of this section shows that only K_3 has this property.

THEOREM 3. *Every uniquely n-line colorable graph $G\neq K_3$ has $\Delta(G)=n$.*

Proof. Suppose G is uniquely n-line colorable. By Vizing's theorem, we know that $n=\Delta(G)$ or $n=\Delta(G)+1$. Assume $n=\Delta(G)+1$. Let v be a point of G having degree $n-1$. Consider an n-line coloring of G with the colors $1, 2, \ldots, n$. We can assume that the color n is not used in coloring the lines incident to v and all other colors are. Let C_i denote the lines of G colored i, $1\leq i\leq n$. As noted in the proof of Corollary 1.2, C_n and C_i, $1\leq i\leq n-1$, together induce either a path or a cycle of even length. Denote this graph by S_i. Since v is a point of S_i and no line incident to v is colored n, S_i must be a path.

We now show that each S_i contains exactly n points. Each line incident to v has to be adjacent to some line colored n. There are $n-1$ such lines, and no two of them can be incident to the same point. Thus there are $(n-1)$-distinct points each incident to some line colored n. All of these points and v are in S_i. Suppose some S_i, say S_1, has $k>n$ points. Denote the points of this path by $v_1, v_2, \ldots, v_k$. Since v is always an endpoint of each S_i, we may assume that $v=v_1$ and at least one of the points $v_2, v_3, \ldots, v_n$ is not an endpoint of any S_i. Call this point w. Since w is incident to a line colored n and is not an endpoint of any S_i, the degree of w in each S_i is two. But this means that w is incident to some line colored i for all $i=1, \ldots, n$. This is impossible, however, since $\Delta(G)=n-1$. Hence $|V(S_i)|=n$. Furthermore, n must be odd; otherwise, there would only be $n-2$ points incident to lines colored n and we have just shown that we need at least $n-1$ such points.

Now since n is odd we have $V(S_i)=\{v\} \cup \{u:u \text{ is incident to a line colored } n\}$ for all i. That is, $V(S_i)=V(S_j)$ for any $i, j=1, \ldots, n-1$. Since $G=\bigcup_{i=1}^{n-1} S_i$, we must have $|V(G)|=n$. Each of the n points of G must be endpoints of some S_i. Otherwise we could show, as we did with w in the preceding paragraph, that its degree was too large. Since v is always an endpoint and there are exactly $n-1$ paths S_i being considered, each point other than v is an endpoint in exactly one of the S_i. So each point other than v has degree two in all but one of the S_i. This forces G to be $(n-1)$-regular; that is, G must be K_n.

It remains to show that G has to be K_3. Suppose $G=K_n$ for n odd, $n\geq 5$. If the points of G are labeled $1, \ldots, n$ then we can obtain two distinct line-colorings as follows: For one line-coloring take as line-color classes $C_P=\{(p-q, p+q):q=1, \ldots, (n-1)/2\}$ for $p=1, \ldots, n$. where each of the numbers $p-q$ and $p+q$ is expressed as one of the numbers $1, 2, \ldots, n$ modulo n. Another distinct line-coloring can be obtained by relabeling the points labeled 1, 2, 3 by 3′, 1′, 2′ respectively and using the same scheme. In this second coloring the line (1′, 3′) is colored 2 and is the same line as (2, 1) in the original labeling. But (2, 1) was not colored 2 in the first coloring since (1, 3) was colored 2. Furthermore $(n, 4)$ is colored 2 in both colorings. Hence we have at least two distinct line-colorings of K_n for odd $n\neq 3$. Hence the only possible graph is K_3 which is uniquely 3-line colorable. This completes the proof of Theorem 3.

COROLLARY 3.1. *Every uniquely n-line colorable regular graph is Hamiltonian.*

Proof. Let G be a uniquely n-line colorable regular graph. If $G=K_3$, then it is Hamiltonian. If $G\neq K_3$, then, by Theorem 3, G is n-regular. Therefore each point of G is incident to n lines all of which have to be colored differently. Hence the union of two line-color classes is a connected spanning 2-regular subgraph; i.e., a Hamiltonian cycle.

3. **Uniquely 3-line colorable cubic graphs.** We now consider briefly the special case of cubic graphs. It follows from Theorem 3 that there does not exist a uniquely

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4-line colorable cubic graph. An infinite family of uniquely 3-line colorable cubic graphs can be constructed by repeatedly applying the next theorem.

THEOREM 4. *If G is a uniquely 3-line colorable cubic graph and H is a cubic graph obtained from G by replacing a point of G with a triangle, then H is uniquely 3-line colorable.*

Proof. This result follows from the observation that each 3-line coloring of H induces a 3-line coloring of G.

As an example, we note that K_4 is uniquely 3-line colorable. Hence, the 3-prism obtained from K_4 by replacing one of its points with a triangle is uniquely 3-line colorable. It is also easy to see that if $G \neq K_4$ is a uniquely 3-line colorable cubic graph and H is a graph obtained from G by identifying three points of G which induce a triangle, then H is uniquely 3-line colorable.

Each uniquely 3-line colorable cubic graph known to the authors is planar and contains a triangle. This leads us to make the following conjecture, which is related to a conjecture of Kotzig (see [1, Problem 1]).

CONJECTURE 1. If G is a uniquely 3-line colorable cubic graph, then G is planar and contains a triangle.

In connection with Conjecture 1, it is not hard to show that if there exists a nonplanar uniquely 3-line colorable cubic graph, then there exists a nonplanar uniquely 3-line colorable cubic graph containing no triangle.

Our final theorem shows that uniquely 3-line colorable cubic graphs are Hamiltonian in a very special way.

THEOREM 5. *Every uniquely 3-line colorable cubic graph contains exactly three Hamiltonian cycles.*

Proof. Let G be a uniquely 3-line colorable graph. As we observed in the proof of Corollary 3.1, the union of any two line-color classes in a 3-line coloring of G induces a Hamiltonian cycle. Hence G contains at least three Hamiltonian cycles. If there were a fourth Hamiltonian cycle in G, then another line-coloring of G could be produced by coloring the lines of the cycle with two colors and the remaining lines with the third color.

Since Kotzig (see [1, Theorem 2]) has shown that every cubic bipartite graph has an even number of Hamiltonian cycles, Theorem 5 implies that every uniquely 3-line colorable cubic graph contains an odd cycle.

We conclude with a conjecture, which we suspect is a good deal easier than Conjecture 1.

CONJECTURE 2. If G is a cubic graph containing exactly three Hamiltonian cycles, then G is uniquely 3-line colorable.

References

1. J. Bosak, *Hamiltonian lines in cubic graphs*, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, New York, (1967), 35–46.

2. D. Cartwright and F. Harary, *On the coloring of signed graphs*, Elem. Math. **23** (1968), 85–89.

3. G. Chartrand and D. P. Geller, *On uniquely colorable planar graphs*, J. Combinatorial Theory, **6** (1969), 271–278.

4. F. Harary, S. T. Hedetniemi, and R. W. Robinson, *Uniquely Colorable Graphs*, J. Combinatorial Theory, **6** (1969), 271–278.

5. ———, Errata, J. Combinatorial Theory, **9** (1970), p. 221.

6. V. G. Vizing, *A bound on the chromatic class of a p-graph*, Diskret Analiz., **3** (1964), 25–30.

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