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ON THE PRINCIPLE OF DEPENDENT CHOICES AND SOME FORMS OF ZORN's LEMMA

BY

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ABSTRACT. The main result of this paper is to prove that a generalization of the Principle of Dependent Choices, introduced by A. Levy [2; see also 1, Chapter 8], is equivalent to a form of Zorn's Lemma.

The *Principle of Dependent Choices* [1, Section 2.4] is a weak version of the Axiom of Choice (AC), and may be stated as follows.

DC: Let X be a non-empty set. If R is a relation with dom $R = X$ and range $R \subseteq X$, then there exists a sequence $\{x_n, n < \omega\}$ such that $x_n R x_{n+1}$ for all $n < \omega$.

We first show, without any use of AC, that the statement DC is equivalent to the following weak form of Zorn's Lemma.

Z_ω: If every chain in a partially ordered set P is finite, then P contains a maximal element.

Proof. $DC \Rightarrow Z_\omega$. Suppose P satisfies the hypothesis of Z_ω but contains no maximal element. Define a relation R on the set P by xRy if and only if $x < y$, for $x \in P$, $y \in P$. By DC, there exists a sequence $\{x_n, n < \omega\}$ with $x_0 < x_1 < x_2 < \cdots$, a contradiction.

$Z_\omega \Rightarrow DC$. Let R be a relation with dom $R = X$ and range $R \subseteq X$. Let S be the set of all finite sequences $s = \{x_0, \ldots, x_k\}$ of elements of X such that $x_0 R x_1 R \cdots R x_k$. Partially order S by defining $s < t$ if and only if dom s is an initial segment of dom t and $s(i) = t(i)$ for all $i \in \text{dom } s$. By the hypothesis of DC, S has no maximal element. Hence by Z_ω there exists an infinite chain C in S. Then $\bigcup\{s : s \in C\}$ is an infinite sequence $\{x_n, n < \omega\}$ with $x_n R x_{n+1}$ for all $n < \omega$.

The purpose of this note is to show that a generalized form of DC, introduced by A. Levy [2; see also 1, Chapter 8], is equivalent to a corresponding generalization of Z_ω. By a *sequence of type* γ, where γ is any ordinal, we mean any function defined on the set γ. The following proposition, as we shall show, may be regarded as a generalisation of DC. Here and throughout this paper λ will denote an initial ordinal (aleph).

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DC_λ: Let X be any non-empty set and $S_\lambda(X)$ the set of all sequences in X of type less than λ. If R is a relation with dom $R = S_\lambda(X)$ and range $R \subseteq X$, then there exists a sequence f of type λ such that, for, all $\alpha < \lambda$,

$$(f \mid \alpha) R f(\alpha).$$

In the statement of DC_λ it is assumed that $S_\lambda(X)$ contains the empty set $\varnothing$, which may be considered as a sequence of type 0. A. Levy has shown that DC_μ implies DC_λ whenever $\lambda < \mu$, and also that the statement $(\forall\lambda)$ DC_λ is equivalent to AC [1, p. 120].

Let us say that a partially ordered set P is *well-founded* if and only if every chain in P is well-ordered. We consider two propositions related to Zorn's Lemma:

Z_λ: Let P be a partially ordered set in which every well-ordered chain has type less than λ. If every well-ordered chain in P has an upper bound in P, then P contains a maximal element.

Z_λ^*: Let P be a well-founded partially ordered set in which every chain has type less than λ. If every chain in P has an upper bound in P, then P contains a maximal element.

Our purpose is to prove (without AC) that for each initial ordinal λ, the statements DC_λ, Z_λ, and Z_λ^* are all equivalent.

If s is a sequence in a partially ordered set P, we say that s is *strictly increasing* if and only if $\alpha < \beta$ implies $s(\alpha) < s(\beta)$ for all α, β in dom s. In this case the range of s is a well-ordered chain in P.

We now prove our main result.

THEOREM 1. *For each initial ordinal λ, the statements DC_λ, Z_λ, and Z_λ^* are equivalent.*

Proof. Since Z_λ trivially implies Z_λ^*, it will be sufficient to prove that $Z_\lambda^* \Rightarrow DC_\lambda \Rightarrow Z_\lambda$.

$Z_\lambda^* \Rightarrow DC_\lambda$. Let X be a non-empty set, λ an initial ordinal, and R a relation with dom $R = S_\lambda(X)$ and range $R \subseteq X$. Let us say that a sequence s in X is *R-admissible* if and only if (i) $(s \mid \alpha) \in S_\lambda(X)$ for all $\alpha \in \text{dom } s$, and (ii) $(s \mid \alpha)$ $Rs(\alpha)$ for all $\alpha \in \text{dom } s$. We must show that the set P of all R-admissible sequences contains a member which is of type λ. For $s \in P$, $t \in P$, define $s < t$ if and only if dom s is an initial segment of dom t, and $s(\alpha) = t(\alpha)$ for all $\alpha \in \text{dom } s$. The set P is a well-founded partially ordered set with respect to the relation $\leq$. Note that for any chain C in P, $\bigcup\{s\colon s \in C\}$ is an R-admissible sequence and hence is an upper bound of C in P. Suppose P contains no member of type λ. Then there is no chain C in P of type λ; because then $\bigcup\{s\colon s \in C\}$ would be a sequence of type λ. So P satisfies the hypothesis of Z_λ^*, and we therefore conclude that P contains a maximal element t, which by assump-

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tion has type $\beta<\lambda$. By the hypothesis of DC_λ there exists $y\in X$ with tRy. Define a sequence t^* by

$$t^*(\alpha)=t(\alpha) \quad \text{for} \quad \alpha<\beta,$$
$$t^*(\beta)=y.$$

Then t^* is R-admissible but $t<t^*$, contradicting the maximality of t.

$DC_\lambda \Rightarrow Z_\lambda$. Suppose P satisfies the hypothesis of Z_λ and P contains no maximal element. Let $B(P)$ be the set of all $s\in S_\lambda(P)$ such that

(i) s is strictly increasing, and

(ii) $P-$range s contains an upper bound of range s.

Define a relation R as follows:

(1) if $s\in B(P)$, then sRy if and only if $s(\alpha)<y$ for all $\alpha\in\text{dom}\, s$,

(2) if $s\in S_\lambda(P)-B(P)$, then sRy for all $y\in P$.

By DC_λ, there exists a sequence f of type λ with $(f\mid\alpha)Rf(\alpha)$ for all $\alpha<\lambda$. We assert that f is strictly increasing. For suppose not. Let β_0 be the first ordinal less than λ for which there exists $\beta<\beta_0$ with $f(\beta)\nless f(\beta_0)$. Then $f\mid\beta_0$ is a strictly increasing sequence and $A=\text{range}(f\mid\beta_0)$ is well-ordered. If A contains a greatest element m, then since m is not a maximal element of P (by our assumption on P), it follows that $f\mid\beta_0\in B(P)$. Hence, by definition of R, we have $f(\beta)<f(\beta_0)$ for all $\beta<\beta_0$: a contradiction. If A has no greatest element, then since A has an upper bound in P, we again have $f\mid\beta_0\in B(P)$ and $f(\beta)<f(\beta_0)$ for all $\beta<\beta_0$: again a contradiction. Hence f is strictly increasing.

However, the above result implies that range f is a well-ordered chain in P of type λ, contradicting the hypothesis on P and completing the proof of the theorem.

As a consequence of Levy's result that the statement $(\forall\lambda)\ DC_\lambda$ is equivalent to AC, we have the following corollary of Theorem 1.

THEOREM 2. *Each of the statements* $(\forall\lambda)Z_\lambda$ *and* $(\forall\lambda)Z^*_\lambda$ *is equivalent to* AC.

As a final comment, it follows that the statement $(\forall\lambda)Z^*_\lambda$ is equivalent to the usual form of Zorn's Lemma.

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