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EXPONENTIAL FAMILIES AND GAME DYNAMICS

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A symmetric game consists of a set of pure strategies indexed by $\{0, \ldots, n\}$ and a real $(n + 1) \times (n + 1)$ payoff matrix (a_{ij}). When two players choose strategies i and j the payoffs are a_{ij} and a_{ji} to the i-player and j-player respectively. In classical game theory of Von Neumann and Morgenstern [**16**] the payoffs are measured in units of utility, i.e., desirability, or in units of some desirable good, e.g. money. The problem of game theory is that of a rational player who seeks to choose a strategy or mixture of strategies which will maximize his return.

In evolutionary game theory of Maynard Smith and Price [**13**] we look at large populations of game players. Each player's opponents are selected randomly from the population, and no information about the opponent is available to the player. For each one the choice of strategy is a fixed inherited characteristic. The payoffs are measured in units of Darwinian fitness, i.e., net reproduction rate. The problem of evolutionary game theory is to describe the strategy or mixture of strategies toward which the population will evolve and which are stable under small perturbations due to mutation and sampling drift.

The state of the population is described by the distribution vector x of the strategies in it. x_i is the proportion of i-players. So x lies in the simplex

$$\Delta = \{x \in \mathbf{R}^{n+1}: x_i \geqq 0 \quad \text{and} \quad \textstyle\sum_i x_i = 1\}.$$

A distribution is called *interior* if all $n + 1$ of the strategies occur. So the set of interior distributions is

$$\mathring{\Delta} = \{x \in \Delta: x_i > 0 \quad \text{for all } i\}.$$

The mean payoff to an i-player when the population is in state x is $a_{ix} = \sum_j a_{ij}x_j$ because his opponent is a j-player with probability x_j. So the mean payoff for the entire population is a_{xx} where, in general, we define

$$a_{xy} = \sum_i x_i a_{iy} = \sum_{i,j} x_i a_{ij} y_j.$$

Maynard Smith and Price called strategy i an evolutionarily stable strategy (ESS) when for all competing j (a) $a_{ii} \geqq a_{ji}$ and if equality holds in (a), then (b) $a_{ij} > a_{jj}$. This means that a population of i-

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strategists is buffered against invasion by or mutation to a small number of j-strategists. This is because most contests will still be against the dominant i-strategy and here the i-players do better by (a). If the mutants do equally well against i then the second order effect of (b) will still serve to eliminate them.

By analogy one can generalize to consider a mixed state which is evolutionarily stable against alteration by invasion, mutation or sampling drift. $x \in \Delta$ is an ESS if for all $y \neq x$

(a) $a_{xx} \geqq a_{yx}$

and if equality holds in (a) for some y then

(b) $a_{xy} > a_{yy}$.

Implicitly we are thinking of some population dynamic with respect to which an ESS is a stable equilibrium. This dynamical system was explicitly defined by Taylor and Jonker [**15**]. Interpreting the payoffs as net reproductive rates they derived the equation:

$$(0.1) \qquad \frac{dx_i}{dt} = [a_{ix} - a_{xx}]x_i.$$

This says that the relative rate of increase of the proportion of i players is the average advantage that an i player has over a random member of the population.

If x is an ESS then for all i either $a_{ix} = a_{xx}$ or $x_i = 0$, for otherwise, $a_{xx} = \sum_i x_i a_{ix} = a_{xx}$. Thus, an ESS is an equilibrium for (0.1). Combined work in [**15**], [**8**] and [**17**] proved that an ESS is indeed locally stable, i.e., an attracting equilibrium. However, the converse is not true. This means that even for a generic class of games obtained by throwing away such structurally unstable examples as the paper-rock-scissors game, there exist examples with no ESS. This can happen because the attracting equilibria fail to be ESS. It can even happen that there are no attracting equilibria but that the attractors which occur are more complicated, e.g. limit cycles (see [**17**]). These results raised the question of the significance of the difference between an attracting equilibrium and an ESS.

In [**6**] Hines attacked a different problem which he then showed in [**7**] to be closely related to this question.

In classical game theory the players may choose to use a randomized mix of strategies rather than some pure strategy. A mixed strategy can also be regarded as an element x of Δ where x_i is now the probability that the player uses strategy i. The pure strategies are then identified with the vertices of the simplex. Suppose that each player of the evolutionary game uses some mixed strategy. Now the state of the population is described by a Borel probability measure π on the simplex Δ. If the number of different strategy types is small compared with the size of the population then π will be concentrated on some finite subset of Δ. On the other hand if each strategy type occurs only a small number of times in a large population then a nonatomic measure or one with atoms only at the vertices will be more appropriate. This is true if we suppose that mutation can act on the mixed strategies to alter slightly the proportions in the mix.

For any Borel probability measure π on Δ we define the mean vector $\hat{x} = \hat{x}(\pi)$ by:

$$(0.2) \qquad \hat{x} = \int_\Delta x\pi(dx).$$

Hines derived a differential equation whose solution is the path of the mean strategy $\hat{x}(\pi_t)$ as the population state π_t evolves. He showed that if x is an equilibrium for the pure strategy dynamic (0.1) then any population state π with mean x is an equilibrium for the mixed strategy dynamic. Furthermore, he proved that the mean x is stable with respect to all mixed strategy perturbations π if and only if x is an ESS.

Hines' dynamic is complicated by the fact that the evolution of the mean of π depends on the second moment or covariance of π whose evolution depends in turn on still higher moments. Zeeman, in [**18**], reinterpreted Hines' work by looking directly at the dynamic on the infinite dimensional space of measures on Δ. His version of Hines' equation is:

$$(0.3) \qquad \frac{d\pi(dx)}{dt} = [a_{x\pi} - a_{\pi\pi}]\pi(dx).$$

Here $a_{x\pi}$ is the average payoff to the x-player when the population is in state π:

$$(0.4) \qquad a_{x\pi} = \int_\Delta a_{xy}\pi(dy) = a_{x\hat{x}}$$

with $\hat{x} = \hat{x}(\pi)$. Similarly, $a_{\pi\pi} = a_{\hat{x}\hat{x}}$.

Heuristically (0.3) means just as before that the relative increase in density of the x strategy is given by its average advantage over the mean. Formally, $d\pi/dt$ is the signed measure with density function relative to π given by the function of x in brackets.

The purpose of this paper is to show the relation between the ostensibly infinite dimensional dynamics of (0.3) and the finite dimensional dynamics of (0.1).

We prove:

(1) The space of measures can be naturally (i.e., independent of a_{ij}) foliated into submanifolds invariant under the flow of (0.3). In statistical terminology each submanifold is an exponential family of probability measures on Δ. The map $\pi \to \hat{x}(\pi)$ is a diffeomorphism of each sub-

manifold onto the interior of a closed convex subset of Δ where the interior is taken with respect to the affine subspace spanned by the convex set. In particular, each submanifold has dimension $\leqq n$.

Thus, the behavior of the mean path $\hat{x}(\pi_t)$ together with the initial distribution π_0 completely determine the state π_t.

(2) Consider the differential form on Δ:

$$\theta = \sum_{i,j} (dx_i)a_{ij}x_j = \sum_i a_{ix}dx_i. \tag{0.5}$$

Each of the invariant submanifolds of (1) carries a natural Riemannian metric with respect to which the vectorfield of (0.3) is dual to θ when the invariant manifold is identified with a subset of Δ by the map $\pi \rightarrow \hat{x}(\pi)$.

To understand the significance of the form θ, we return to the pure strategy equation (0.1). The associated vectorfield X_0 on Δ is defined by:

$$X_0(x)_i = x_i[a_{ix} - a_{xx}]. \tag{0.6}$$

Note that X_0 maps into $(\mathbf{R}^{n+1})_0 = \{v \in \mathbf{R}^{n+1}: \sum_i v_i = 0\}$ and so is tangent to Δ at every point.

On $\mathring{\Delta}$ Shahshahani [**14**] defined a Riemannian metric by:

$$(u, v)_x = \sum_i x_i^{-1}u_iv_i. \tag{0.7}$$

If v is in the tangent space of $\mathring{\Delta}$, i.e., $v \in (\mathbf{R}^{n+1})_0$, then

$$(X_0(x), v)_x = \sum_i (a_{ix} - a_{xx})v_i = \sum_{ij} v_ia_{ij}x_j = \theta_x(v). \tag{0.8}$$

So the vectorfield X_0 is dual to θ with respect to the Shahshahani metric.

Thus, before the payoff matrix a_{ij} is chosen the space of measures is foliated by finite dimensional Riemannian submanifolds. The different leaves can be identified with a subset of Δ and so with one another by the map $\hat{x}$. However, the Riemannian metrics vary from leaf to leaf. The payoff matrix then defines the form θ. The vectorfield of (0.3) is dual to θ with respect to the metric on each leaf. Given a fixed form the vectorfields dual with respect to various Riemannian metrics can have rather different dynamical behavior. We will see this when we study Hines' characterization of ESS.

In the next two sections we describe the differential geometry of spaces of measures related by bounded density functions. I see this work as part of a growing bridge between differential geometry and statistics, compare [**3**] and [**4**]. In the third section we erect our study of evolutionary games on these foundations.

For the measure theory needed below we will follow [**5**]. In general, we look only at finite measures and signed measures, using π_1, $\tilde{\pi}$ etc. to denote measures (not necessarily probability measures) and μ, ν etc. to denote possibly signed measures. For the definitions from differential topology needed below, e.g. tangent space, foliation etc. we will follow [**10**]. However, a less forbidding treatment of these elementary definitions occurs in [**1**, Section I.3].

1. Manifolds of measures. Consider a fixed measurable space, M. Two measures π_1 and π_2 are called equivalent (written $\pi_1 \sim \pi_2$) if they have the same sets of measure 0. In general, if π and μ are a measure and a signed measure on M then $\pi > \mu$ means π measure zero implies μ measure zero. If $\pi > \mu$ then the Radon-Nikodym derivative $d\mu/d\pi$ is defined [**5**, Section 31]. This is the unique (a.e. π) measurable real function such that for any bounded measurable $f: M \to \mathbf{R}$

$$\int f(x)\mu(dx) = \int f(x) \frac{d\mu}{d\pi}(x)\pi(dx). \tag{1.1}$$

From the uniqueness easily follows the chain rule:

$$\frac{d\mu}{d\pi_1} = \frac{d\mu}{d\pi_2} \cdot \frac{d\pi_2}{d\pi_1} \text{ (a.e. } \pi_1) \tag{1.2}$$

where $\pi_1 \sim \pi_2 > \mu$.

Let C be a locally compact metric space. If $F: M \to C$ is Borel measurable then F is called *essentially bounded* (rel π) if there exists a compact subset C_0 of C such that

$$\pi(M - F^{-1}(C_0)) = 0,$$

i.e., $F(x) \in C_0$ a.e. For an essentially bounded map the *essential image*, ess. Im F, is:

$$\begin{aligned}\text{ess. Im } F &= \cap \{C_0 \subset C: C_0 \text{ is compact and } F(x) \in C_0 \text{ a.e.}\}\\ &= \text{the smallest compact subset } C_0 \text{ of } C \text{ such that } F(x) \in C_0 \text{ a.e.}\end{aligned}$$

In particular, if $M = C$ is compact then the essential image of the identity map is called the *support* of π.

If $C = \mathbf{R}$ the vector space of essentially bounded functions is denoted $L^\infty(\pi)$. It is a Banach space with respect to the norm:

$$\|f\| = \text{ess. sup } |f| = \sup (\text{ess. Im } |f|).$$

If $\pi_1 \sim \pi_2$ then the concepts of essential boundedness and essential image rel π_1 and π_2 agree. In particular, the support of equivalent measures on a compact metric space agree. Also the Banach space $L^\infty(\pi)$ is independent of the choice of measure π in an equivalence class.

Now we sharpen the ordering $>$ by defining

$$\pi \gg \mu \quad \text{if } \pi > \mu \quad \text{and} \quad \frac{d\mu}{d\pi} \in L^\infty(\pi).$$

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The corresponding equivalence relation is:

$$\pi_1 \approx \pi_2 \quad \text{if } \pi_1 \sim \pi_2 \quad \text{and} \quad \frac{d\pi_1}{d\pi_2}, \frac{d\pi_2}{d\pi_1} = \left(\frac{d\pi_1}{d\pi_2}\right)^{-1} \in L^\infty$$

or equivalently

$$\pi_1 \approx \pi_2 \quad \text{if } \pi_1 \sim \pi_2 \quad \text{and} \quad \ln \frac{d\pi_1}{d\pi_2} \in L^\infty.$$

Here ln is the natural logarithm. L^∞ denotes $L^\infty(\pi_1) = L^\infty(\pi_2)$. Note that transitivity of $\gg$ and $\approx$ follow from the chain rule (1.2).

From now on let $\mathfrak{P}$ be a fixed $\approx$ equivalence class of measures on M, L^∞ be the associated Banach space of essentially bounded real valued functions and

$$\mathfrak{M} = \{\mu : \mu \text{ is a signed measure with } \pi \equiv \mu \text{ for } \pi \in \mathfrak{P}\}.$$

Note that $\mathfrak{P} \subset \mathfrak{M}$.

$\mathfrak{P}$ is a complete metric space with respect to the distance:

$$(1.4) \qquad d(\pi_1, \pi_2) = \left\| \ln \frac{d\pi_1}{d\pi_2} \right\|.$$

To see that d is a metric we define, for $\pi_0 \in \mathfrak{P}$:

$$(1.5) \qquad \begin{aligned} &J_{\pi_0} : \mathfrak{P} \rightarrow L^\infty \quad J_{\pi_0}(\pi) = \ln \frac{d\pi}{d\pi_0} \\ &F_{\pi_0} : \mathfrak{M} \rightarrow L^\infty \quad F_{\pi_0}(\mu) = \frac{d\mu}{d\pi_0}. \end{aligned}$$

1. PROPOSITION. *For every $\pi \in \mathfrak{P}$, J_π is an isometry of $\mathfrak{P}$ onto L^∞. The set of charts $\{(L^\infty, J_\pi^{-1}) : \pi \in \mathfrak{P}\}$ is an atlas with respect to which $\mathfrak{P}$ becomes a smooth manifold modeled on L^∞. In other words, we regard the map J_π^{-1} as a global coordinate chart on the manifold $\mathfrak{P}$.*

For every $\pi \in \mathfrak{P}$, F_π is a linear isomorphism of $\mathfrak{M}$ onto L^∞. The maps F_π induce equivalent norms on $\mathfrak{M}$ with respect to which $\mathfrak{M}$ becomes a Banach space.

The inclusion map $i: \mathfrak{P} \rightarrow \mathfrak{M}$ is a smooth diffeomorphism onto an open subset.

Identify the linear spaces $\mathfrak{M}$ and L^∞ with their own tangent spaces at each point. At each point $\pi \in \mathfrak{P}$ identify the tangent space of $\mathfrak{P}$ at π, i.e., $T_\pi\mathfrak{P}$ with $\mathfrak{M}$ via the inclusion map i. With these identifications we have, for π, $\pi_0 \in \mathfrak{P}$:

$$(1.6) \qquad F_\pi = (T_\pi J_{\pi_0})^{-1} : \mathfrak{M} \rightarrow L^\infty,$$

where $T_\pi J_{\pi_0}$ is the tangent map of J_{π_0} at π.

Proof. By the chain rule:

$$(1.7) \qquad J_{\pi_0}(\pi_1) - J_{\pi_0}(\pi_2) = \ln \frac{d\pi_1}{d\pi_2} \qquad \pi_0, \pi_1, \pi_2 \in \mathfrak{P}.$$

From this the isometry result is clear. The inverse map $J_{\pi_0}{}^{-1}: L^\infty \to \mathfrak{P}$ is given by

$$(1.8) \qquad J_{\pi_0}{}^{-1}(f) = e^f \pi_0.$$

Consequently,

$$J_{\pi_1} \circ J_{\pi_0}{}^{-1}(f) = f + \ln \frac{d\pi_0}{d\pi_1}.$$

So the transition map $J_{\pi_1} \circ J_{\pi_0}{}^{-1}: L^\infty \to L^\infty$ is translation by $\ln (d\pi_0/d\pi_1)$. Its tangent map is the identity and $\mathfrak{P}$ is an affine manifold modeled on L^∞.

$F_{\pi_0}: \mathfrak{M} \to L^\infty$ is an isomorphism with inverse:

$$(1.9) \qquad F_{\pi_0}{}^{-1}(f) = f\pi_0.$$

This identification of $\mathfrak{M}$ with L^∞ maps $\mathfrak{P}$ onto

$$L_+ = \{f \in L^\infty : f \geqq \epsilon \text{ a.e. for some } \epsilon > 0\}.$$

L^∞ is a Banach algebra and we claim that L_+ is the component of the identity, 1, in the group of units of L^∞. L_+ is clearly open and convex. It is a subgroup of the group of units. Since the union of the other cosets is open, L_+ is open and closed in the group of units.

$$(1.10) \qquad F_\pi \circ i \circ J_\pi{}^{-1}(f) = e^f = \sum f^n/n!$$

So for any $\pi \in \mathfrak{P}$, $F_\pi \circ i \circ J_\pi{}^{-1}$ is the exponential map for the group of units. It is clearly C^∞ with inverse $f \to \ln f$ mapping L_+ onto L^∞. On the other hand it is the coordinate version of i. Since its tangent map at zero is the identity we have that the diagram of tangent maps:

$$\begin{tikzcd}
T_\pi \mathfrak{P} \arrow[rr, "T_\pi J_{\pi_0}"] \arrow[dr, "T_\pi i"'] & & L^\infty \\
& \mathfrak{M} \arrow[ur, "F_\pi"'] &
\end{tikzcd}$$

commutes, at least when $\pi_0 = \pi$. Since the tangent map of $J_\pi \circ J_{\pi_0}{}^{-1}$ is always the identity the above diagram always commutes and implies (1.6).

Finally, $F_\pi \circ F_{\pi_0}{}^{-1}$ is multiplication by the unit $d\pi/d\pi_0 \in L_+$ and so is a linear automorphism of the Banach space L^∞.

Remark. By (1.6) $F_\pi{}^{-1}$ is the tangent map at π of the diffeomorphism

$i \circ J_{\pi_0}{}^{-1}$ and so is smooth in π, i.e., the functions:

$$F\colon \mathfrak{P} \to L(\mathfrak{M}; L^\infty)$$
$$F^{-1}\colon \mathfrak{P} \to L(L^\infty; \mathfrak{M})$$

taking π to F_π and $(F_\pi)^{-1}$ respectively are smooth maps. In practice, this means that a smooth vectorfield on $\mathfrak{P}$ can be regarded as a smooth map from $\mathfrak{P}$ to $\mathfrak{M}$ or as a smooth map from $\mathfrak{P}$ to L^∞ with the two versions related by F_π at π.

Integration is a continuous bilinear pairing of $\mathfrak{M}$ and L^∞ which we will call the *Kronecker pairing*:

$$(1.11)\quad \langle\ ,\ \rangle\colon \mathfrak{M} \times L^\infty \to \mathbf{R}$$
$$\langle \mu, f\rangle = \int f(x)\mu(dx).$$

For each $\pi \in \mathfrak{P}$ we can define inner products on L^∞ and $\mathfrak{M}$. First, the $L^2(\pi)$ inner product restricted to L^∞:

$$(1.12)\quad {}_\pi(\ ,\ )\colon L^\infty \times L^\infty \to \mathbf{R}$$
$${}_\pi(f_1, f_2) = \int f_1(x) f_2(x)\pi(dx).$$

${}_\pi(\ ,\ )$ makes L^∞ a pre-Hilbert space with a weaker topology than that induced from $\|\ \ \|$. Its completion is $L^2(\pi)$. Different choices of π induce equivalent although not identical inner products on L^∞.

On $\mathfrak{M}$ we define $(\ ,\ )_\pi\colon \mathfrak{M} \times \mathfrak{M} \to \mathbf{R}$ by

$$(1.13)\quad (\mu_1, \mu_2)_\pi = \langle \mu_1, F_\pi(\mu_2)\rangle = \int \frac{d\mu_2}{d\pi}(x)\mu_1(dx)$$
$$= \int \frac{d\mu_1}{d\pi}(x)\frac{d\mu_2}{d\pi}(x)\pi(dx) = {}_\pi(F_\pi(\mu_1), F_\pi(\mu_2)).$$

So F_π is an isometry between $(\ ,\ )_\pi$ and ${}_\pi(\ ,\ )$. Since $T_\pi\mathfrak{P} = \mathfrak{M}$, $(\ ,\ )_\pi$ is a weak Riemannian metric on $\mathfrak{P}$. By (1.6) the local version of $(\ ,\ )_\pi$ with respect to the chart $(L^\infty, J_{\pi_0}{}^{-1})$ is ${}_\pi(\ ,\ )$. So it is given by the map

$$g\colon L^\infty \to L_{\mathrm{sym}}{}^2(L^\infty; \mathbf{R})$$
$$g(f)(f_1, f_2) = \int f_1(x) f_2(x) e^{f(x)}\pi_0(dx).$$

Smoothness of (1.10) implies that g is smooth and so ${}_\pi(\ ,\ )$ is a weak Riemannian metric on $\mathfrak{P}$ which we will call the Shahshahani metric (c.f. [**1**]). It is called weak because it induces an incomplete topology weaker, i.e., coarser, than the norm topology on $\mathfrak{M}$. So a smooth real valued function on $\mathfrak{P}$ need not have a gradient vectorfield with respect to this inner product. Two important classes of functions do have gradients.

2. Proposition. *For $\pi_0 \in \mathfrak{P}$, $\mu \in \mathfrak{M}$ and $f_0 \in L^\infty$ define the maps*

$L_{\pi_0}{}^{\mu}$, E^{f_0}: $\mathfrak{P} \to \mathbf{R}$ *by:*

$$(1.14)\quad L_{\pi_0}{}^{\mu}(\pi) = \langle \mu, J_{\pi_0}(\pi)\rangle = \int \ln \frac{d\pi}{d\pi_0}(x)\mu(dx)$$

$$E^{f_0}(\pi) = \langle \pi, f_0\rangle = \int f_0(x)\pi(dx).$$

$L_{\pi_0}{}^{\mu}$ and E^{f_0} are smooth maps admitting gradient vectorfields. The gradients $\nabla L_{\pi_0}{}^{\mu}$ and ∇E^{f_0} are the smooth maps from $\mathfrak{P}$ to $\mathfrak{M}$ given by:

$$(1.15)\quad \nabla_{\pi} L_{\pi_0}{}^{\mu} = \mu$$

$$\nabla_{\pi} E^{f_0} = f_0\pi = F_{\pi}{}^{-1}(f_0).$$

Proof. $L_{\pi_0}{}^{\mu}$ is the composition of J_{π_0} with the continuous linear map ω^{μ}: $L^{\infty} \to \mathbf{R}$ ($\omega^{\mu}(f) = \langle \mu, f\rangle$). So $L_{\pi_0}{}^{\mu}$ is smooth and its tangent map satisfies (by (1.6)):

$$T_{\pi}(L_{\pi_0}{}^{\mu})(T_{\pi}J_{\pi_0}{}^{-1}(f)) = \omega^{\mu}(f) = \langle \mu, f\rangle = (\mu, F_{\pi}{}^{-1}(f))_{\pi}$$
$$= (\mu, T_{\pi}J_{\pi_0}{}^{-1}(f))_{\pi}.$$

E^{f_0} is the restriction to $\mathfrak{P}$ of the continuous linear map ω^{f_0}: $\mathfrak{M} \to \mathbf{R}$ ($\omega^{f_0}(\mu) = \langle \mu, f_0\rangle$). So E^{f_0} is smooth and its tangent map satisfies:

$$T_{\pi}E^{f_0}(\mu) = \omega^{f_0}(\mu) = \langle \mu, f_0\rangle = (f_0\pi, \mu)_{\pi}.$$

Note that by (1.7) the difference $L_{\pi_0}{}^{\mu} - L_{\pi_1}{}^{\mu}$ is the constant $L_{\pi_0}{}^{\mu}(\pi_1)$ and so the gradient is independent of the choice of π_0.

Now suppose A is a linear subspace of L^{∞}. We can sharpen the equivalence relation $\approx$ on $\mathfrak{P}$ by defining:

$$\pi_1 \approx_A \pi_2 \quad \text{if} \quad \ln \frac{d\pi_1}{d\pi_2} \in A \quad \text{for } \pi_1, \pi_2 \in \mathfrak{P}.$$

Let $\mathfrak{F}^A$ denote the set of equivalence classes in $\mathfrak{P}$ and $\mathfrak{F}^A(\pi)$ denote the equivalence class of π. By (1.7) it is clear that for $\pi_0 \in \mathfrak{P}$:

$$\pi_1 \approx_A \pi_2 \text{ if and only if } J_{\pi_0}(\pi_1) - J_{\pi_0}(\pi_2) \in A.$$

So J_{π_0} maps the set of equivalence classes $\mathfrak{F}^A$ to the set of translates of the subspace A in L^{∞}. If P^A: $L^{\infty} \to L^{\infty}/A$ is the quotient map of congruence mod A, we can define:

$$(1.16)\quad L_{\pi_0}{}^{A} = P^A \circ J_{\pi_0}: \mathfrak{P} \to L^{\infty}/A.$$

Just as with $L_{\pi_0}{}^{\mu}$ a different choice of π_0 changes the map $L_{\pi_0}{}^{A}$ by a translation on L^{∞}/A.

If A is a closed subspace of L^{∞} then L^{∞}/A is a Banach space, P^A is a continuous linear map and $L_{\pi_0}{}^{A}$ is a surjective smooth map whose point inverses are the equivalence classes $\mathfrak{F}^A$. $\mathfrak{F}^A$ is a foliation of $\mathfrak{P}$ by smooth submanifolds modeled on A.

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On the other hand, if B is a closed subspace of $\mathfrak{M}$ then we let $\mathfrak{D}^B$ denote the equivalence classes of congruence mod B in $\mathfrak{M}$ restricted to $\mathfrak{P}$ via i. So

$$\pi_1 \in \mathfrak{D}^B(\pi_2) \text{ if and only if } \pi_1 - \pi_2 \in B.$$

If P^B is the quotient map of $\mathfrak{M}$ onto $\mathfrak{M}/B$ then $\mathfrak{D}^B$ consists of the point inverses of the smooth map:

$$(1.17) \quad E^B = P^B \circ i: \mathfrak{P} \to \mathfrak{M}/B.$$

3. PROPOSITION. *Let A be a closed subspace of L^∞ and B be a closed subspace of $\mathfrak{M}$. Fix $\pi_0 \in \mathfrak{P}$. The tangent distributions of the foliations $\mathfrak{J}^A$ and $\mathfrak{D}^B$ satisfy:*

$$(1.18) \quad \begin{aligned} T_\pi \mathfrak{J}^A &= \{\nabla_\pi E^f: f \in A\} = F_\pi^{-1}(A) \\ T_\pi \mathfrak{D}^B &= \{\nabla_\pi L_{\pi_0}{}^\mu: \mu \in B\} = B. \end{aligned}$$

By $T_\pi \mathfrak{J}^A$, for example, we mean the tangent space of the leaf of $\mathfrak{J}^A$ containing π. Here the gradients are taken with respect to the Shahshahani metric on $\mathfrak{P}$.

Proof. The tangent space of $\mathfrak{D}^B(\pi)$ is just the subspace B since $\mathfrak{D}^B$ is the intersection with $\mathfrak{P}$ of translates of B. So

$$T_\pi \mathfrak{D}^B = \{\mu \in B\} = \{\nabla_\pi L_{\pi_0}{}^\mu: \mu \in B\} \quad \text{by (1.15).}$$

Similarly, $T_\pi \mathfrak{J}^A = T_\pi J_{\pi_0}{}^{-1}(A)$ and so by (1.6)

$$T_\pi \mathfrak{J}^A = F_\pi^{-1}(A) = \{f\pi: f \in A\}.$$

In a number of important examples these two foliations are orthogonal complements. This requires very strong conditions on the pair.

4. *Definition.* Linear subspaces $A \subset L^\infty$ and $B \subset \mathfrak{M}$ are called *totally complementary* if they satisfy:

(CI) $f \in A$ and $\mu \in B$ imply $\langle \mu, f \rangle = 0$.

(CII) For any $\pi \in \mathfrak{P}$ and $\nu \in \mathfrak{M}$ there exist $f \in A$ and $\mu \in B$ such that $\nu = f\pi + \mu$.

5. LEMMA. (a) *If A and B are totally complementary then A and B are closed subspaces. In fact,*

$$B = \{\mu \in \mathfrak{M}: \langle \mu, f \rangle = 0 \textit{ for all } f \in A\}$$
$$A = \{f \in L^\infty: \langle \mu, f \rangle = 0 \textit{ for all } \mu \in B\}$$

i.e., A and B are the annihilators of one another with respect to the Kronecker pairing.

Furthermore, for any π and ν the decomposition of ν in CII *is unique.*

(b) *If A is a closed subspace of L^∞ and $B \subset \mathfrak{M}$ is its annihilator then (A, B) is a totally complementary pair if and only if for every $\pi \in \mathfrak{P}$ A*

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admits a closed complement in L^∞ *orthogonal to* A *with respect to the* $L^2(\pi)$ *metric* ${}_\pi(\ ,\ )$. *In other words, if and only if for every* π *there exists a continuous projection* $P_\pi: L^\infty \to A$ *whose kernel is orthogonal to* A *with respect to* ${}_\pi(\ ,\ )$.

Proof. (a) Uniqueness follows from uniqueness for $\nu = 0$. If $0 = f\pi + \mu$ with $f \in A$ and $\mu \in B$ then by CI:

$$0 = \langle f\pi, f\rangle + \langle \mu, f\rangle = {}_\pi(f, f).$$

So $f = 0$ and $\mu = -f\pi = 0$.

If $\nu = f\pi + \mu$ annihilates A then it annihilates f and so ${}_\pi(f, f) = 0$. Thus, $\nu = \mu \in B$. On the other hand if g annihilates B let $g\pi = f\pi + \mu$ with $f \in A$ and $\mu \in B$:

$$0 = \langle \mu, g\rangle = (\mu, g\pi)_\pi = \langle \mu, f\rangle + (\mu, \mu)_\pi = (\mu, \mu)_\pi.$$

So $\mu = 0$ and $g\pi = f\pi$. Hence, $g = f \in A$.

(b) If B is the annihilator of A then $F_\pi(B)$ consists of the vectors orthogonal to A with respect to ${}_\pi(\ ,\ )$. Furthermore, $\nu = f\pi + \mu$ if and only if $F_\pi(\nu) = f + F_\pi(\mu)$. So (A, B) is totally complementary if and only if L^∞ is the direct sum of A and $F_\pi(B)$ for all π.

Remark. Uniqueness, as opposed to existence, of the decomposition $\nu = f\pi + \mu$ required only CI.

6. The Product Theorem. *Let* A, B *be closed subspaces of* L^∞ *and* $\mathfrak{M}$ *respectively. Suppose that* CI *holds. Fix* $\pi_0 \in \mathfrak{P}$ *and define* $L_{\pi_0}{}^A: \mathfrak{P} \to L^\infty/A$ *and* $E^B: \mathfrak{P} \to \mathfrak{M}/B$ *as in* (1.16) *and* (1.17). *The product map*

$$E^B \times L_{\pi_0}{}^A: \mathfrak{P} \to \mathfrak{M}/B \times L^\infty/A$$

is a smooth injective immersion, i.e., a smooth injective map whose tangent map is injective at each point.

If (A, B) *is a totally complementary pair then* $E^B \times L_{\pi_0}{}^A$ *is a diffeomorphism of* $\mathfrak{P}$ *onto an open subset of* $\mathfrak{M}/B \times L^\infty/A$. $T_\pi\mathfrak{P}$ *is then the direct sum of the closed subspaces* $T_\pi\mathfrak{J}^A$ *and* $T_\pi\mathfrak{D}^B$. *This direct sum decomposition is orthogonal with respect to the Shahshahani metric.*

Proof. The tangent map at π is the product

$$P^B \circ T_\pi i \times P^A \circ T_\pi J_{\pi_0}{}^{-1}$$

or (1.6):

$$P^B \times P^A \circ F_\pi: \mathfrak{M} \to \mathfrak{M}/B \times L^\infty/A.$$

So $\mu \in \mathfrak{M}$ is in the kernel of the tangent map if and only if $\mu \in B$ and $F_\pi(\mu) = f \in A$, i.e., $0 = -f\pi + \mu$. By CI and the remark after Lemma 5, $\mu = 0$. So CI implies that the tangent map is injective.

Suppose that CII also holds. To show that the tangent map is surjective we start with $\nu \in \mathfrak{M}$ and $g \in L^\infty$ and construct $\mu \in \mathfrak{M}$ such that $\mu - \nu \in B$ and $F_\pi(\mu) - g \in A$. Decompose $\nu = f_1\pi + \mu_1$ and $g\pi = f_2\pi + \mu_2$ with $f_1, f_2 \in A$, $\mu_1, \mu_2 \in B$. Let $\mu = f_1\pi + \mu_2$.

$$\mu - \nu = \mu_2 - \mu_1 \in B \quad \text{and}$$
$$F_\pi(\mu) - g = f_1 - f_2 \in A.$$

So the tangent map is an isomorphism and $E^B \times L_{\pi_0}{}^A$ is a local diffeomorphism by the inverse function theorem. In particular, it has an open image.

To prove that the map is injective suppose $\pi_1, \pi_2 \in \mathfrak{P}$ with

$$E^B(\pi_1) = E^B(\pi_2) \quad \text{and} \quad L_{\pi_0}{}^A(\pi_1) = L_{\pi_0}{}^A(\pi_1).$$

Let $f_i = d\pi_i/d\pi_0$. So $\ln f_i = J_{\pi_0}(\pi_i)$. Since $L_{\pi_0}{}^A(\pi_1) = L_{\pi_0}{}^A(\pi_2)$, $\ln f_1 - \ln f_2 \in A$. Since $E^B(\pi_1) = E^B(\pi_2)$ the signed measure $\pi_1 - \pi_2 \in B$. CI implies

$$0 = \int (\ln f_1(x) - \ln f_2(x))(\pi_1 - \pi_2)(dx)$$
$$= \int (\ln f_1(x) - \ln f_2(x)(f_1(x) - f_2(x))\pi_0(dx).$$

But the function of two positive real variables:

$$(\ln x - \ln y)(x - y)$$

is nonnegative and vanishes only when $x = y$. So $f_1 = f_2$ a.e. (π_0) i.e., they are the same element of L^∞. Thus,

$$\pi_1 = f_1\pi_0 = f_2\pi_0 = \pi_2.$$

Finally, the decomposition of $T_\pi\mathfrak{P}$ identified with $\mathfrak{M}$ is into the subspaces $T_\pi\mathfrak{D}^B = B$ and $T_\pi\mathfrak{J}^A = F_\pi^{-1}(A)$ which are orthogonal complements in $\mathfrak{M}$ with respect to $(\, , \,)_\pi$ by Lemma 5 (b) and its proof. Orthogonality also follows from Proposition 3 and the equation:

$$(1.19) \quad (\nabla_\pi L_{\pi_0}{}^\mu, \nabla_\pi E^f)_\pi = \langle \mu, f \rangle.$$

Remark. $L_{\pi_0}{}^A$ maps $\mathfrak{P}$ onto L^∞/A. The image of E^B is an open convex subset of $\mathfrak{M}/B$ which we will denote $\mathfrak{O}^B$. I conjecture that in the totally complementary case $E^B \times L_{\pi_0}{}^A$ maps onto $\mathfrak{O}^B \times L^\infty/A$. Call this the "Image Conjecture". When true it implies that E^B restricts to a diffeomorphism of each leaf of $\mathfrak{J}^A$ onto $\mathfrak{O}^B$ and $L_{\pi_0}{}^A$ restricts to a diffeomorphism of each leaf of $\mathfrak{D}^B$ onto L^∞/A.

In passing we notice that the inclusion of $\mathfrak{P}$ in $\mathfrak{M}$ on the one hand and the atlas of maps J_π on the other induce two different parallelisms on $\mathfrak{P}$, i.e., maps identifying the tangent spaces of $\mathfrak{P}$ with one another. These are related by the linear maps F_π as shown in Proposition 1. The Shahshahani Riemannian metric also induces a parallelism as shown by:

7. Proposition. *Fix* $\pi_0 \in \mathfrak{P}$. *The map* $K_{\pi_0}\colon L^\infty \to \mathfrak{M}$ *defined by:*

$$(1.20) \quad K_{\pi_0}(f) = 4^{-1}f^2\pi_0$$

is a smooth map and its restriction to L_+ *is a diffeomorphism onto* $\mathfrak{P}$. *This restriction is an isometry between* L_+ *with the constant Riemannian metric* ${}_{\pi_0}(\ ,\ )$ *and* $\mathfrak{P}$ *with the Shahshahani metric.*

Proof. $F_{\pi_0}K_{\pi_0}\colon L^\infty \to L^\infty$ maps f to $4^{-1}f^2$ which is clearly smooth. Its restriction to L_+ is a diffeomorphism onto L_+ with inverse: $g \to 2g^{1/2}$.

If $K_{\pi_0}(f) = \pi$ then

$$T_fK_{\pi_0}(f_1) = 2^{-1}f_1f\pi_0.$$

If $f \in L_+$ then this is $(2f_1/f)\pi$ and

$$(T_fK_{\pi_0}(f_1), T_fK_{\pi_0}(f_2))_\pi = \int(4f_1f_2/f^2)\pi = \int f_1f_2\pi_0 = {}_{\pi_0}(f_1, f_2).$$

Remark. The image of K_{π_0} is the closure of $\mathfrak{P}$ in $\mathfrak{M}$. In fact, K_{π_0} is a homeomorphism of the closure of L_+ in L^∞, which is the set of nonnegative bounded functions, onto the closure of $\mathfrak{P}$. On the other hand, the image of K_{π_0} is the same as the image of K_{π_0} on the nonnegatives. In fact, K_{π_0} is really the quotient map of a group action. For the additive group of the measure algebra of subsets of M acts on L^∞ by $(M_0, f) \to M_0 \cdot f$ where $M_0 \cdot f$ agrees with $-f$ on M_0 and with f on $M - M_0$. $f_1^2 = f_2^2$ if and only if $|f_1| = |f_2|$ if and only if f_1 and f_2 lie in the same orbit of this action.

In applications we are frequently interested only in probability measures. So we define

$$\mathfrak{P}_1 = \{\pi \in \mathfrak{P}\colon \int\pi(dx) = 1\},$$

i.e., the set of probability measures in $\mathfrak{P}$. Define

$$\mathfrak{M}_0 = \{\mu \in \mathfrak{M}\colon \int\mu(dx) = 0\}.$$

For the constant function 1, the map $E^1\colon \mathfrak{P} \to \mathbf{R}$ defined by (1.14) sends π to $\int\pi(dx)$. So E^1 maps $\mathfrak{P}$ onto the positive reals $\mathbf{R}^+$. Let $P_1\colon \mathfrak{P} \to \mathfrak{P}_1$ be defined by

$$(1.21) \quad P_1(\pi) = \pi/E^1(\pi).$$

Clearly, $E^1 \times P_1$ is a diffeomorphism of $\mathfrak{P}$ onto $\mathbf{R}^+ \times \mathfrak{P}_1$.

Because E^1 is constant on $\mathfrak{P}_1$ its gradient defined by $\nabla_\pi E^1 = \pi$ is normal to $\mathfrak{P}_1$. ∇E^1 is easily seen to be of unit length on $\mathfrak{P}_1$ and so is the unit normal vectorfield for the submanifold $\mathfrak{P}_1$ of $\mathfrak{P}$.

If a function $f\colon \mathfrak{P} \to \mathbf{R}$ has a gradient field ∇f then the restriction to $\mathfrak{P}_1$ has a gradient which we denote $\overline{\nabla}f$. It is the normal projection to $\mathfrak{M}_0$. Since ∇E^1 is the unit normal field this projection is given by:

$$(1.22) \quad \overline{\nabla}_\pi f = \nabla_\pi f - (\nabla_\pi f, \nabla_\pi E^1)_\pi \nabla_\pi E^1.$$

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Regarding $\nabla_\pi f$ and $\nabla_\pi E^1 = \pi$ as elements of $\mathfrak{M}$ this says:

$$(1.23)\quad \overline{\nabla}_\pi f = \nabla_\pi f - \langle \nabla_\pi f, 1\rangle \pi.$$

In particular, from (1.15) we have for $\pi \in \mathfrak{P}_1$:

$$(1.24)\quad \begin{aligned} \overline{\nabla}_\pi L_{\pi_0}{}^\mu &= \mu - \langle \mu, 1\rangle \pi \\ \overline{\nabla}_\pi E^{f_0} &= (f_0 - \langle \pi, f_0\rangle)\pi. \end{aligned}$$

Note that if $\mu \in \mathfrak{M}_0$,

$$\overline{\nabla}_\pi L_{\pi_0}{}^\mu = \nabla_\pi L_{\pi_0}{}^\mu = \mu.$$

If (A, B) is a totally complementary pair then $1 \in A$ if and only if $B \subset \mathfrak{M}_0$. In that case every leaf of $\mathfrak{D}^B$ which meets $\mathfrak{P}_1$ lies completely in $\mathfrak{P}_1$, i.e., E^1 is constant on the leaves of $\mathfrak{D}^B$ because $1 \in A$. We will then denote by $\overline{\mathfrak{D}}^B$ the restriction of the $\mathfrak{D}^B$ foliation to $\mathfrak{P}_1$. On the other hand, each leaf of $\mathfrak{J}^A$ is closed under multiplication by positive scalars. Thus every leaf of $\mathfrak{J}^A$ intersects $\mathfrak{P}_1$ in a codimension one submanifold. We let $\overline{\mathfrak{J}}^A$ denote the induced foliation of $\mathfrak{P}_1$ consisting of the point inverses of the restriction of $L_{\pi_0}{}^A$ to $\mathfrak{P}_1$. (1.18) becomes for $\pi \in \mathfrak{P}_1$:

$$(1.25)\quad \begin{aligned} T_\pi \overline{\mathfrak{J}}^A &= \{\overline{\nabla}_\pi E^f : f \in A\} \\ T_\pi \overline{\mathfrak{D}}^B &= \{\overline{\nabla}_\pi L_{\pi_0}{}^\mu : \mu \in B\} = B \subset \mathfrak{M}_0. \end{aligned}$$

The map $f \to \nabla_\pi E^f$ is the isomorphism F_π^{-1} from A to $T_\pi \mathfrak{J}^A$. Since $\overline{\nabla}_\pi E^1 = 0, f \to \overline{\nabla}_\pi E^f$ is an isomorphism of any complement of $[1]$ in A onto $T_\pi \overline{\mathfrak{J}}^A$. $F_\pi(T_\pi \overline{\mathfrak{J}}^A)$ is the ${}_\pi(\ ,\ )$ orthogonal complement of $[1]$ in A.

Theorem 1.6 implies:

8. The Product Theorem. *If (A, B) is a totally complementary pair with $1 \in A$ then $E^B \times L_{\pi_0}{}^A$ is a diffeomorphism of $\mathfrak{P}_1$ onto an open subset of $(\mathfrak{M}/B)_1 \times L^\infty/A$ where*

$$(\mathfrak{M}/B)_1 = \{\mu + B \in \mathfrak{M}/B : \textstyle\int \mu(dx) = 1\}.$$

In other words, $(\mathfrak{M}/B)_1$ is that translate of the subspace $\mathfrak{M}_0/B \subset \mathfrak{M}/B$ which maps to 1 under E^1.

Remark. The image of $\mathfrak{P}_1$ under $L_{\pi_0}{}^A$ is still all of L^∞/A. The image of $\mathfrak{P}_1$ under E^B is the intersection of $\mathfrak{O}^B \equiv E^B(\mathfrak{P})$ with $(\mathfrak{M}/B)_1$. So $\mathfrak{O}_1{}^B \equiv E^B(\mathfrak{P}_1)$ is an open convex subset of the affine subspace $(\mathfrak{M}/B)_1$. $E^B \times L_{\pi_0}{}^A$ maps $\mathfrak{P}$ onto $\mathfrak{O}_1{}^B \times L^\infty/A$ if and only if it maps $\mathfrak{P}_1$ onto $\mathfrak{O}_1{}^B \times L^\infty/A$ (cf. Remark after Theorem 1.6).

Finally, K_{π_0} of Proposition 1.7 restricts to a diffeomorphism of

$$\{f \in L_+ : \textstyle\int f^2(x)\pi_0(dx) = 4\}$$

onto $\mathfrak{P}_1$.

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2. Exponential families. Suppose A is a finite dimensional subspace of L^∞ containing the constant functions. If $\{p_0, \ldots, p_n\}$ is a basis for A then the map

$$(2.1)\qquad \alpha: \mathbf{R}^{n+1} \to A$$
$$\alpha(u_0, \ldots, u_n) = \sum u_i p_i$$

is a linear isomorphism.

The annihilator B of A consists of those $\mu \in \mathfrak{M}$ such that $\langle \mu, p_i \rangle = 0$ for $i = 0, \ldots, n$. So

$$(2.2)\qquad \beta: \mathfrak{M}/B \to \mathbf{R}^{n+1}$$
$$\beta(\mu + B) = (\langle \mu, p_0 \rangle, \ldots, \langle \mu, p_n \rangle)$$

is an injective linear map. That it is in fact a linear isomorphism follows from the fact that (A, B) is a totally complementary pair. For the isomorphism $F_\pi: \mathfrak{M} \to L^\infty$ maps B onto the vectors perpendicular to A with respect to ${}_\pi(\ , \ )$. If the pair is totally complementary then $L^\infty/F_\pi(B)$ is isomorphic to A. So $L^\infty/F_\pi(B)$ and its isomorph $\mathfrak{M}/B$ are $n + 1$ dimensional. β is then an injective linear map between vector spaces of the same finite dimension and so is an isomorphism.

For any measure $\pi \in \mathfrak{P}$ we can choose a basis $\{e_0^\pi, \ldots, e_n^\pi\}$ to be orthonormal with respect to the inner product ${}_\pi(\ , \ )$ restricted to A. Then the ${}_\pi(\ , \ )$ orthogonal projection $P_\pi: L^\infty \to A$ is defined by:

$$(2.3)\qquad P_\pi(f) = \sum {}_\pi(f, e_i^\pi) e_i^\pi.$$

So the pair is totally complementary by Lemma 1.5.

Identifying $\mathfrak{M}/B$ with $\mathbf{R}^{n+1}$ via β identifies the map E^B with:

$$(2.4)\qquad E^B: \mathfrak{P} \to \mathbf{R}^{n+1}$$
$$E^B(\pi) = (\langle \pi, p_0 \rangle, \ldots, \langle \pi, p_n \rangle) = (E^{p_0}(\pi), \ldots, E^{p_n}(\pi)).$$

Since J_π^{-1} maps A onto the leaf $\mathfrak{F}^A(\pi)$ of the $\mathfrak{F}^A$ foliation,

$$\mathfrak{F}^A(\pi) = \{e^f \pi : f \in A\}.$$

Using α we can coordinatize $\mathfrak{F}^A(\pi)$ by $\mathbf{R}^{n+1}$:

$$(2.3)\qquad \mathfrak{F}^A(\pi) = \left\{ \exp\left(\sum_{i=0}^n u_a p_i \right) \pi : (u_0, \ldots, u_n) \in \mathbf{R}^{n+1} \right\}.$$

As $1 \in A$, the leaf $\bar{\mathfrak{F}}^A(\pi)$ of the foliation $\bar{\mathfrak{F}}^A$ on probability distributions consist of the $\mathfrak{F}^A(\pi)$ measures normalized to measure 1. $\bar{\mathfrak{F}}^A(\pi)$ is n dimensional and if the basis is such that $p_0 = 1$ then we can coordinatize $\bar{\mathfrak{F}}^A(\pi)$ by $\mathbf{R}^n$ for $\pi \in \mathfrak{P}_1$:

$$(2.4)\qquad \bar{\mathfrak{F}}^A(\pi) = \left\{ C(u_1, \ldots, u_n) \exp\left(\sum_{i=1}^n u_i p_i \right) \pi : (u_1, \ldots, u_n) \in \mathbf{R}^n \right\}$$
$$C(u_1, \ldots, u_n)^{-1} = \int \exp\left(\sum_{i=1}^n u_i p_i(x) \right) \pi(dx).$$

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This is because two measures

$$\exp\left(\sum_{i=0}^{n} u_i p_i\right)\pi \quad \text{and} \quad \exp\left(\sum_{i=0}^{n} \bar{u}_i p_i\right)\pi$$

normalize to the same probability measure if and only if $\sum_{i=0}^{n} (u_i - \bar{u}_i)p_i(x)$ is constant, i.e., a multiple of 1. Since $p_0 = 1$ this is true if and only if $u_i = \bar{u}_i$ for $i = 1, \ldots, n$.

Thus the leaves of $\bar{\mathfrak{F}}^A$ are n dimensional exponential families of probability measures (cf. [**11**, Section 2.7]).

To prove the Image Conjecture in this case we need an analytic result.

1. LEMMA. *Fix $\pi \in \mathfrak{P}_1$ and $f_0 \in L^\infty$. Then as $K \to \infty$ in $\mathbf{R}$ and $f \to f_0$ in L^∞*

$$\int f(x)e^{Kf(x)}\pi(dx)/\int e^{Kf(x)}\pi(dx) \to S$$

where S is the essential supremum of f_0.

Proof. Let π_f^K be the probability measure $e^{Kf}\pi/\int e^{Kf}\pi$. The result says

$$\operatorname{Lim} \int f(x)\pi_f^K(dx) = S \quad \text{as } K \to \infty \text{ and } f \to f_0.$$

If $\|f - f_0\| \leqq \epsilon$ then $f \leqq S + \epsilon$ and so

$$\int f\pi_f^K \leqq S + \epsilon.$$

Thus, Lim sup $\leqq S$.

For the other direction define

$$M_f(\epsilon) = \{x: f(x) \geqq S - \epsilon\}.$$

Clearly, $\|f_1 - f_2\| \leqq \delta/2$ implies

$$M_{f_1}(\delta/2) \subset M_{f_2}(\delta).$$

Also since S is the essential supremum of f_0,

$$\pi(M_{f_0}(\epsilon)) > 0 \quad \text{for all } \epsilon > 0.$$

We now claim that for any fixed $\epsilon > 0$

$$\operatorname{Lim} \pi_f^K(\tilde{M}_f(\epsilon)) = 0 \tag{2.5}$$

as $f \to f_0$ and $K \to \infty$, where $\tilde{M}_f(\epsilon) = M - M_f(\epsilon)$.

In fact, if $\|f - f_0\| \leqq \epsilon/4$ then

$$\pi_f^K(\tilde{M}_f(\epsilon)) \leqq \frac{\int_{M_f(\epsilon)} e^{Kf}\pi}{\int_{\tilde{M}_f(\epsilon/2)} e^{Kf}\pi} \leqq \frac{e^{K(S-\epsilon)}}{e^{K(S-\epsilon/2)}\pi(M_f(\epsilon/2))}$$

$$\leqq e^{-K\epsilon/2}/\pi(M_{f_0}(\epsilon/4)).$$

Since ϵ and f_0 are fixed this approaches 0 as $K \to \infty$.

That Lim inf $\int f\pi_f^K \geqq S$ now follows from

$$\int f\pi_f^K \geqq \int_{Mf(\epsilon)} f\pi_f^K \geqq (S-\epsilon)(1-\pi_f^K(\tilde{M}_f(\epsilon)))$$

because the last term approaches $S-\epsilon$ as $K \to \infty$, $f \to f_0$.

2. THEOREM. *Define* $p: X \to \mathbf{R}^{n+1}$ *by*

$$p(x) = (p_0(x), \ldots, p_n(x)).$$

Let C *be the convex hull of the essential image of* p, *i.e.,* C *is the smallest compact convex set such that* $p(x) \in C$ *a.e. Define* $E^B: \mathfrak{P} \to \mathbf{R}^{n+1}$ *by*

$$E^B(\pi) = (\langle \pi, p_0 \rangle, \ldots, \langle \pi, p_n \rangle).$$

The image of E^B, *denoted* $\mathfrak{D}^B$, *is the interior of the positive cone on* C, *i.e.,* $\mathfrak{D}^B =$ *Interior* C_+ *with*

$$C_+ = \{tv: t \geqq 0 \text{ and } v \in C\}.$$

$E^B \times L_{\pi_0}{}^A$ *is a diffeomorphism of* $\mathfrak{P}$ *onto* $\mathfrak{D}^B \times L^\infty/A$.

Suppose $p_0 = 1$ *and define* $\bar{p}: X \to \mathbf{R}^n$ *by*

$$\bar{p}(x) = (p_1(x), \ldots, p_n(x)).$$

Let $\bar{C}$ *be the convex hull of the essential image of* $\bar{p}$. *Define* $\bar{E}^B: \mathfrak{P}_1 \to \mathbf{R}^n$ *by*

$$\bar{E}^B(\pi) = (\langle \pi, p_1 \rangle, \ldots, \langle \pi, p_n \rangle).$$

The image of $\bar{E}^B$, *denoted* $\mathfrak{D}^B$, *is the interior of* $\bar{C}$. $\bar{E}^B \times L_{\pi_0}{}^A$ *is a diffeomorphism of* $\mathfrak{P}_1$ *onto* $\bar{\mathfrak{D}}^B \times L^\infty/A$.

Proof. Since a change of basis doesn't affect the result we can assume $p_0 = 1$. Then the first paragraph follows from the second because $p = j \circ \bar{p}$ and $E^B|\mathfrak{P}_1 = j \circ \bar{E}^B$ where $j: \mathbf{R}^n \to \mathbf{R}^{n+1}$ is defined by

$$j(u_1, \ldots, u_n) = (1, u_1, \ldots, u_n).$$

Multiplying π by a positive scalar t doesn't affect $L_{\pi_0}{}^A$ and multiplies $E^B(\pi)$ by t.

Now from the Product Theorem we know that $\mathfrak{D}^B$ is open and that $E^B \times L_{\pi_0}{}^A$ is a diffeomorphism of $\mathfrak{P}$ onto an open subset of $\mathfrak{D}^B \times L^\infty/A$. $\bar{\mathfrak{D}}^B = j^{-1}(\mathfrak{D}^B)$ is open and $\bar{E}^B \times L_{\pi_0}{}^A$ is a diffeomorphism onto an open subset of $\bar{\mathfrak{D}}^B \times L^\infty(A)$ by Theorem 1.8. To prove $\bar{\mathfrak{D}}^B = \operatorname{Int} \bar{C}$ and the Image Conjecture it is enough to prove:

(1) $\bar{E}^B(\mathfrak{P}_1) \subset \bar{C}$.

(2) $\bar{E}^B$ maps $\bar{\mathfrak{F}}^A(\pi)$ onto Int $\bar{C}$ for any $\pi \in \mathfrak{P}_1$.

The key facts that we need are:

(2.6) $\bar{C} = \{v \in \mathbf{R}^n\colon$ For all $(u_1, \ldots, u_n) \in \mathbf{R}^n$

$$\sum u_i v_i \leqq \text{ess. sup} \sum u_i p_i\}.$$

(2.7) Boundary $\bar{C} = \{v \in \bar{C}\colon$ For some $(u_1, \ldots, u_n)$

$$\sum u_i v_i = \text{ess. sup} \sum u_i p_i\}.$$

These are true because the hyperplanes of support for the convex hull of the essential image of p consist of

$$\{v \in \mathbf{R}^n\colon \sum u_i v_i = \text{ess. sup} \sum u_i p_i\}$$

for $(u_1, \ldots, u_n) \in \mathbf{R}^n$.

(1) If $v = \bar{E}^B(\pi) = (\langle \pi, p_1 \rangle, \ldots, \langle \pi, p_n \rangle)$ then

$$\sum u_i v_i = \langle \pi, \sum u_i p_i \rangle \leqq \text{ess. sup} \sum u_i p_i$$

because $\pi \in \mathfrak{P}_1$. So (1) follows from (2.6).

(2) We use the coordinatization of $\bar{\mathfrak{F}}^A(\pi)$ by $\mathbf{R}^n$ given by (2.4). So the coordinate version of $\bar{E}^B|\bar{\mathfrak{F}}^A(\pi)$ becomes the map:

(2.8) $\bar{g}\colon \mathbf{R}^n \to \mathbf{R}^n$

$$\bar{g}_i(u) = \int p_i(x) \exp\left(\sum_{j=1}^n u_j p_j(x)\right) \pi(dx) \Big/ \int \exp\left(\sum_{j=1}^n u_j p(x)\right) \pi(dx) \quad \text{for } i = 1, \ldots, n.$$

We show that $\bar{g}$ maps onto Int $\bar{C}$.

Suppose that $u^{(k)}$ is a sequence in $\mathbf{R}^n$ such that $\bar{g}^{(k)} = \bar{g}(u^{(k)})$ converges to $\bar{g}^{(\infty)}$ not in the image of $\bar{g}$. It suffices to show that $\bar{g}^{(\infty)}$ is a boundary point of $\bar{C}$.

Since $u^{(k)}$ can have no convergent subsequence $K_k = \|u^{(k)}\|$ must tend to ∞. Let $\bar{u}^{(k)} = u^{(k)}/K_k$. By going to a subsequence we can assume that K_k is never zero and $\bar{u}^{(k)}$ converges to $\bar{u}^{(\infty)}$ in the unit sphere of $\mathbf{R}^n$. Let

$$p^{(k)} = \sum \bar{u}_j^{(k)} p_j \ (k \leqq \infty).$$

Then $p^{(k)} \to p^{(\infty)}$ in L^∞ and $K_k \to \infty$. Apply Lemma 1 with $f_0 = p^{(\infty)}$:

$$\sum \bar{u}_j^{(\infty)} \bar{g}_j^{(\infty)} = \text{Lim} \sum \bar{u}_j^{(k)} \bar{g}_j^{(k)} = \text{ess. sup. } p^{(\infty)} = \text{ess. sup} \sum \bar{u}_j^{(\infty)} p_j.$$

So by (2.7) $\bar{g}^{(\infty)}$ is a boundary point of $\bar{C}$.

Remark. In applications A will usually be defined by a list of functions $\{p_i\colon i = 0, \ldots, n\}$ which therefore spans A but which need not be linearly independent in L^∞. We can still define $E^B\colon \mathfrak{M} \to \mathbf{R}^{n+1}$ to be the linear map of (2.4) but now it need not be onto. The vector function

$p: M \to \mathbf{R}^{n+1}$ is essentially bounded and the image $E^B(\mathfrak{M})$ is the subspace spanned by the essential image of p. The image $\mathfrak{O}^B = E^B(\mathfrak{P})$ is the interior in $E^B(\mathfrak{M})$ of the positive convex cone spanned by ess. Im (p). Since 1 is assumed to lie in A, $E^B(\mathfrak{M}_0)$ will be a codimension one subspace of $E^B(\mathfrak{M})$. Some translate of this subspace will be the affine subspace spanned by ess. Im (p). For example, if $1 = \sum_{i=0}^n p_i$ then

$$E^B(\mathfrak{M}_0) = E^B(\mathfrak{M}) \cap (\mathbf{R}^{n+1})_0 = \{v \in E^B(\mathfrak{M}): \textstyle\sum_i v_i = 0\}.$$

The affine subspace spanned by ess. Im (p) is

$$\{v \in E^B(\mathfrak{M}): \textstyle\sum_i v_i = 1\}.$$

Finally the image $\mathfrak{O}_1{}^B = E^B(\mathfrak{P}_1)$ is the intersection of $\mathfrak{O}^B$ with this affine subspace. Equivalently, $\mathfrak{O}_1{}^B$ is the interior of the convex hull of ess. Im (p) in the affine subspace it spans.

3. Game dynamics. The purpose of the smooth structure on $\mathfrak{P}$ is to make sense of differential equations of measures. A vectorfield on $\mathfrak{P}$ can be regarded as a smooth map $X: \mathfrak{P} \to \mathfrak{M}$ or $\xi: \mathfrak{P} \to L^\infty$ related at π by F_π, i.e., $F_\pi(X(\pi)) = \xi(\pi)$. So the corresponding differential equation is:

$$(3.1) \qquad d\pi/dt = X(\pi) = \xi(\pi)\pi.$$

By analogy with the finite dimensional case we can think of $X(\pi)$ as the absolute rate of change at π and $\xi(\pi)$ as the relative rate.

Restricting to probability measures a $\mathfrak{P}_1$ vectorfield is a map $X: \mathfrak{P}_1 \to \mathfrak{M}_0$, i.e.,:

$$(3.2) \qquad 0 = \langle X(\pi), 1\rangle = (X(\pi), \pi)_\pi = {}_\pi(\xi(\pi), 1) \quad (\pi \in \mathfrak{P}_1).$$

From the Product Theorem we get the following:

1. PROPOSITION. *Let (A, B) be a totally complementary pair with $1 \in A$. Suppose $X: \mathfrak{P}_1 \to \mathfrak{M}_0$ is a smooth vectorfield.*

(a) *The following conditions are equivalent and define "X is a horizontal field":*

(i) *$\xi(\pi) \in A$ for all $\pi \in \mathfrak{P}_1$.*

(ii) *The leaves of the foliation $\mathfrak{F}^A$ of $\mathfrak{P}_1$ are invariant manifolds for the local flow associated with X.*

(iii) *If π_t is a solution of equation* (3.1) *defined for t in some interval containing* 0, *then $\pi_t \approx_A \pi_0$ for all t, i.e.,*

$$\ln (d\pi_t/d\pi_0) \in A \quad \textit{for all } t.$$

(iv) *For all $\mu \in B$ and $\pi_0 \in \mathfrak{P}_1$ the functions $L_{\pi_0}{}^\mu: \mathfrak{P}_1 \to \mathbf{R}$ are integrals of the motion associated with X, i.e., each $L_{\pi_0}{}^\mu$ is constant on solutions of* (3.1).

(b) *The following conditions are equivalent and define "X is a vertical field":*

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(i) $X(\pi) \in B$ *for all* $\pi \in \mathfrak{P}_1$.

(ii) *The leaves of the foliation* $\overline{\mathfrak{D}}^B$ *of* $\mathfrak{P}_1$ *are invariant manifolds for the local flow associated with* X.

(iii) *If* π_t *is a solution of equation* (3.1) *defined for* t *in some interval containing* 0, *then* $\pi_t - \pi_0 \in B$ *for all* t.

(iv) *For all* $f \in A$ *the functions* $E^f\colon \mathfrak{P}_1 \to \mathbf{R}$ *are integrals of the motion associated with* X.

Proof. In each of the two cases, condition (i) says that the vectorfield is everywhere tangent to the corresponding foliation and so restricts to a vectorfield on each leaf. So condition (i) is equivalent to condition (ii) by uniqueness of solutions for the differential equations (3.1) restricted to the leaves [**10**, Chapter IV]. Condition (iii) is a restatement of condition (ii). Condition (iv) says that the vectorfield is orthogonal to the complementary foliation and so is equivalent to condition (i) because (A, B) is a totally complementary pair.

Remark. In each case the equivalence of (i), (ii) and (iii) and their implying (iv) requires only that A and B be closed subspaces of L^∞ and $\mathfrak{M}$ respectively satisfying condition (CI) of Definition 1.4.

We now return to game dynamics. The underlying measurable space is $\Delta \subset \mathbf{R}^{n+1}$ where $\{0, 1, \ldots, n\}$ indexes the strategies in the game. Let $p_i\colon \Delta \to [0, 1]$ $i = 0, \ldots, n$ be the ith coordinate function so that $p\colon \Delta \to \mathbf{R}^{n+1}$ is the inclusion map of Δ into $\mathbf{R}^{n+1}$. For $\mu \in \mathfrak{M}$ define

$$(3.3)\qquad E^i(\mu) = \int_\Delta p_i\mu = \langle \mu, p_i\rangle \quad i = 0, \ldots, n.$$

So $E^i\colon \mathfrak{P}_1 \to \mathbf{R}$ is the map E^f with $f = p_i$. Clearly, E^i maps $\mathfrak{P}_1$ to $[0, 1]$ and $\sum_i E^i = 1$. So $E\colon \mathfrak{P}_1 \to \mathbf{R}^{n+1}$ with coordinate functions E^i maps into Δ and in the notation of the introduction $E(\pi)$ is the mean value of the measure π.

$$(3.4)\qquad E(\pi) = \int_\Delta p\pi = \hat{x}(\pi).$$

Now let A be the subspace of L^∞ spanned by $p_0, \ldots, p_n$ and let B be the subspace of $\mathfrak{M}$ annihilating A, so

$$(3.5)\qquad \mu \in B \Leftrightarrow \langle \mu, p_i\rangle = 0 \quad i = 0, \ldots, n \Leftrightarrow \int_\Delta p\mu = 0.$$

So the dimension of $A \leqq n + 1$ and by Section 2, (A, B) is a totally complementary pair. The leaves of the corresponding foliation $\overline{\mathfrak{F}}$ of $\mathfrak{P}_1$ have dimension equal to dim $A - 1$. The leaf through π is the exponential family of dimension $\leqq n$ given by (cf. (2.3) and (2.4)):

$$(3.6) \quad \overline{\mathfrak{F}}(\pi) = \left\{ C(u_0, \ldots, u_n) \exp \left(\sum_{i=0}^{n} u_i p_i \right) \pi : (u_0, \ldots, u_n) \in \mathbf{R}^{n+1} \right\}$$

$$C(u_0, \ldots, u_n)^{-1} = \int_{\Delta} \exp \left(\sum_{i=0}^{n} u_i p_i \right) \pi.$$

We get a coordinate system on $\overline{\mathfrak{F}}(\pi)$ by choosing dim $A - 1$ linearly independent p_i's such that they together with the constant function $1 = \sum_i p_i$ form a basis for A in L^∞.

With (a_{ij}) a given payoff matrix we define the vector-field $X: \mathfrak{P}_1 \to \mathfrak{M}_0$ by $X(\pi) = \xi(\pi)\pi$ with $\xi(\pi) \in L^\infty$ given by (Cf. (0.3)):

$$(3.7) \quad \xi(\pi)(x) = a_{x\pi} - a_{\pi\pi} \quad x \in \Delta.$$

Here we follow the notation of (0.4) with $a_{xy} = \sum x_i a_{ij} y_j$. The subscript π stands for $\hat{x}(\pi) = E(\pi)$.

2. THEOREM. *Let $\mathfrak{P}_1$ be a fixed $\approx$ equivalence class of probability measures on Δ. The vectorfield $X: \mathfrak{P}_1 \to \mathfrak{M}_0$ defined by $X(\pi) = \xi(\pi)\pi$ with $\xi(\pi)$ given by* (3.7) *is a smooth, complete vectorfield on $\mathfrak{P}_1$. With respect to the Shahshahani metric on $\mathfrak{P}_1$ it is dual to the one-form:*

$$(3.8) \quad \sum_{i,j} (dE^i) a_{ij} E^j = E^*\theta$$

where θ is the one form on Δ given by (0.5).

$$(3.9) \quad X(\pi) = \sum_{i,j} (a_{ij} E^j(\pi)) \overline{\nabla}_\pi E^i$$

where $\overline{\nabla} E^i$ is the gradient of E^i with respect to the Shahshahani metric on $\mathfrak{P}_1$.

X is a horizontal vectorfield so that each leaf of the foliation $\overline{\mathfrak{F}}$ of $\mathfrak{P}_1$ is an invariant submanifold of the flow.

Let $\bar{C}$ be the compact convex subset of Δ which is the convex hull of the support of π for $\pi \in \mathfrak{P}_1$. $\bar{C}$ is independent of the choice of $\pi \in \mathfrak{P}_1$. Let $\mathfrak{D}_1$ be the interior of $\bar{C}$ in the affine subspace spanned by $\bar{C}$. E restricts to a diffeomorphism of each leaf of $\overline{\mathfrak{F}}$ onto $\mathfrak{D}_1$.

Proof. Note first that

$$_\pi(\xi(\pi), 1) = \int \xi(\pi)\pi = a_{\pi\pi} - a_{\pi\pi} = 0$$

and so $X(\pi) \in \mathfrak{M}_0$. Thus, the vectorfield X is tangent to $\mathfrak{P}_1$.

The form on the left in (3.8) defines the pullback of θ by E, i.e., $E^*\theta$. The proof that it is dual to X is identical to (0.8): Let $\mu \in \mathfrak{M}_0$, so $\langle \mu, 1 \rangle = 0$.

$$(X(\pi), \mu)_\pi = \langle \mu, \xi(\pi) \rangle = \int_{\Delta} [a_{x\pi} - a_{\pi\pi}] \mu(dx)$$
$$= \sum_{ij} \langle \mu, p_i \rangle a_{ij} \langle \pi, p_j \rangle - 0 = \sum_{ij} d_\pi(E^i)(\mu) a_{ij} E^j(\pi)$$
$$= \left(\sum_{ij} d_\pi E^i a_{ij} E^j(\pi) \right)(\mu).$$

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Here we use the fact that E^i is the restriction to $\mathfrak{P}_1$ of a linear map and so its differential is this same linear map.

(3.9) now follows because the dual of the differential of E^i is the gradient of E^i. It also follows by direct computation using (1.24).

From (3.9) follows the smoothness of X. To see that it is complete first extend X to a vectorfield on $\mathfrak{P}$ by preceding ξ by the smooth projection P_1: $\mathfrak{P} \to \mathfrak{P}_1$ given by (1.21). Now fix $\pi_0 \in \mathfrak{P}_1$ and use the chart $J_{\pi_0}{}^{-1}$: $L^\infty \to \mathfrak{P}$. The local representative of X with respect to this chart is the map $\tilde{\xi}$: $L^\infty \to L^\infty$ defined by

$$\tilde{\xi}(f) = \xi(P_1(J_{\pi_0}{}^{-1}(f))) = a_{x\pi} - a_{\pi\pi}$$

where $\pi = P_1(J_{\pi_0}{}^{-1}(f))$. Now let

$$M = \sup \{|a_{xy}|: x, y \in \Delta\}.$$

$M < \infty$ by compactness of Δ. Clearly, the L^∞ norm of $\tilde{\xi}(f)$ is at most $2M$. So $\tilde{\xi}$ is a smooth uniformly bounded vectorfield defined everywhere on the Banach space L^∞. It follows from the usual estimates that it is complete, i.e., initial value problems have unique solutions defined for all t.

X is a horizontal vectorfield by Proposition 1 (ai) since the gradients $\overline{\nabla}E^i$ are everywhere tangent to $\overline{\mathfrak{F}}$ (cf. (1.25)). By Proposition 1(a) each leaf of $\overline{\mathfrak{F}}$ is an invariant submanifold. E is a diffeomorphism of each leaf onto $\mathfrak{D}_1$ by Theorem 2.2 and the Remark thereafter.

3. Corollary. *A solution path* $\{\pi_t: t \in \mathbf{R}\}$ *of equation* (3.1) *with* X *given by* (3.9) *is completely determined by the initial distribution*, π_0, *and the mean path* $\hat{x}(\pi_t) = E(\pi_t)$.

Proof. π_t lies in the leaf $\overline{\mathfrak{F}}(\pi_0)$ and $E|\overline{\mathfrak{F}}(\pi_0)$ is invertible. So

$$\pi_t = (E|\overline{\mathfrak{F}}(\pi_0))^{-1}(\hat{x}(\pi_t)).$$

Before applying these results, we must interpret the dimension of $A = 1 +$ the dimension of $\bar{C}$ geometrically. Clearly, the $n + 1$ functions $p_0, \ldots, p_n$ are linearly independent on Δ. But they need not be linearly independent in L^∞. For example, if the measures of $\mathfrak{P}_1$ have support contained in the face of Δ defined by $p_0 = 0$ then $p_0 = 0$ a.e. π and so $p_0 = 0$ in L^∞.

4. Lemma. (a) *The mean values of the measures in* $\mathfrak{P}_1$ *are all contained in some open face of the simplex* Δ, *possibly* $\mathring{\Delta}$ *itself. Call this face* $\Delta(\mathfrak{P})$ *i.e.*, $\mathfrak{D}_1 \subset \mathring{\Delta}(\mathfrak{P})$ *and so*

$$\text{(3.10)} \quad \dim A - 1 = \dim \bar{C} \leqq \dim \Delta(\mathfrak{P}).$$

The measures in the family $\mathfrak{P}_1$ *are called "full" if the inequality in* (3.10) *is an equality, i.e., if the affine subspace spanned by the support of* π *for* π *in* $\mathfrak{P}_1$ *is the same as the affine subspace spanned by* $\Delta(\mathfrak{P})$.

(b) *Let* $F_{ij}(\pi)$ $i, j = 0, \ldots, n$ *denote the covariance matrix of the functions* $\{p_i\}$ *with respect to the measure* π *in* $\mathfrak{P}_1$, *i.e.*,

$$(3.11) \quad F_{ij}(\pi) = \int_\Delta p_i p_j \pi - E^i(\pi)E^j(\pi).$$

The rank of the matrix $F_{ij}(\pi) = \dim A - 1$ *for all* $\pi \in \mathfrak{P}_1$. *So the measures in* $\mathfrak{P}_1$ *are full if and only if*

$$\text{rank } F_{ij}(\pi) = \dim \Delta(\mathfrak{P}).$$

Proof. (a). Let J be a proper subset of $\{0, \ldots, n\}$ and let $\Delta(J)$ be the face of Δ defined by $p_i = 0$ for $i \in J$. Clearly, if the support of $\pi \subset \Delta(J)$ then $p_i = 0$ a.e. (π) for $i \in J$ and so the mean $\hat{x}(\pi)$ satisfies

$$\hat{x}(\pi)_i = \int p_i \pi(dp) = 0 \quad \text{for } i \in J.$$

Conversely, if $\hat{x}(\pi)_i = 0$ for $i \in J$ then since $p_i \geqq 0$ and has zero integral, $p_i = 0$ a.e. (π). So the support of J is in $\Delta(J)$.

Now all of the measures in $\mathfrak{P}$ have the same support. Let $\Delta(\mathfrak{P})$ be the smallest face of Δ containing this support. The mean values of the measures in $\mathfrak{P}_1$ are contained in this face by the preceding paragraph. None of them lie in any proper face of $\Delta(\mathfrak{P})$ for then the entire support would lie in this smaller face. So $\mathfrak{O}_1 \subset \mathring{\Delta}(\mathfrak{P})$. Now $\dim A - 1 = \dim \bar{C}$ which is at most the dimension of $\Delta(\mathfrak{P})$ since $\bar{C} \subset \Delta(\mathfrak{P})$. $\dim \bar{C}$ is strictly less than $\dim \Delta(\mathfrak{P})$ if and only if the affine space spanned by $\bar{C}$ (= the affine space spanned by the support) is a proper subspace of the affine space spanned by $\Delta(\mathfrak{P})$.

(b). Consider the linear map $\bar{\alpha}$: $\mathbf{R}^{n+1} \to A/[1]$ where $[1]$ is the subspace of constants in L^∞, defined by

$$\alpha(u_0, \ldots, u_n) = \sum_{i=0}^n u_i p_i.$$

Now $\sum_{ij} u_i F_{ij} u_j$ is the variance of $\sum u_i p_i$ which is $\geqq 0$ and equals zero if and only if $\sum u_i p_i$ is a.e. constant, i.e., if and only if the vector u is in the kernel of $\bar{\alpha}$. Hence the symmetric matrix F_{ij} is positive semidefinite and its rank is the codimension of its annihilator, i.e., $n + 1 - \dim K$ where K is the kernel of $\bar{\alpha}$. But since $\bar{\alpha}$ is onto,

$$\dim A - 1 = \dim A/[1] = n + 1 - \dim K = \text{rank } F_{ij}.$$

Remarks. (a) The covariance matrix is closely related to the Riemannian metric on the leaves. In fact, from (2.4) it follows that:

$$(3.12) \quad (\bar{\nabla}_\pi E^i, \bar{\nabla}_\pi E^j)_\pi = F_{ij}(\pi).$$

From this equation we can derive Hines' original dynamic for the mean:

$$(3.13) \quad \frac{d\hat{x}(\pi)_i}{dt} = \sum_{j,k} F_{ij}(\pi) a_{jk} \hat{x}(\pi)_k.$$

To see this recall that $\hat{x}(\pi)_i = E^i(\pi)$ and so the left side is $(\overline{\nabla}_\pi E^i, X(\pi))_\pi$. (3.13) then follows from (3.9) and (3.12).

(b) Note that if π is concentrated on the vertices of Δ with $\pi(\{i\}) = x_i$ then

$$\hat{x}(\pi) = x \quad \text{and} \quad F_{ij} = x_i\delta_{ij} - x_i x_j.$$

(c) We call the measures of the family $\mathfrak{P}_1$ *interior* if $\Delta(\mathfrak{P}) = \Delta$, i.e., if the means $\hat{x}(\pi)$ lie in $\mathring{\Delta}$. So the measures are full and interior if and only if $\mathfrak{D}_1$ is an open subset of $\mathring{\Delta}$.

5. Proposition. *π is an equilibrium for the vectorfield X, i.e., $X(\pi) = 0$ if and only if the restriction of the form θ to $\mathfrak{D}_1$ vanishes at $\hat{x}(\pi)$. In that case, the entire leaf $\{\pi_1 \in \mathfrak{P}_1: \hat{x}(\pi_1) = \hat{x}(\pi)\}$ of the vertical foliation $\overline{\mathfrak{D}}$ consists of equilibria for X.*

Proof. X is dual to $E^*(\theta|\mathfrak{D}_1)$ with E a submersion of $\mathfrak{P}_1$ onto $\mathfrak{D}_1$. So $X(\pi) = 0$ if and only if the form $E^*(\theta|\mathfrak{D}_1)$ vanishes at π if and only if $\theta|\mathfrak{D}_1$ vanishes at $\hat{x}(\pi) = E(\pi)$. Since this condition depends only on the mean $\hat{x}(\pi)$ the whole leaf $\overline{\mathfrak{D}}(\pi)$ consists of equilibria.

6. Corollary. *If $\hat{x}(\pi)$ is an equilibrium for X_0 of* (0.6) *then π is an equilibrium for X. Conversely, if the measures of π are full and π is an equilibrium for X then $\hat{x}(\pi)$ is an equilibrium for X_0.*

Proof. If $\mathfrak{D}_1 \subset \mathring{\Delta}(\mathfrak{P})$ then X_0 is dual to $\theta|\mathring{\Delta}(\mathfrak{P})$ with respect to the Shahshahani metric on $\mathring{\Delta}(\mathfrak{P})$. So if $X_0(e) = 0$ with $e = \hat{x}(\pi)$ then $\theta|\mathring{\Delta}(\mathfrak{P}) = 0$ at e and a fortiori $\theta|\mathfrak{D}_1 = 0$ at e. $X(\pi) = 0$ by Proposition 5. If the measures are full then $\mathfrak{D}_1$ is open in $\mathring{\Delta}(\mathfrak{P})$ and so the reasoning can be reversed.

To relate the details of equilibrium behavior of X for full, interior families to that of X_0 on $\mathring{\Delta}$ we recall:

7. Proposition. (a) *Suppose $e \in \mathring{\Delta}$ and $X_0(e) = 0$. Then e is an* ESS *if and only if the quadratic form a_{xx} for $x \in (\mathbf{R}^{n+1})_0$ is negative definite. If we define*

$$I_e(x) = -\sum_i e_i \ln (x_i/e_i) \quad \textit{for } x \in \mathring{\Delta}$$

then $I_e(x)$ is a smooth nonnegative function of x vanishing only at e. e is an ESS *if and only if*

(3.14) $\quad d_x I_e(X_0(x)) < 0 \quad e \neq x \in \mathring{\Delta}.$

(b) *If X_0 never vanishes on $\mathring{\Delta}$, i.e., there are no interior equilibria, then there exists $b \in (\mathbf{R}^{n+1})_0$ such that with $L^b(x) = \sum_i b_i \ln x_i$ for $x \in \mathring{\Delta}$,*

(3.15) $\quad d_x L^b(X_0(x)) > 0 \quad x \in \mathring{\Delta}.$

Proof. The first part of (a) is proved in [**1**, Appendix 3]. By strict convexity of ln,

$$I_e(x) \geqq -\ln \sum e_i(x_i/e_i) = 0$$

with equality only when all the x_i/e_i's are equal, in which case they are all 1. By direct computation

$$d_x I_e(X_0(x)) = a_{xx} - a_{ex} = a_{(x-e)(x-e)}.$$

Note that because e is an interior equilibrium $a_{ie} = a_{ee}$ for all i and so $a_{xe} = a_{ee}$.

$$a_{(x-e)(x-e)} < 0$$

for all $x \neq e$ in $\mathring{\Delta}$ if and only if the quadratic form is negative definite on $(\mathbf{R}^{n+1})_0$. See [**8**].

(b) is [**2**, Theorem 3].

Remark. If the quadratic form a_{xx} on $(\mathbf{R}^{n+1})_0$ is negative semidefinite instead of definite, then (3.14) is still true if $<$ is replaced by $\leqq$. So e is stable although not necessarily asymptotically stable. In such a case we call e a *weak* ESS.

The main tool in relating these X_0 results to X is:

8. LEMMA. *Let $b \in \mathbf{R}^{n+1}$. There exists $\mu \in \mathfrak{M}$ such that:*

$$(3.16) \quad \langle \mu, p_i \rangle = b_i \quad i = 0, \dots, n$$

if and only if b is in the vector subspace of $\mathbf{R}^{n+1}$ spanned by $\bar{C}$. In particular if the measures of $\mathfrak{P}_1$ are full and interior such μ's exist for all b in $\mathbf{R}^{n+1}$.

Define L^b: $\mathring{\Delta} \to \mathbf{R}$ by $L^b(x) = \sum_i b_i \ln x_i$. Fix $\pi_0 \in \mathfrak{P}_1$ and recall that $L_{\pi_0}{}^{\mu}$: $\mathfrak{P} \to \mathbf{R}$ is defined by

$$L_{\pi_0}{}^{\mu}(\pi) = \langle \mu, \ln (d\pi/d\pi_0) \rangle.$$

If μ and b are related by (3.16) *then*

$$(3.17) \quad d_\pi L_{\pi_0}{}^{\mu}(X(\pi)) = d_{\hat{x}} L^b(X_0(\hat{x}))$$

for all $\pi \in \mathfrak{P}_1$. Here $\hat{x}$ is $\hat{x}(\pi)$.

Proof. $\mathfrak{P}$ is open in $\mathfrak{M}$ and E: $\mathfrak{P}_1 \to \Delta$ is the restriction of the linear map: $\mu \to \langle \mu, p \rangle$ from $\mathfrak{M}$ to $\mathbf{R}^{n+1}$. By the remark after Theorem 2.6 the image of $\mathfrak{M}$ is the subspace spanned by $E(\mathfrak{P}_1) = \mathfrak{O}_1$. This proves the first paragraph.

(3.17) is a direct computation using (3.9) and (1.24):

$$d_\pi L_{\pi_0}{}^{\mu}(X(\pi)) = (\bar{\nabla}_\pi L_{\pi_0}{}^{\mu}, X(\pi))_\pi = \sum_{ij} (\bar{\nabla}_\pi L_{\pi_0}{}^{\mu}, \bar{\nabla}_\pi E^i)_\pi a_{ij} E^j(\pi)$$
$$= \sum_{ij} (\langle \mu, p_i \rangle - \langle \mu, 1 \rangle \langle \pi, p_i \rangle) a_{ij} E^j(\pi).$$

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But $\langle \mu, p_i \rangle = b_i$ so that $\langle \mu, 1 \rangle = \sum_k b_k$. Also, $\langle \pi, p_i \rangle = E^i(\pi) = \hat{x}_i$. So we continue:

$$= \sum_{ij} \left(b_i - \left(\sum_k b_k\right)\hat{x}_i\right) a_{ij}\hat{x}_j = \sum_{ij} b_i(a_{i\hat{x}} - a_{\hat{x}\hat{x}})$$

$$= d_{\hat{x}}L^b(X_0(\hat{x})).$$

9. THEOREM. *Let $\mathfrak{P}_1$ be a full, interior family of probability measures.*

(a) *If $e = \hat{x}(\pi_1)$ is an ESS for X_0 then π_1 is a globally attracting equilibrium for the restriction of X to the invariant manifold $\overline{\mathfrak{F}}(\pi_1)$. In fact, for $\pi \in \mathfrak{P}_1$ define π_e to be the unique element of $\overline{\mathfrak{F}}(\pi)$ such that $\hat{x}(\pi_e) = e$. Define the function I_e: $\mathfrak{P}_1 \to \mathbf{R}$ by:*

$$(3.18) \quad I_e(\pi) = -\left\langle \pi_e, \ln\left(\frac{d\pi}{d\pi_e}\right)\right\rangle.$$

I_e is a smooth nonnegative function with $I_e(\pi) = 0$ if and only if $\hat{x}(\pi) = e$, i.e., $\pi = \pi_e$. I_e is a global Lyapunov function for X, i.e.,

$$(3.19) \quad d_\pi I_e(X(\pi)) < 0 \quad \hat{x}(\pi) \neq e.$$

(b) *Suppose X_0 has no interior equilibrium. Then there exists $\mu \in \mathfrak{M}_0$ such that with π_0 a fixed element of $\mathfrak{P}_1$:*

$$(3.20) \quad d_\pi L_{\pi_0}{}^\mu(X(\pi)) > 0 \quad \pi \in \mathfrak{P}_1.$$

Proof. (a) Theorem 2.2 gives a diffeomorphism

$$E \times L: \mathfrak{P}_1 \to \mathfrak{O}_1 \times L^\infty/A.$$

The map $\pi \to \pi_e$ from $\mathfrak{P}_1$ to itself is given by

$$\pi \to (\hat{x}(\pi), f) \to (e, f) \to \pi_e$$

where the first map is $E \times L$ and the third is $(E \times L)^{-1}$. Clearly, $\pi \to \pi_e$ is a smooth map.

Fix $\pi_0 \in \mathfrak{P}_1$ with $\hat{x}(\pi_0) = e$. Since $\pi_e \in \overline{\mathfrak{F}}(\pi)$,

$$\ln\frac{d\pi}{d\pi_e} \in A.$$

Since $\hat{x}(\pi_0) = e = \hat{x}(\pi_e)$, $\pi_e - \pi_0 \in B$ which annihilates A. Hence,

$$(3.21) \quad I_e(\pi) = -\left\langle \pi_e, \ln\left(\frac{d\pi}{d\pi_e}\right)\right\rangle = -\left\langle \pi_0, \ln\left(\frac{d\pi}{d\pi_e}\right)\right\rangle$$

$$= -\left\langle \pi_0, \ln\left(\frac{d\pi}{d\pi_0}\right)\right\rangle + \left\langle \pi_0, \ln\left(\frac{d\pi_e}{d\pi_0}\right)\right\rangle$$

$$= -L_{\pi_0}{}^{\pi_0}(\pi) + L_{\pi_0}{}^{\pi_0}(\pi_e).$$

So $I_e(\pi)$ is a smooth function of π. Note that the second term which we

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will call $L_2(\pi)$ is constant on leaves of $\bar{\mathfrak{F}}$. Strict convexity of the log again implies

$$I_e(\pi) \geqq -\ln\left\langle \pi_e, \frac{d\pi}{d\pi_e}\right\rangle = 0$$

with strict inequality unless $d\pi/d\pi_e$ is constant a.e. in which case it is 1 and $\pi = \pi_e$. So $I_e(\pi) > 0$ unless $\pi = \pi_e$, i.e., $\hat{x}(\pi) = e$ and in that case $I_e(\pi) = 0$.

Finally, by Lemma 8 and (3.21):

$$d_\pi I_e(X(\pi)) = -d_{\hat{x}} L^e(X_0(\hat{x})) + d_\pi L_2(X(\pi)).$$

Since L_2 is constant on the leaves of $\bar{\mathfrak{F}}$ and $X(\pi)$ is tangent to $\bar{\mathfrak{F}}$ the second term vanishes. On $\mathring{\Delta}$,

$$-L^e(x) = -\sum e_i \ln x_i$$

differs by a constant from

$$I_e(x) = -\sum e_i \ln (x_i/e_i).$$

So we have

$$(3.22) \quad d_\pi I_e(X(\pi)) = d_{\hat{x}} I_e(X_0(\hat{x})).$$

(3.19) follows from (3.14).

(b) Choose $b \in (\mathbf{R}^{n+1})_0$ satisfying (3.15). By Lemma 8 we can choose $\mu \in \mathfrak{M}$ satisfying (3.16). Since $\langle \mu, 1\rangle = \sum b_i = 0$, $\mu \in \mathfrak{M}_0$. (3.20) follows from (3.17) and (3.16).

Remarks. (a) If e is a weak ESS (cf. the remark after Proposition 7) then (3.19) is true with $<$ replaced by $\leqq$. So each π with $\hat{x}(\pi) = e$ is stable in its leaf.

(b) The function of π: $-L_{\pi_0}{}^{\pi_0}(\pi)$ is nonnegative and vanishes only at π_0. It is the information function of Kullback [**9**].

There is a special case where the entire pattern of behavior of the vectorfield X mimics that of X_0. We call the game *totally cooperative* if the payoff matrix is symmetric, i.e.,

$$(3.23) \quad a_{ij} = a_{ji}.$$

In this case the payoffs to the two players are always the same so there is a common interest in finding the largest mean payoff. As pointed out in [**1**, Appendix 3] this is the genetic model of natural selection and in this case X_0 is the gradient on $\mathring{\Delta}$ of $\frac{1}{2}a_{xx}$ with respect to the Shahshahani metric.

10. Theorem. *If the game is totally cooperative then the vectorfield X on*

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$\mathfrak{P}_1$ is a gradient vectorfield with respect to the Shahshahani metric. It is the gradient of

$$\frac{1}{2}a_{\hat{x}\hat{x}} = \frac{1}{2}\sum_{i,j} E^i(\pi)a_{ij}E^j(\pi).$$

Proof. Since a_{ij} is symmetric, direct computation of the gradient of $\frac{1}{2}a_{\hat{x}\hat{x}}$ yields the right side of (3.9).

We now describe a situation where the behavior about an equilibrium for X_0 can be quite different from the behavior about the equilibria for X lying over it. We begin examining the possible values that the covariance matrices F_{ij} can have.

Let Sym denote the vector space of symmetric $(n + 1) \times (n + 1)$ matrices. Define the map S: $\mathbf{R}^{n+1} \to \text{Sym}$ by:

$$(3.24) \quad S(v)_{ij} = v_i v_j.$$

11. LEMMA. *Let G be an open subset of $\mathring{\Delta}$. For $p \in G$ define $\delta(p)$ to be the Euclidean distance from p to the closed set $\Delta - G$.*

Let $q_{ij} \in$ Sym such that $\sum_i q_{ij} = 0$ for all j. Let $p \in G$. If q_{ij} has non-negative eigenvalues, i.e., the associated form is positive semi-definite and

$$(3.25) \quad \sup\left\{\sum_{ij} v_i q_{ij} v_j\colon \|v\| = 1\right\} < \delta(p)^2/n$$

then there exist sequences $\{t^{(k)}: k = 1, \ldots, N\}$ in $[0, 1]$ and

$$\{x^{(k)}: k = 1, \ldots, N\}$$

in G such that:

$$(3.26) \quad \sum_k t^{(k)} = 1$$

$$\sum_k t^{(k)} S(x^{(k)})_{ij} = q_{ij} + p_i p_j.$$

Proof. The result is clear if q_{ij} is zero (use $x = p$), so assume it is not. The symmetric matrix q_{ij} can be diagonalized by an orthonormal basis in $\mathbf{R}^{n+1}$. Since q_{ij} annihilates the constant vector 1, we can find a sequence $\{v^{(k)}: k = 1, \ldots, N_1\}$ with $N_1 \leqq n$ of vectors in $(\mathbf{R}^{n+1})_0$ having unit length which are the eigenvectors with eigenvalues $\lambda_k > 0$. This means

$$q_{ij} = \sum_{k=1}^{N_1} \lambda_k S(v^{(k)})_{ij}.$$

Furthermore,

$$0 < \lambda^2 \equiv \sum_{k=1}^{N_1} \lambda_k \leqq N_1 \sup\{q_{vv}: \|v\| = 1\} < \delta(\text{p})^2.$$

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Now let $t_k = \lambda_k/\lambda^2$ and $u^{(k)} = \lambda v^{(k)}$. Then $\|u^{(k)}\| < \delta(p)$, $\sum t_k = 1$ and

$$q_{ij} = \sum_{k=1}^{N_1} t_k S(u^{(k)})_{ij}.$$

It is easy to check that for $x, y \in \mathbf{R}^{n+1}$:

$$S(x) + S(y) = \tfrac{1}{2}(S(x+y) + S(x-y)).$$

So we get:

$$\begin{aligned} q_{ij} + p_ip_j &= \sum t_k(S(u^{(k)})_{ij} + S(p)_{ij}) \\ &= \sum \tfrac{1}{2}t_k(S(p+u^{(k)})_{ij} + S(p-u^{(k)})_{ij}). \end{aligned}$$

Since $\|u^{(k)}\| < \delta(p)$, $p \pm u^{(k)} \in G$.

We saw that the measures of $\mathfrak{P}$ are full and interior if the convex hull of their support has nonempty interior in Δ. We now make a stronger demand; namely that the support itself have nonempty interior.

12. PROPOSITION. *Suppose that G is an open subset of $\mathring{\Delta}$ which is contained in the support of the measures in $\mathfrak{P}$. In particular, the measures in $\mathfrak{P}$ are full and interior.*

Let q_{ij} be a symmetric matrix of rank n, with nonnegative eigenvalues and such that $\sum_i q_{ij} = 0$. Let $p \in G$.

There exists $\epsilon_0 > 0$ depending on q_{ij}, p and G such that for all positive $\epsilon < \epsilon_0$ one can choose $\pi \in \mathfrak{P}_1$ satisfying:

$$(3.26) \quad \hat{x}(\pi) = p \quad \textit{and} \quad F_{ij}(\pi) = \epsilon q_{ij}.$$

Proof. Let

$$\epsilon_0 = \delta(p)^2/n \sup\{q_{vv}: \|v\| = 1\}.$$

Associated with $S: \Delta \to \mathrm{Sym}$ we define $E^S: \mathfrak{P} \to \mathrm{Sym}$ by:

$$(3.27) \quad E^S(\pi)_{ij} = \int_\Delta S(x)_{ij}\pi(dx) = \int_\Delta x_ix_j\pi(dx).$$

When we apply Theorem 2.2 and the remark thereafter to this map, we get that the image $E^S(\mathfrak{P}_1)$ is the interior of the convex hull of the essential image of S, where the interior is taken in an affine subspace of Sym. Now the essential image of S is just the image of S on the support because S is continuous. Let Sym_1 be the affine subspace of Sym defined by:

$$\mathrm{Sym}_1 = \{a_{ij} \in \mathrm{Sym}: \textstyle\sum_{i,j} a_{ij} = 1\}.$$

Note that $S(\Delta) \subset \mathrm{Sym}_1$.

If q_{ij} is positive semidefinite and annihilates 1, $p \in G$ and $\epsilon < \epsilon_0$ then by Lemma 11, $\epsilon q_{ij} + p_ip_j$ is in the convex hull of $S(G)$. If, in addition, q_{ij} has rank n and so is positive definite on $(\mathbf{R}^{n+1})_0$ then any near enough element of Sym_1 is still of the form $\tilde{\epsilon}\tilde{q}_{ij} + \tilde{p}_i\tilde{p}_j$ and so is still in the convex hull of $S(G)$. So if the rank condition is satisfied $\epsilon q_{ij} + p_i p_j$ is in the Sym_1 interior of the convex hull and so lies in $E^S(\mathfrak{P}_1)$. This means that for some $\pi \in \mathfrak{P}_1$,

$$E^S(\pi)_{ij} = \epsilon q_{ij} + p_i p_j.$$

This completes the proof since

$$F_{ij}(\pi) = E^S(\pi)_{ij} - p_i p_j.$$

We now apply these results to examine Hines' characterization of ESS.

13. PROPOSITION. *Let V be a Euclidean vector space, i.e., a finite dimensional space with inner product $(\, , \,)$. Let $L: V \to V$ be a linear map. If for every symmetric positive definite linear map $P: V \to V$ the composite map $P \circ L$ has only nonpositive eigenvalues, then the quadratic form (Lv, v) is negative semidefinite.*

Proof. See the Appendix of [**7**].

14. THEOREM. *Let $e \in \mathring{\Delta}$ be a linearly asymptotically stable equilibrium which is not a weak* ESS *(for example, see* [**17**, *Example* 1]*). If e is in the interior of the support of the measures of $\mathfrak{P}$, then there exist $\pi_1, \pi_2 \in \mathfrak{P}_1$ with $\hat{x}(\pi_1) = \hat{x}(\pi_2) = e$ such that π_1 is an asymptotically stable equilibrium for the restriction of X to $\overline{\mathfrak{F}}(\pi_1)$ while π_2 is an unstable equilibrium for the restriction of X to $\overline{\mathfrak{F}}(\pi_2)$.*

Proof. Fix a leaf $\overline{\mathfrak{F}}(\pi_0)$ of the foliation. $x = \hat{x}(\pi)$ can be regarded as a coordinatization of $\overline{\mathfrak{F}}(\pi_0)$ with $x \in \mathfrak{D}$, i.e., for $x \in \mathfrak{D}$ let

$$\pi(x) = (E|\overline{\mathfrak{F}}(\pi_0))^{-1}(x)$$

be the unique element of the leaf with mean x. Let

$$F_{ij}(x) = F_{ij}(\pi(x)).$$

In this coordinate system, the restriction of X to $\overline{\mathfrak{F}}(\pi_0)$ yields the equation:

$$\frac{dx_i}{dt} = \sum_k F_{ik}(x)a_{kx} = \sum_k F_{ik}(x)(a_{kx} - a_{*x}) \tag{3.28}$$

where

$$a_{*x} = (n+1)^{-1} \sum_k a_{kx}.$$

The first form is a rewriting of (3.13). The second form is equivalent because $\sum_k F_{ik} = 0$.

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At the equilibrium $x = e$, the parenthesized term vanishes. So the linearization of (3.2) for $x = e + v, v \in (\mathbf{R}^{n+1})_0$ is given by

$$(3.29) \quad \frac{dv_i}{dt} = \sum_k F_{ik}(e)(a_{kv} - a_{*v}).$$

To determine the stability of $\pi(e)$ in $\overline{\mathfrak{F}}(\pi_0)$ we have to look at the eigenvalues of the matrix $\sum_k F_{ik}(a_{kj} - a_{*j})$ regarded as a linear operator on $(\mathbf{R}^{n+1})_0$. Proposition 12 says that we can choose the leaf so that up to a positive scalar multiple F_{ij} is any positive definite operator on $(\mathbf{R}^{n+1})_0$ that we want.

Choose π_1 so that $\hat{x}(\pi_1) = e$ and F_{ij} is a multiple of $e_i\delta_{ij} - e_ie_j$ (cf. Remark (b) after Lemma 4). (3.29) is then the linearization of X_0 at e (up to positive multiple). So the eigenvalues have negative real parts by hypothesis and π_1 is asymptotically stable in its leaf.

Since e is not a weak ESS, Proposition 13 implies that we can choose π_2 so that $\hat{x}(\pi_2) = e$ and $\sum_k F_{ik}(\pi_2)(a_{kj} - a_{*j})$ has some eigenvalues with positive real parts. So π_2 is unstable in its leaf.

Remark. By Proposition 13 we see that by varying P we can change the spectrum of $P \circ L$ from stable to unstable. Since the kernel of the operator remains unchanged the only way that such instability can arise is by a pair of complex conjugate eigenvalues crossing the imaginary axis. This means that as we vary π in the leaf of $\overline{\mathfrak{D}}$ over the equilibrium e in such a way that π changes from a stable to an unstable equilibrium, a (possibly degenerate) Hopf bifurcation occurs [**12**; Theorem 3.15]. So there are small cycles in some of the leaves whose means stay entirely in the basin of the attractor e of X_0.

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