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ARTICLE

Regular Schur labeled skew shape posets and their 0-Hecke modules

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Abstract
Assuming Stanley's P-partitions conjecture holds, the regular Schur labeled skew shape posets are precisely the finite posets P with underlying set $\{1, 2, \ldots, |P|\}$ such that the P-partition generating function is symmetric and the set of linear extensions of P, denoted $\Sigma_L(P)$, is a left weak Bruhat interval in the symmetric group $\mathfrak{S}_{|P|}$. We describe the permutations in $\Sigma_L(P)$ in terms of reading words of standard Young tableaux when P is a regular Schur labeled skew shape poset, and classify $\Sigma_L(P)$'s up to descent-preserving isomorphism as P ranges over regular Schur labeled skew shape posets. The results obtained are then applied to classify the 0-Hecke modules M_P associated with regular Schur labeled skew shape posets P up to isomorphism. Then we characterize regular Schur labeled skew shape posets as the finite posets P whose linear extensions form a dual plactic-closed subset of $\mathfrak{S}_{|P|}$. Using this characterization, we construct distinguished filtrations of M_P with respect to the Schur basis when P is a regular Schur labeled skew shape poset. Further issues concerned with the classification and decomposition of the 0-Hecke modules M_P are also discussed.

Contents

1. Introduction

Schur labeled skew shape posets naturally appear in the context of the celebrated Stanley's P-partitions conjecture. Let P_n be the set of posets with underlying set $[n] := \{1, 2, \ldots, n\}$. Each poset $P \in \mathsf{P}_n$ can be identified with the labeled poset (P, ω) with the labeling $\omega : P \to [n]$ given by $\omega(i) = i$. Consequently, to each poset $P \in \mathsf{P}_n$, one can associate the following generating function for its P-partitions:

$$K_P := \sum_{f : P\text{-partition}} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots .$$

In 1972, Stanley [34, p. 81] proposed a conjecture stating that K_P is a symmetric function if and only if P is a Schur labeled skew shape poset. For the precise definition of Schur labeled skew shape posets, refer to Section 2.3. While this conjecture has been verified to be true for all posets P with $|P| \le 8$, it remains an open question in the general case (see [28]). We denote by SP_n the set of all Schur labeled skew shape posets in P_n.

Regular posets were introduced by Björner–Wachs [8] during their investigation of the convex subsets of the symmetric group $\mathfrak{S}_n$ on $\{1, 2, \ldots n\}$ under the right weak Bruhat order. For $P \in \mathsf{P}_n$ with the partial order $\preceq$, let $\Sigma_R(P)$ be the set of permutations $\pi \in \mathfrak{S}_n$ satisfying that if $x \preceq y$, then $\pi^{-1}(x) \le \pi^{-1}(y)$. They observed that every convex subset of $\mathfrak{S}_n$ under the right weak Bruhat order appears as $\Sigma_R(P)$ for some $P \in \mathsf{P}_n$, and every right weak Bruhat interval in $\mathfrak{S}_n$ is convex. This observation led them to characterize the posets $P \in \mathsf{P}_n$ satisfying that $\Sigma_R(P)$ is a right weak Bruhat interval. They introduced the notion of regular posets and proved that $P \in \mathsf{P}_n$ is a regular poset if and only if $\Sigma_R(P)$ is a right weak Bruhat interval in $\mathfrak{S}_n$. For the definition of regular posets, refer to Definition 2.3. We denote by RP_n the set of all regular posets in P_n.

Let $\mathsf{RSP}_n := \mathsf{RP}_n \cap \mathsf{SP}_n$. In the following, we explain the reason why we consider regular Schur labeled skew shape posets from the perspective of the representation theory of the 0-Hecke algebra.

In 1996, Duchamp, Krob, Leclerc and Thibon [12] showed that the Grothendieck ring of the tower of 0-Hecke algebras $\bigoplus_{n\ge 0} H_n(0)$, when equipped with addition and multiplication from direct sum and induction product, is isomorphic to the ring QSym of quasisymmetric functions. To be precise, they showed that the map

$$\mathrm{ch} : \bigoplus_{n\ge 0} \mathcal{G}_0(H_n(0)\textbf{-mod}) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha,$$

called the *quasisymmetric characteristic*, is a ring isomorphism. Here, $\mathcal{G}_0(H_n(0)\textbf{-mod})$ is the Grothendieck group of the category $H_n(0)$**-mod** of finitely generated left $H_n(0)$-modules, α is a composition, $\mathbf{F}_\alpha$ is the irreducible $H_n(0)$-module attached to α, and F_α is the fundamental quasisymmetric function attached to α (for more details, see Section 2.4). Afterwards, Bergeron–Li [5] showed that the map ch is not just a ring isomorphism but also a Hopf algebra isomorphism. In 2002, Duchamp–Hivert–Thibon [11] associated a right $H_n(0)$-module M_P with each poset $P \in \mathsf{P}_n$, such that the image of M_P under the quasisymmetric characteristic is K_P. This was achieved by defining a suitable right $H_n(0)$-action on $\Sigma_R(P)$.

Since the middle of 2010, various left 0-Hecke modules, each equipped with a tableau basis and yielding an important quasisymmetric characteristic image, have been constructed ([2, 4, 32, 37, 38]).

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In order to handle these modules in a uniform manner, Jung–Kim–Lee–Oh [19] introduced a left $H_n(0)$-module $\mathsf{B}(I)$, referred to as *the weak Bruhat interval module associated with* I, for each left weak Bruhat interval I in $\mathfrak{S}_n$. Furthermore, they showed that $\bigoplus_{n\geq 0} \mathcal{G}_0(\mathscr{B}_n)$ is isomorphic to QSym as Hopf algebras, where $\mathscr{B}_n$ is the full subcategory of $H_n(0)$-**mod** consisting of objects that are direct sums of finitely many isomorphic copies of weak Bruhat interval modules of $H_n(0)$. Recently, Choi–Kim–Oh [9] clarified the exact relationship between the weak Bruhat interval modules and the 0-Hecke modules M_P, using Björner–Wachs' characterization. More precisely, they constructed a contravariant functor $\mathcal{F} : H_n(0)$-**mod** $\to$ **mod**-$H_n(0)$ that preserves the quasisymmetric characteristic and showed that $M_P = \mathcal{F}(\mathsf{B}(\Sigma_L(P)))$, where **mod**-$H_n(0)$ is the category of finitely generated right $H_n(0)$-modules and $\Sigma_L(P) := \{\gamma^{-1} \mid \gamma \in \Sigma_R(P)\}$ for $P \in \mathsf{RP}_n$. For technical reasons, we use a slightly different 0-Hecke module, denoted as M_P, instead of Duchamp, Hivert and Thibon's module M_P. This module is a left $H_n(0)$-module with the basis $\Sigma_L(P)$. For the detailed definition of M_P, refer to Definition 2.8.

The aim of this paper is to give a comprehensive investigation of regular Schur labeled skew shape posets and their associated 0-Hecke modules.

In Section 3, we provide an explicit description of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$. We first introduce a Schur labeling τ_P, which is a bijective tableau uniquely determined by suitable conditions. For details, see Equation (3.2). Let λ/μ be the shape of τ_P. Then τ_P gives rise to a reading, denoted read_{τ_P}, on the set $\mathrm{SYT}(\lambda/\mu)$ of standard Young tableaux of shape λ/μ. We show that all permutations in $\Sigma_L(P)$ appear as reading words of standard Young tableaux of shape λ/μ, i.e., $\Sigma_L(P) = \mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$ (Lemma 3.2). Then, we derive that

$$\Sigma_L(P) = [\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})]_L,$$

where $T_{\lambda/\mu}$(resp. $T'_{\lambda/\mu}$) is the standard Young tableau obtained by filling the Young diagram of shape λ/μ by $1, 2, \dots, n$ from left to right starting with the top row (resp. from top to bottom starting with leftmost column) (Theorem 3.9).

In Section 4, we introduce an equivalence relation $\overset{D}{\simeq}$ on the set $\mathrm{Int}(n)$ of left weak Bruhat intervals in $\mathfrak{S}_n$. This relation is defined by $I_1 \overset{D}{\simeq} I_2$ if there is a descent-preserving poset isomorphism between I_1 and I_2. We show that every equivalence class C is of the form

$$\{[\gamma, \xi_C\gamma]_L \mid \gamma \in [\sigma_0, \sigma_1]_R\},$$

where σ_0 and σ_1 are the minimal and maximal elements in $\{\sigma \mid [\sigma, \rho]_L \in C\}$, respectively, and $\xi_C = \rho\sigma^{-1}$ for any $[\sigma, \rho]_L \in C$ (Theorem 4.6). In the case where $P \in \mathsf{RSP}_n$, we show in Theorem 4.7 that the equivalence class of $\Sigma_L(P)$ is given by

$$\{\Sigma_L(Q) \mid Q \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_Q) = \mathrm{sh}(\tau_P)\}. \tag{1.1}$$

In Section 5, we classify the $H_n(0)$-modules M_P up to isomorphism as P ranges over RSP_n. We show in Theorem 5.5 that for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q).$$

The 'if' part is straightforward and can be derived from Equation (1.1). As for the 'only if' part, it can be verified by showing that when τ_P and τ_Q have different shapes, it results in either nonisomorphic projective covers or nonisomorphic injective hulls of M_P and M_Q. To accomplish this, we compute both a projective cover and an injective hull of M_P for $P \in \mathsf{RSP}_n$ (Lemma 5.4).

In Section 6, we first prove that a poset $P \in \mathsf{P}_n$ is a regular Schur labeled skew shape poset if and only if $\Sigma_L(P)$ is dual plactic-closed (Theorem 6.4). This improves Malvenuto's result [27, Theorem 1], which states that if $\Sigma_L(P)$ is dual plactic-closed, then $P \in \mathsf{SP}_n$. Then, we introduce the notion of a *distinguished filtration* of an $H_n(0)$-module M with respect to a linearly independent subset $\mathcal{B}$ of QSym_n (Definition 6.5). If such a filtration is available, we have a representation theoretic interpretation of the expansion of $\mathrm{ch}([M])$ in $\mathcal{B}$. The existence of a distinguished filtration is quite nontrivial as seen in Example 6.6. However, using the characterization given in Theorem 6.4, we show that M_P admits a distinguished filtration with respect to the Schur basis when $P \in \mathsf{RSP}_n$ (Theorem 6.7).

The final section is mainly devoted to further issues concerned with the classification and decomposition of the 0-Hecke modules M_P. We discuss the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$ and $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$. In particular, we expect that for $P, Q \in \mathsf{RP}_n$, $\mathsf{M}_P \cong \mathsf{M}_Q$ if and only if $\Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q)$ (Conjecture 7.2). The decomposition problem is also discussed for the 0-Hecke modules M_P when $P \in \mathsf{RSP}_n$. Based on experimental data, we expect that for $P \in \mathsf{RSP}_n$, M_P is indecomposable if and only if $\mathrm{sh}(\tau_P)$ is disconnected and does not contain any disconnected ribbon (Conjecture 7.5). At the end of this section, we provide a remark on how to recover M_P for $P \in \mathsf{RSP}_n$ from a module of the generic Hecke algebra $H_n(q)$ by specializing q to 0.

In the appendix, we give a tableau description of M_P for $P \in \mathsf{RSP}_n$. For a skew partition λ/μ of size n, we construct an $H_n(0)$-module $X_{\lambda/\mu}$ with standard Young tableaux of shape λ/μ as basis elements. This module can be viewed as a representative of the isomorphism class of M_P in the category $H_n(0)$-**mod** for every $P \in \mathsf{RSP}_n$ with $\mathrm{sh}(\tau_P) = \lambda/\mu$.

2. Preliminaries

For integers m and n, we define $[m, n]$ and $[n]$ to be the intervals $\{t \in \mathbb{Z} \mid m \le t \le n\}$ and $\{t \in \mathbb{Z} \mid 1 \le t \le n\}$, respectively. Throughout this paper, n will denote a nonnegative integer unless otherwise stated.

2.1. Compositions, Young diagrams and bijective tableaux

A *composition* α of n, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^k \alpha_i = n$. We call α_i $(1 \le i \le k)$ a *part* of α, $k =: \ell(\alpha)$ the *length* of α, and $n =: |\alpha|$ the *size* of α. And we define the empty composition $\varnothing$ to be the unique composition of size and length 0. Whenever necessary, we set $\alpha_i = 0$ for all $i > \ell(\alpha)$.

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$ and $I = \{i_1 < i_2 < \cdots < i_l\} \subseteq [n-1]$, let

$$\mathrm{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}, \text{ and}$$
$$\mathrm{comp}(I) := (i_1, i_2 - i_1, i_3 - i_2, \ldots, n - i_l).$$

The set of compositions of n is in bijection with the set of subsets of $[n-1]$ under the correspondence $\alpha \mapsto \mathrm{set}(\alpha)$ (or $I \mapsto \mathrm{comp}(I)$). The *reverse composition* α^{r} *of* α is defined to be the composition $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$, and the *complement* α^{c} *of* α is defined to be the unique composition satisfying $\mathrm{set}(\alpha^c) = [n-1] \setminus \mathrm{set}(\alpha)$.

If a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \models n$ satisfies $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$, then it is called a *partition* of n and denoted as $\lambda \vdash n$. Given two partitions λ and μ with $\ell(\lambda) \ge \ell(\mu)$, we write $\lambda \supseteq \mu$ if $\lambda_i \ge \mu_i$ for all $1 \le i \le \ell(\mu)$. A *skew partition* λ/μ is a pair (λ, μ) of partitions with $\lambda \supseteq \mu$. We call $|\lambda/\mu| := |\lambda| - |\mu|$ the *size* of λ/μ. In the case where $\lambda \supset \mu \supset \nu$, we say that λ/μ *extends* μ/ν.

Given a partition λ, we define the *Young diagram* $\mathtt{yd}(\lambda)$ *of* λ to be the left-justified array of n boxes, where the ith row from the top has λ_i boxes for $1 \le i \le k$. Similarly, given a skew partition λ/μ, we define the *Young diagram* $\mathtt{yd}(\lambda/\mu)$ *of* λ/μ to be the Young diagram $\mathtt{yd}(\lambda)$ with all boxes belonging to $\mathtt{yd}(\mu)$ removed. A Young diagram is called *connected* if for each pair of consecutive rows, there are at least two boxes (one in each row) which have a common edge. A skew partition is called *connected* if the corresponding Young diagram is connected, and it is called *basic* if the corresponding Young diagram contains neither empty rows nor empty columns. In this paper, every skew partition is assumed to be basic unless otherwise stated.

For two skew partitions λ/μ and ν/κ, we define $\lambda/\mu \star \nu/\kappa$ to be the skew partition whose Young diagram is obtained by taking a rectangle of empty squares with the same number of rows as $\mathtt{yd}(\lambda/\mu)$

and the same number of columns as $\mathtt{yd}(\nu/\kappa)$, and putting $\mathtt{yd}(\nu/\kappa)$ below and $\mathtt{yd}(\lambda/\mu)$ to the right of this rectangle. For instance, if $\lambda/\mu = (2,2)$ and $\nu/\kappa = (3,2)/(1)$, then $\lambda/\mu \star \nu/\kappa = (5,5,3,2)/(3,3,1)$ and

$$\mathtt{yd}(\lambda/\mu \star \nu/\kappa) = \begin{array}{ccccc} & & & \square & \square \\ & & & \square & \square \\ & \square & \square & & \\ \square & \square & & & \end{array}.$$

Given a skew partition λ/μ of size n, a *bijective tableau* of shape λ/μ is a filling of $\mathtt{yd}(\lambda/\mu)$ with distinct entries in $[n]$. For later use, we denote by $\tau_0^{\lambda/\mu}$ (resp. $\tau_1^{\lambda/\mu}$) the bijective tableau of shape λ/μ obtained by filling $1, 2, \ldots, n$ from right to left starting with the top row (resp. from top to bottom starting with the rightmost column). If λ/μ is clear in the context, we will drop the superscript λ/μ from $\tau_0^{\lambda/\mu}$ and $\tau_1^{\lambda/\mu}$. Again, letting $\lambda/\mu = (2,2) \star (3,2)/(1)$, we have

$$\tau_0 = \begin{array}{ccccc} & & & 2 & 1 \\ & & & 4 & 3 \\ & 6 & 5 & & \\ 8 & 7 & & & \end{array} \quad \text{and} \quad \tau_1 = \begin{array}{ccccc} & & & 3 & 1 \\ & & & 4 & 2 \\ & 6 & 5 & & \\ 8 & 7 & & & \end{array}.$$

A bijective tableau is referred to as a *standard Young tableau* if the elements in each row are arranged in increasing order from left to right, and the elements in each column are arranged in increasing order from top to bottom. We denote by $\mathrm{SYT}(\lambda/\mu)$ the set of all standard Young tableaux of shape λ/μ. And we let $\mathrm{SYT}_n := \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda)$.

2.2. *Weak Bruhat orders on the symmetric group*

Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. Every permutation $\sigma \in \mathfrak{S}_n$ can be expressed as a product of simple transpositions $s_i := (i, i+1)$ for $1 \le i \le n-1$. A *reduced expression for* σ is an expression that represents σ in the shortest possible length, and the *length* $\ell(\sigma)$ *of* σ is the number of simple transpositions in any reduced expression for σ. Let

$$\mathrm{Des}_L(\sigma) := \{i \in [n-1] \mid \ell(s_i\sigma) < \ell(\sigma)\} \text{ and } \mathrm{Des}_R(\sigma) := \{i \in [n-1] \mid \ell(\sigma s_i) < \ell(\sigma)\}.$$

It is well known that if $\sigma = w_1 w_2 \cdots w_n$ in one-line notation, then

$$\begin{aligned} &\mathrm{Des}_L(\sigma) = \{i \in [n-1] \mid i \text{ is right of } i+1 \text{ in } w_1 w_2 \cdots w_n\} \quad \text{and} \\ &\mathrm{Des}_R(\sigma) = \{i \in [n-1] \mid w_i > w_{i+1}\}. \end{aligned}$$

The *left weak Bruhat order* $\preceq_L$ (resp. *right weak Bruhat order* $\preceq_R$) on $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ whose covering relation $\preceq_L^{\mathrm{c}}$ (resp. $\preceq_R^{\mathrm{c}}$) is given as follows:

$$\sigma \preceq_L^{\mathrm{c}} s_i\sigma \text{ if } i \notin \mathrm{Des}_L(\sigma) \quad (\text{resp. } \sigma \preceq_R^{\mathrm{c}} \sigma s_i \text{ if } i \notin \mathrm{Des}_R(\sigma)).$$

Although these two weak Bruhat orders are not identical, there exists a poset isomorphism

$$(\mathfrak{S}_n, \preceq_L) \to (\mathfrak{S}_n, \preceq_R), \quad \sigma \mapsto \sigma^{-1}.$$

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For each $\gamma \in \mathfrak{S}_n$, let

$$\mathrm{Inv}_L(\gamma) := \{(i, j) \mid 1 \le i < j \le n \text{ and } \gamma(i) > \gamma(j)\} \quad \text{and}$$
$$\mathrm{Inv}_R(\gamma) := \{(\gamma(i), \gamma(j)) \mid 1 \le i < j \le n \text{ and } \gamma(i) > \gamma(j)\}.$$

Then, for $\sigma, \rho \in \mathfrak{S}_n$,

$$\sigma \preceq_L \rho \quad \text{if and only if} \quad \mathrm{Inv}_L(\sigma) \subseteq \mathrm{Inv}_L(\rho) \quad \text{and}$$
$$\sigma \preceq_R \rho \quad \text{if and only if} \quad \mathrm{Inv}_R(\sigma) \subseteq \mathrm{Inv}_R(\rho).$$

Given $\sigma, \rho \in \mathfrak{S}_n$, the *left weak Bruhat interval* $[\sigma, \rho]_L$ (resp. the *right weak Bruhat interval* $[\sigma, \rho]_R$) denotes the closed interval $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$ (resp. $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_R \gamma \preceq_R \rho\}$) with respect to the left weak Bruhat order (resp. the right weak Bruhat order).

For later use, we introduce the following lemma.

Lemma 2.1 [6, Proposition 3.1.6]. *For $\sigma, \rho \in \mathfrak{S}_n$ with $\sigma \preceq_R \rho$, the map $[\sigma, \rho]_R \to [\mathrm{id}, \sigma^{-1}\rho]_R, \gamma \mapsto \sigma^{-1}\gamma$ is a poset isomorphism. Equivalently, for $\sigma, \rho \in \mathfrak{S}_n$ with $\sigma \preceq_L \rho$, the map $[\sigma, \rho]_L \to [\mathrm{id}, \rho\sigma^{-1}]_L, \gamma \mapsto \gamma\sigma^{-1}$ is a poset isomorphism.*

Let us collect notations which will be used later. For $S \subseteq \mathfrak{S}_n$ and $\xi \in \mathfrak{S}_n$, let

$$S \cdot \xi := \{\gamma\xi \mid \gamma \in S\} \quad \text{and} \quad \xi \cdot S := \{\xi\gamma \mid \gamma \in S\}.$$

We use w_0 to denote the longest element in $\mathfrak{S}_n$. For $I \subseteq [n-1]$, let $\mathfrak{S}_I$ be the parabolic subgroup of $\mathfrak{S}_n$ generated by $\{s_i \mid i \in I\}$ and $w_0(I)$ the longest element in $\mathfrak{S}_I$. For $\alpha \models n$, let $w_0(\alpha) := w_0(\mathrm{set}(\alpha))$. Finally, for $\sigma \in \mathfrak{S}_n$, we let $\sigma^{w_0} := w_0\sigma w_0$.

Lemma 2.2 [7, Theorem 6.2]. *For $I \subseteq J \subseteq [n-1]$, we have*

$$\{\sigma \in \mathfrak{S}_n \mid I \subseteq \mathrm{Des}_L(\sigma) \subseteq J\} = [w_0(I), w_0(J^{\mathrm{c}})w_0]_R.$$

2.3. Regular posets and Schur labeled skew shape posets

Let P_n be the set of posets whose underlying set is $[n]$. Given $P \in \mathsf{P}_n$, we write the partial order of P as $\preceq_P$.

Definition 2.3 [8, p. 110]. A poset $P \in \mathsf{P}_n$ is said to be *regular* if the following holds: for all $x, y, z \in [n]$ with $x \preceq_P z$, if $x < y < z$ or $z < y < x$, then $x \preceq_P y$ or $y \preceq_P z$.

We denote by RP_n the set of all regular posets in P_n. In the following, we will explain how regular posets can be characterized in terms of left weak Bruhat intervals.

Given $P \in \mathsf{P}_n$, let

$$\Sigma_L(P) := \{\sigma \in \mathfrak{S}_n \mid \sigma(i) \le \sigma(j) \text{ for all } i, j \in [n] \text{ with } i \preceq_P j\}.$$

Throughout this paper, $\Sigma_L(P)$ is considered as the set of all linear extensions of P under the correspondence $\sigma \mapsto ([n], \preceq_E)$, where $\preceq_E$ is the total order on $[n]$ given by $\sigma^{-1}(1) \preceq_E \sigma^{-1}(2) \preceq_E \cdots \preceq_E \sigma^{-1}(n)$.

Theorem 2.4 [8, Theorem 6.8]. *Let $U \subseteq \mathfrak{S}_n$ with $|U| > 1$. The following conditions are equivalent:*

(1) *U is a left weak Bruhat interval.*
(2) *$U = \Sigma_L(P)$ for some $P \in \mathsf{RP}_n$.*

Consider the map

$$\eta : \mathsf{P}_n \to \mathscr{P}(\mathfrak{S}_n), \quad P \mapsto \Sigma_L(P),$$

where $\mathscr{P}(\mathfrak{S}_n)$ is the power set of $\mathfrak{S}_n$. One can see that η is injective. Combining this with Theorem 2.4, we obtain a one-to-one correspondence

$$\eta|_{\mathsf{RP}_n} : \mathsf{RP}_n \to \mathrm{Int}(n), \quad P \mapsto \Sigma_L(P),$$

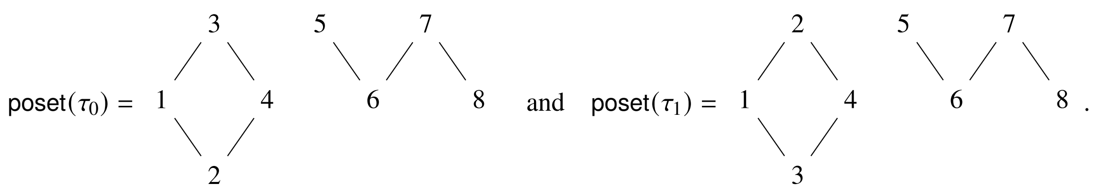
where $\mathrm{Int}(n)$ is the set of nonempty left weak Bruhat intervals in $\mathfrak{S}_n$.

Next, let us introduce Schur labeled skew shape posets. Let λ/μ be a skew partition of size n. Given a bijective tableau τ of shape λ/μ, we define $\mathsf{poset}(\tau)$ to be the poset $([n], \preceq_\tau)$, where

$$i \preceq_\tau j \text{ if and only if } i \text{ lies weakly upper-left of } j \text{ in } \tau. \tag{2.1}$$

The Hasse diagram of $\mathsf{poset}(\tau)$ can be obtained by rotating τ 135° counterclockwise.[1]

Example 2.5. Let $\lambda/\mu = (2,2) \star (3,2)/(1)$. For the bijective tableaux τ_0 and τ_1 of shape λ/μ introduced in Section 2.1, we have



A *Schur labeling of shape λ/μ* is a bijective tableau of shape λ/μ such that the entries in each row decrease from left to right and the entries in each column increase from top to bottom. Let $\mathsf{S}(\lambda/\mu)$ be the set of all Schur labelings of shape λ/μ. Since τ_0 and τ_1 are Schur labelings of shape λ/μ, $\mathsf{S}(\lambda/\mu)$ is nonempty. Set

$$\mathsf{SP}(\lambda/\mu) := \{\mathsf{poset}(\tau) \mid \tau \in \mathsf{S}(\lambda/\mu)\} \quad \text{and} \quad \mathsf{SP}_n := \bigcup_{|\lambda/\mu|=n} \mathsf{SP}(\lambda/\mu).$$

Definition 2.6. A poset $P \in \mathsf{P}_n$ is said to be a *Schur labeled skew shape poset* if it is contained in SP_n.

Remark 2.7. In some papers, for instance [27, 28], authors used a different convention than ours for Schur labeling. We adopt the definition of Schur labeling used in Stanley's paper [34].

For simplicity, we set $\mathsf{RSP}_n := \mathsf{RP}_n \cap \mathsf{SP}_n$.

2.4. The 0-Hecke algebra and the quasisymmetric characteristic

The 0-Hecke algebra $H_n(0)$ is the associative $\mathbb{C}$-algebra with 1 generated by $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the following relations:

[1]Note that $\mathsf{poset}(\tau) \in \mathsf{P}_n$. Following our convention, the partial order $\preceq_\tau$ can also be written as $\preceq_{\mathsf{poset}(\tau)}$.

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$$\pi_i^2 = \pi_i \quad \text{for } 1 \le i \le n-1,$$
$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} \quad \text{for } 1 \le i \le n-2,$$
$$\pi_i\pi_j = \pi_j\pi_i \quad \text{if } |i-j| \ge 2.$$

For each $1 \le i \le n-1$, let $\overline{\pi}_i := \pi_i - 1$. Then, $\{\overline{\pi}_i \mid i = 1, 2, \ldots, n-1\}$ is also a generating set of $H_n(0)$.

For any reduced expression $s_{i_1}s_{i_2}\cdots s_{i_p}$ for $\sigma \in \mathfrak{S}_n$, let

$$\pi_\sigma := \pi_{i_1}\pi_{i_2}\cdots\pi_{i_p} \quad \text{and} \quad \overline{\pi}_\sigma := \overline{\pi}_{i_1}\overline{\pi}_{i_2}\cdots\overline{\pi}_{i_p}.$$

It is well known that these elements are independent of the choices of reduced expressions, and both $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ and $\{\overline{\pi}_\sigma \mid \sigma \in \mathfrak{S}_n\}$ are $\mathbb{C}$-bases for $H_n(0)$.

According to [30], there are 2^{n-1} pairwise nonisomorphic irreducible $H_n(0)$-modules which are naturally indexed by compositions of n. To be precise, for each composition α of n, there exists an irreducible $H_n(0)$-module $\mathbf{F}_\alpha := \mathbb{C}v_\alpha$ endowed with the $H_n(0)$-action defined as follows: for each $1 \le i \le n-1$,

$$\pi_i \cdot v_\alpha = \begin{cases} 0 & i \in \mathrm{set}(\alpha), \\ v_\alpha & i \notin \mathrm{set}(\alpha). \end{cases}$$

Let $H_n(0)$-**mod** be the category of finite dimensional left $H_n(0)$-modules and $\mathcal{R}(H_n(0))$ the $\mathbb{Z}$-span of the set of (representatives of) isomorphism classes of modules in $H_n(0)$-**mod**. We denote by $[M]$ the isomorphism class corresponding to an $H_n(0)$-module M. The *Grothendieck group* $\mathcal{G}_0(H_n(0))$ *of* $H_n(0)$-**mod** is the quotient of $\mathcal{R}(H_n(0))$ modulo the relations $[M] = [M'] + [M'']$ whenever there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$. The equivalence classes of the irreducible $H_n(0)$-modules form a $\mathbb{Z}$-basis for $\mathcal{G}_0(H_n(0))$. Let

$$\mathcal{G} := \bigoplus_{n \ge 0} \mathcal{G}_0(H_n(0)).$$

Let us review the connection between $\mathcal{G}$ and the ring QSym of quasisymmetric functions. For the definition of quasisymmetric functions, see [35, Section 7.19]. For a composition α, the *fundamental quasisymmetric function* F_α, which was firstly introduced in [16], is defined by

$$F_\varnothing = 1 \quad \text{and} \quad F_\alpha = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{set}(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} \quad \text{if } \alpha \ne \varnothing.$$

It is known that $\{F_\alpha \mid \alpha \text{ is a composition}\}$ is a $\mathbb{Z}$-basis for QSym. When M is an $H_m(0)$-module and N is an $H_n(0)$-module, we write $M \boxtimes N$ for the induction product of M and N; that is,

$$M \boxtimes N := M \otimes N \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}.$$

Here, $H_m(0) \otimes H_n(0)$ is viewed as the subalgebra of $H_{m+n}(0)$ generated by $\{\pi_i \mid i \in [m+n-1] \setminus \{m\}\}$. The induction product induces a multiplication on $\mathcal{G}$. It was shown in [12] that the linear map

$$\mathrm{ch} : \mathcal{G} \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha,$$

called *quasisymmetric characteristic*, is a ring isomorphism. Indeed, it turns out to be a Hopf algebra isomorphism when $\mathcal{G}$ has the comultiplication induced from restriction.

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It is well known that $H_n(0)$ has the automorphisms θ and ϕ, as well as the anti-automorphism χ, defined in the following manner:

$$\begin{aligned}
&\phi : H_n(0) \to H_n(0), \quad \pi_i \mapsto \pi_{n-i} \quad \text{for } 1 \le i \le n-1,\\
&\theta : H_n(0) \to H_n(0), \quad \pi_i \mapsto -\overline{\pi}_i \quad \text{for } 1 \le i \le n-1,\\
&\chi : H_n(0) \to H_n(0), \quad \pi_i \mapsto \pi_i \quad \text{for } 1 \le i \le n-1
\end{aligned}$$

(for instance, see [13, 22, 24, 25]). These maps commute with each other.

Note that an automorphism μ of $H_n(0)$ induces a covariant functor

$$\mathbf{T}^+_\mu : H_n(0)\text{-}\mathbf{mod} \to H_n(0)\text{-}\mathbf{mod}$$

called the *μ-twist*. Similarly, an anti-automorphism ν of $H_n(0)$ induces a contravariant functor

$$\mathbf{T}^-_\nu : H_n(0)\text{-}\mathbf{mod} \to H_n(0)\text{-}\mathbf{mod}$$

called the *ν-twist*. For the precise definitions of $\mathbf{T}^+_\mu$ and $\mathbf{T}^-_\nu$, see [19, Subsection 3.4]. In [13, Proposition 3.3.], it was shown that

$$\mathbf{T}^+_\phi(\mathbf{F}_\alpha) = \mathbf{F}_{\alpha^{\mathrm{r}}}, \quad \mathbf{T}^+_\theta(\mathbf{F}_\alpha) = \mathbf{F}_{\alpha^{\mathrm{c}}}, \quad \text{and} \quad \mathbf{T}^-_\chi(\mathbf{F}_\alpha) = \mathbf{F}_\alpha$$

for $\alpha \models n$. Let ρ and ψ be the automorphisms of QSym defined by

$$\rho(F_\alpha) = F_{\alpha^{\mathrm{r}}} \quad \text{and} \quad \psi(F_\alpha) = F_{\alpha^{\mathrm{c}}}$$

for every composition α. For a finite dimensional $H_n(0)$-module M, it holds that

$$\begin{gathered}
\mathrm{ch}([\mathbf{T}^+_\phi(M)]) = \rho \circ \mathrm{ch}([M]), \quad \mathrm{ch}([\mathbf{T}^+_\theta(M)]) = \psi \circ \mathrm{ch}([M]),\\
\text{and} \quad \mathrm{ch}([\mathbf{T}^-_\chi(M)]) = \mathrm{ch}([M]).
\end{gathered} \tag{2.2}$$

2.5. *Modules arising from posets and weak Bruhat interval modules*

Let $P \in \mathsf{P}_n$. In [11, Definition 3.18], Duchamp, Hivert and Thibon defined a right $H_n(0)$-module M_P associated with P. In this paper, we are primarily concerned with left modules, so we introduce a left $H_n(0)$-module, denoted as M_P, associated with P.

Definition 2.8. Let $P \in \mathsf{P}_n$. Define M_P to be the left $H_n(0)$-module with $\mathbb{C}\Sigma_L(P)$ as the underlying space and with the $H_n(0)$-action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma),\\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin \Sigma_L(P),\\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in \Sigma_L(P). \end{cases} \tag{2.3}$$

One can see that the $H_n(0)$-action provided in Equation (2.3) is well-defined through a slight modification of the proof in [11, Subsection 3.9] that M_P is a well-defined right $H_n(0)$-module. Indeed, there is a close connection between M_P and M_P. Let $\mathbf{mod}\text{-}H_n(0)$ be the category of finite dimensional right $H_n(0)$-modules. In [9, Subsection 4.3], the authors introduced a contravariant functor

$$\mathcal{F}_n : H_n(0)\text{-}\mathbf{mod} \to \mathbf{mod}\text{-}H_n(0)$$

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that preserves quasisymmetric characteristics.[2] Using this functor, it is not difficult to see that

$$\mathsf{M}_P \cong \mathbf{T}^+_\theta \circ \mathbf{T}^-_\chi \circ \mathcal{F}_n^{-1}(M_P).$$

Since the underlying set of P is $[n]$, we can regard P as the labeled poset (P,ω) with the labeling $\omega : P \longrightarrow [n]$ given by $\omega(i) = i$. Under this consideration, a map $f : [n] \longrightarrow \mathbb{Z}_{\geq 0}$ is called a *P-partition* if it satisfies the following conditions:

(1) If $i \preceq_P j$, then $f(i) \leq f(j)$.
(2) If $i \preceq_P j$ and $i > j$, then $f(i) < f(j)$.

We define the *P-partition generating function* K_P of P by

$$K_P := \sum_{f:P\text{-partition}} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots .$$

Theorem 2.9. *For $P \in \mathsf{P}_n$, the following hold.*

(1) $\mathrm{ch}([\mathsf{M}_P]) = \psi(K_P)$.
(2) *If $P \in \mathsf{SP}(\lambda/\mu)$ for a skew partition λ/μ, then* $\mathrm{ch}([\mathsf{M}_P]) = s_{\lambda/\mu}$.

Proof. (1) It was shown in [11, Theorem 3.21(i)] that the (right) quasisymmetric characteristic of M_P is given by K_P. Since $\mathsf{M}_P \cong \mathbf{T}^+_\theta \circ \mathbf{T}^-_\chi \circ \mathcal{F}_n^{-1}(M_P)$, the assertion follows from Equation (2.2).

(2) In the same manner as in [35, Subsection 7.19], one can see that $K_P = s_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}}$. Here, λ^{t} and μ^{t} are the transposes of λ and μ, respectively. Now the assertion can be derived from the well-known identity $\psi(s_{\lambda/\mu}) = s_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}}$ (for instance, see [26, Subsection 3.6]). □

Next, let us introduce weak Bruhat interval modules, which were introduced by Jung, Kim, Lee and Oh [19] to provide a unified method for dealing with $H_n(0)$-modules constructed using tableaux.

Definition 2.10 [19, Definition 1]. For each left weak Bruhat interval $[\sigma,\rho]_L$ in $\mathfrak{S}_n$, define $\mathsf{B}([\sigma,\rho]_L)$ (simply, $\mathsf{B}(\sigma,\rho)$) to be the $H_n(0)$-module with $\mathbb{C}[\sigma,\rho]_L$ as the underlying space and with the $H_n(0)$-action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma), \\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin [\sigma,\rho]_L, \\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in [\sigma,\rho]_L. \end{cases}$$

This module is called the *weak Bruhat interval module associated to* $[\sigma,\rho]_L$.

We can deduce from Theorem 2.4 that for every $[\sigma,\rho]_L \in \mathrm{Int}(n)$, there exists a unique poset $P \in \mathsf{P}_n$ such that $\Sigma_L(P) = [\sigma,\rho]_L$. Since both $\mathsf{B}(\sigma,\rho)$ and M_P share $[\sigma,\rho]_L$ as their basis and exhibit identical $H_n(0)$-actions on this set, we can conclude that $\mathsf{B}(\sigma,\rho)$ is indeed equal to M_P.

Remark 2.11. The weak Bruhat interval modules are equipped with the structure of semi-combinatorial $H_n(0)$-modules due to Hivert, Novelli and Thibon [17] and also that of diagram modules due to Searles [33]. More precisely,

(1) $\mathsf{B}(\sigma,\rho)$ is the semi-combinatorial $H_n(0)$-module associated to the Yang-Baxter interval $[Y_\sigma(\mathrm{id}), Y_\rho(\mathrm{id})]$, and
(2) it was shown in [33, Subsection 7.3] that $\mathsf{B}(\sigma,\rho)$ is isomorphic to a diagram module $\widehat{\mathbf{N}}_{\mathsf{StdTab}(D)}$.

[2]In [9, Subsection 4.3], the authors considered both left and right quasisymmetric characteristics because they were simultaneously working with two categories, $H_n(0)$-**mod** and **mod**-$H_n(0)$.

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3. The weak Bruhat interval structure of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$

Let $P \in \mathsf{RSP}_n$. In this section, we explicitly describe the left weak Bruhat interval $\Sigma_L(P)$ in terms of reading words of standard Young tableaux. To begin with, we introduce readings for bijective tableaux.

Definition 3.1. Let τ be a bijective tableau of shape λ/μ. The τ-*reading* is the map

$$\mathsf{read}_\tau : \{\text{bijective tableaux of shape } \lambda/\mu\} \to \mathfrak{S}_n, \quad T \mapsto \mathsf{read}_\tau(T),$$

where $\mathsf{read}_\tau(T)$ is the permutation in $\mathfrak{S}_n$ given by $\mathsf{read}_\tau(T)(k) = T_{\tau^{-1}(k)}$ for $1 \le k \le n$. We call $\mathsf{read}_\tau(T)$ the τ-*reading word of* T.

Given a bijective tableaux T of shape λ/μ, the permutation $\mathsf{read}_\tau(T)$ in one-line notation can be obtained by reading the entries of T in the order given by $\tau^{-1}(1), \tau^{-1}(2), \ldots, \tau^{-1}(n)$. For instance, if $\tau = \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$ and $T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$, then $\mathsf{read}_\tau(T) = 53412$. With this definition, we have the following lemma.

Lemma 3.2. *For any bijective tableau τ of shape λ/μ, $\Sigma_L(\mathsf{poset}(\tau)) = \mathsf{read}_\tau(\mathrm{SYT}(\lambda/\mu))$.*

Proof. We first show that $\mathsf{read}_\tau(\mathrm{SYT}(\lambda/\mu)) \subseteq \Sigma_L(\mathsf{poset}(\tau))$. To do this, take any $T \in \mathrm{SYT}(\lambda/\mu)$ and $i, j \in [n]$ with $i \preceq_\tau j$ (for the definition of $\preceq_\tau$, see Equation (2.1). Let B_1 and B_2 be the boxes in $\mathsf{yd}(\lambda/\mu)$ such that $\tau_{B_1} = i$ and $\tau_{B_2} = j$. Since $i \preceq_\tau j$, B_1 is weakly upper-left of B_2 in $\mathsf{yd}(\lambda/\mu)$. This implies that

$$\mathsf{read}_\tau(T)(i) = T_{B_1} \le T_{B_2} = \mathsf{read}_\tau(T)(j).$$

Therefore, $\mathsf{read}_\tau(T) \in \Sigma_L(\mathsf{poset}(\tau))$.

To complete the proof, let us show that $|\Sigma_L(\mathsf{poset}(\tau))| = |\mathsf{read}_\tau(\mathrm{SYT}(\lambda/\mu))|$. Note that

$$\mathrm{ch}([\mathsf{M}_{\mathsf{poset}(\tau_0^{\lambda/\mu})}]) = \sum_{\gamma \in \Sigma_L(\mathsf{poset}(\tau_0^{\lambda/\mu}))} F_{\mathrm{comp}(\mathrm{Des}_L(\gamma))^{\mathrm{c}}} \quad \text{and} \quad \mathrm{ch}([\mathsf{M}_{\mathsf{poset}(\tau_0^{\lambda/\mu})}]) = s_{\lambda/\mu},$$

where the second equality follows from Theorem 2.9(2). Putting these together with the well-known equality $s_{\lambda/\mu} = \sum_{T \in \mathrm{SYT}(\lambda/\mu)} F_{\mathrm{comp}(T)}$, we have

$$\sum_{\gamma \in \Sigma_L(\mathsf{poset}(\tau_0^{\lambda/\mu}))} F_{\mathrm{comp}(\mathrm{Des}_L(\gamma))^{\mathrm{c}}} = \sum_{T \in \mathrm{SYT}(\lambda/\mu)} F_{\mathrm{comp}(T)}. \tag{3.1}$$

Here, $\mathrm{comp}(T) = \mathrm{comp}(\{i \in [n-1] \mid i \text{ is weakly right of } i+1 \text{ in } T\})$. As a consequence of Equation (3.1), we have

$$|\Sigma_L(\mathsf{poset}(\tau))| = |\Sigma_L(\mathsf{poset}(\tau_0^{\lambda/\mu}))| = |\mathrm{SYT}(\lambda/\mu)| = |\mathsf{read}_\tau(\mathrm{SYT}(\lambda/\mu))|. \qquad \square$$

The purpose of the remainder of this section is to describe the minimum and maximum of $\Sigma_L(P)$ with respect to $\preceq_L$.

As a first step, we deal with a characterization of regular Schur labeled skew shape posets. For this purpose, the following definition is necessary.

Definition 3.3. Let λ/μ be a skew partition of size n. A Schur labeling τ of shape λ/μ is said to be *distinguished* if $\tau_B \ge \tau_{B'}$ whenever B is weakly below and weakly left of B' for boxes $B, B' \in \mathsf{yd}(\lambda/\mu)$.

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Example 3.4. Consider the Schur labelings of shape $(3, 2, 2)/(1)$

$$\tau_0^{(3,2,2)/(1)} = \begin{ytableau} \none & \none & 1 \\ 3 & 2 \\ 5 & 4 \end{ytableau}, \quad \tau_1^{(3,2,2)/(1)} = \begin{ytableau} \none & \none & 1 \\ 4 & 2 \\ 5 & 3 \end{ytableau}, \quad \text{and} \quad \tau = \begin{ytableau} \none & \none & 2 \\ 3 & 1 \\ 5 & 4 \end{ytableau}.$$

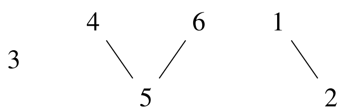
One sees that $\tau_0^{(3,2,2)/(1)}$ and $\tau_1^{(3,2,2)/(1)}$ are distinguished, whereas τ is non-distinguished since 1 appears weakly below and weakly left of 2 in τ.

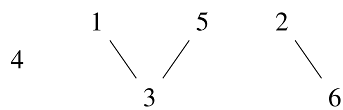
Let $\mathsf{DS}(\lambda/\mu)$ be the set of all distinguished Schur labelings of shape λ/μ. For any Schur labeling τ, let $\mathsf{cnt}_i(\tau)$ be the set of entries in the ith connected component of τ from the top. For each $P \in \mathsf{SP}_n$, there exists a unique Schur labeling τ such that

(i) $\mathrm{sh}(\tau)$ is basic,
(ii) $\mathsf{poset}(\tau) = P$, and (3.2)
(iii) $\min(\mathsf{cnt}_i(\tau)) < \min(\mathsf{cnt}_j(\tau))$ for $1 \le i < j \le k$, where k is the number of connected components of P.

We denote this Schur labeling as τ_P. One can easily see that for $P \in \mathsf{SP}_n$, τ_P is distinguished if and only if every connected component of τ_P is filled with consecutive integers.

Example 3.5. Given two Schur labeled skew shape posets

$P =$ 

and $Q =$ 

,

we have that

$$\tau_P = \begin{ytableau} \none & \none & \none & 2 & 1 \\ \none & \none & 3 \\ 5 & 4 \\ 6 \end{ytableau} \quad \text{and} \quad \tau_Q = \begin{ytableau} \none & \none & \none & 3 & 1 \\ \none & \none & \none & 5 \\ \none & 6 & 2 \\ 4 \end{ytableau}.$$

Lemma 3.6. *For $P \in \mathsf{SP}_n$, P is regular if and only if τ_P is distinguished.*

Proof. To prove the 'only if' part, assume that P is a regular Schur labeled skew shape poset and λ/μ is the shape of τ_P. We claim that every connected component of τ_P is filled with consecutive integers. Take an arbitrary connected component C of τ_P. Let B_1 be the box at the top of the rightmost column of C and B_2 the box at the bottom of the leftmost column of C. Then, we may choose boxes $A_0 := B_1, A_2, A_3, \ldots, A_k := B_2$ satisfying that for all $1 \le i \le k-1$, A_{i+1} is weakly below and weakly left of A_i and A_i, A_{i+1} are in the same row or in the same column. Let $m \in [(\tau_P)_{B_1}, (\tau_P)_{B_2}]$. Then, there exists a unique index $1 \le i \le k-1$ such that $(\tau_P)_{A_i} \le m \le (\tau_P)_{A_{i+1}}$. Since P is regular, one of the following holds:

(i) If $(\tau_P)_{A_i} \preceq_P (\tau_P)_{A_{i+1}}$, then $(\tau_P)_{A_i} \preceq_P m$ or $m \preceq_P (\tau_P)_{A_{i+1}}$.
(ii) If $(\tau_P)_{A_{i+1}} \preceq_P (\tau_P)_{A_i}$, then $(\tau_P)_{A_{i+1}} \preceq_P m$ or $m \preceq_P (\tau_P)_{A_i}$.

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It follows that $(\tau_P)_{A_i}$, m and $(\tau_P)_{A_{i+1}}$ appear in the same connected component; that is, m appears in C. Thus, τ_P is distinguished.

Next, to prove the 'if' part, assume that τ_P is a distinguished Schur labeling and λ/μ is the shape of τ_P. Let $B_1, B_2 \in \mathsf{yd}(\lambda/\mu)$ with $(\tau_P)_{B_1} \prec_{\tau_P} (\tau_P)_{B_2}$. By the definition of $\prec_{\tau_P}$, B_1 and B_2 are in the same connected component. In order to establish the regularity of P, we need to prove that either $(\tau_P)_{B_1} \preceq_{\tau_P} (\tau_P)_C$ or $(\tau_P)_C \preceq_{\tau_P} (\tau_P)_{B_2}$ for all $C \in \mathsf{yd}(\lambda/\mu)$ satisfying $(\tau_P)_{B_1} < (\tau_P)_C < (\tau_P)_{B_2}$ or $(\tau_P)_{B_1} > (\tau_P)_C > (\tau_P)_{B_2}$.

Assume that there exists $C \in \mathsf{yd}(\lambda/\mu)$ such that $(\tau_P)_{B_1} < (\tau_P)_C < (\tau_P)_{B_2}$. Since τ_P is a Schur labeling and $(\tau_P)_{B_1} \prec_{\tau_P} (\tau_P)_{B_2}$, the inequality $(\tau_P)_{B_1} < (\tau_P)_{B_2}$ implies that B_2 is strictly below B_1. In addition, since τ_P is distinguished and $(\tau_P)_{B_1}, (\tau_P)_{B_2}$ appear in the same connected component in τ_P, $(\tau_P)_C$ appears in the same connected component with them in τ_P. Suppose for the sake of contradiction that $(\tau_P)_{B_1} \not\preceq_{\tau_P} (\tau_P)_C$ and $(\tau_P)_C \not\preceq_{\tau_P} (\tau_P)_{B_2}$. Then C satisfies one of the following conditions:

(i) C is strictly above B_1 and strictly right of B_2.
(ii) C is strictly left of B_1 and strictly below B_2.

However, since τ_P is a Schur labeling and $(\tau_P)_{B_1} < (\tau_P)_C$, C cannot satisfy (i). Similarly, since τ_P is a Schur labeling and $(\tau_P)_C < (\tau_P)_{B_2}$, C cannot satisfy (ii). Therefore, $(\tau_P)_{B_1} \preceq_{\tau_P} (\tau_P)_C$ or $(\tau_P)_C \preceq_{\tau_P} (\tau_P)_{B_2}$. In a similar way, one can show that if there exists $C \in \mathsf{yd}(\lambda/\mu)$ such that $(\tau_P)_{B_1} > (\tau_P)_C > (\tau_P)_{B_2}$, then $(\tau_P)_{B_1} \preceq_{\tau_P} (\tau_P)_C$ or $(\tau_P)_C \preceq_{\tau_P} (\tau_P)_{B_2}$. Thus, P is regular. □

Note that $\tau_{\mathsf{poset}(\tau)} = \tau$ for any $\tau \in \mathsf{DS}(\lambda/\mu)$. Considering this property together with Lemma 3.6, one can see that the map

$$\Phi : \mathsf{RSP}_n \to \bigcup_{|\lambda/\mu|=n} \mathsf{DS}(\lambda/\mu), \quad P \mapsto \tau_P \tag{3.3}$$

is a bijection and its inverse is given by $\tau \mapsto \mathsf{poset}(\tau)$.

As a second step, we provide a lemma that will be used throughout this paper.

Lemma 3.7. *For $T \in \mathrm{SYT}(\lambda/\mu)$, $\{\mathsf{read}_\tau(T) \mid \tau \in \mathsf{DS}(\lambda/\mu)\} = [\mathsf{read}_{\tau_0}(T), \mathsf{read}_{\tau_1}(T)]_R$.*

Proof. Let us show the inclusion $\{\mathsf{read}_\tau(T) \mid \tau \in \mathsf{DS}(\lambda/\mu)\} \subseteq [\mathsf{read}_{\tau_0}(T), \mathsf{read}_{\tau_1}(T)]_R$. This can be done by proving $\mathsf{read}_{\tau_0}(T) \preceq_R \mathsf{read}_\tau(T)$ and $\mathsf{read}_\tau(T) \preceq_R \mathsf{read}_{\tau_1}(T)$ for all $\tau \in \mathsf{DS}(\lambda/\mu)$. Since the method of proof for the latter inequality is essentially the same as that for the former one, we omit the proof for the latter inequality. Let $\tau \in \mathsf{DS}(\lambda/\mu)$ and $(i, j) \in \mathrm{Inv}_R(\mathsf{read}_{\tau_0}(T))$. Since $i > j$ and $\mathsf{read}_{\tau_0}(T)^{-1}(i) < \mathsf{read}_{\tau_0}(T)^{-1}(j)$, the box $T^{-1}(j)$ is placed strictly left and weakly below $T^{-1}(i)$. This, together with the definition of distinguished Schur labeling, implies that $\tau_{T^{-1}(i)} < \tau_{T^{-1}(j)}$; equivalently, $\mathsf{read}_\tau(T)^{-1}(i) < \mathsf{read}_\tau(T)^{-1}(j)$. It follows that $(i, j) \in \mathrm{Inv}_R(\mathsf{read}_\tau(T))$. Since we chose an arbitrary $(i, j) \in \mathrm{Inv}_R(\mathsf{read}_{\tau_0}(T))$, we have the inclusion $\mathrm{Inv}_R(\mathsf{read}_{\tau_0}(T)) \subseteq \mathrm{Inv}_R(\mathsf{read}_\tau(T))$. Therefore, $\mathsf{read}_{\tau_0}(T) \preceq_R \mathsf{read}_\tau(T)$.

Let us show the opposite inclusion $\{\mathsf{read}_\tau(T) \mid \tau \in \mathsf{DS}(\lambda/\mu)\} \supseteq [\mathsf{read}_{\tau_0}(T), \mathsf{read}_{\tau_1}(T)]_R$. Since $\{\mathsf{read}_\tau(T) \mid \tau \text{is a bijective tableau of shape} \lambda/\mu\}$ is equal to $\mathfrak{S}_n$ as a set, the inclusion can be obtained by proving that $\mathsf{read}_\tau(T) \notin [\mathsf{read}_{\tau_0}(T), \mathsf{read}_{\tau_1}(T)]_R$ for all bijective tableaux τ of shape λ/μ with $\tau \notin \mathsf{DS}(\lambda/\mu)$. To prove it, choose an arbitrary bijective tableau τ of shape λ/μ with $\tau \notin \mathsf{DS}(\lambda/\mu)$. Since $\tau \notin \mathsf{DS}(\lambda/\mu)$, there exists $1 \le i < j \le n$ such that i is weakly below and weakly left of j in τ. Set $x := T_{\tau^{-1}(i)}$ and $y := T_{\tau^{-1}(j)}$. Then, x appears left of y in $\mathsf{read}_\tau(T)$. However, since $\tau_0, \tau_1 \in \mathsf{DS}(\lambda/\mu)$, we have $(\tau_0)_{\tau^{-1}(i)} > (\tau_0)_{\tau^{-1}(j)}$ and $(\tau_1)_{\tau^{-1}(i)} > (\tau_1)_{\tau^{-1}(j)}$, which implies that x appears right of y in both $\mathsf{read}_{\tau_0}(T)$ and $\mathsf{read}_{\tau_1}(T)$. If $x < y$, then $(y, x) \in \mathrm{Inv}_R(\mathsf{read}_{\tau_0}(T))$ and $(y, x) \notin \mathrm{Inv}_R(\mathsf{read}_\tau(T))$, thus $\mathrm{Inv}_R(\mathsf{read}_{\tau_0}(T)) \not\subseteq \mathrm{Inv}_R(\mathsf{read}_\tau(T))$. Similarly, if $x > y$, then $\mathrm{Inv}_R(\mathsf{read}_\tau(T)) \not\subseteq \mathrm{Inv}_R(\mathsf{read}_{\tau_1}(T))$. Hence, $\mathsf{read}_\tau(T) \notin [\mathsf{read}_{\tau_0}(T), \mathsf{read}_{\tau_1}(T)]_R$. □

As a last step, we define two specific standard Young tableaux. For a skew partition λ/μ of size n, let $T_{\lambda/\mu}$(resp. $T'_{\lambda/\mu}$) be the standard Young tableau obtained by filling $\mathsf{yd}(\lambda/\mu)$ by $1, 2, \dots, n$ from left to right starting with the top row (resp. from top to bottom starting with leftmost column).

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Example 3.8. Let $\lambda/\mu = (2,2) \star (3,2)/(1)$. Then

$$T_{\lambda/\mu} = \begin{array}{ccccc} & & & 1 & 2 \\ & & & 3 & 4 \\ & 5 & 6 & & \\ 7 & 8 & & & \end{array} \quad \text{and} \quad T'_{\lambda/\mu} = \begin{array}{ccccc} & & & 5 & 7 \\ & & & 6 & 8 \\ & 2 & 4 & & \\ 1 & 3 & & & \end{array}.$$

Now, we are ready prove the main theorem of this section.

Theorem 3.9. *Let* $P \in \mathsf{RSP}_n$ *and* $\lambda/\mu = \mathrm{sh}(\tau_P)$. *Then*

$$\Sigma_L(P) = [\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})]_L.$$

Proof. Due to Theorem 2.4 and Lemma 3.2, it suffices to show that $\mathsf{read}_{\tau_P}(T_{\lambda/\mu})$ is minimal and $\mathsf{read}_{\tau_P}(T'_{\lambda/\mu})$ is maximal in $\mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$ with respect to $\preceq_L$. Let $T \in \mathrm{SYT}(\lambda/\mu)$. In the case where $\tau_P = \tau_0$, one can easily see that

$$\mathrm{Inv}_L(\mathsf{read}_{\tau_0}(T_{\lambda/\mu})) \subseteq \mathrm{Inv}_L(\mathsf{read}_{\tau_0}(T)) \subseteq \mathrm{Inv}_L(\mathsf{read}_{\tau_0}(T'_{\lambda/\mu})). \tag{3.4}$$

In the case where $\tau_P \neq \tau_0$, we consider the equality

$$\mathsf{read}_{\tau_P}(T) = \mathsf{read}_{\tau_0}(T)\mathsf{read}_{\tau_0}(\tau_P)^{-1} \tag{3.5}$$

which follows from Definition 3.1. By Lemma 3.6, we have $\tau_P \in \mathsf{DS}(\lambda/\mu)$; therefore, combining Lemma 3.7 with Equation (3.5) yields that

$$\ell(\mathsf{read}_{\tau_P}(T)) = \ell(\mathsf{read}_{\tau_0}(T)) + \ell(\mathsf{read}_{\tau_0}(\tau_P)^{-1}).$$

Now, we have that

$$\begin{aligned} \ell(\mathsf{read}_{\tau_P}(T)) - \ell(\mathsf{read}_{\tau_P}(T_{\lambda/\mu})) &= \ell(\mathsf{read}_{\tau_0}(T)) - \ell(\mathsf{read}_{\tau_0}(T_{\lambda/\mu})) \\ &= \ell(\mathsf{read}_{\tau_0}(T)\mathsf{read}_{\tau_0}(T_{\lambda/\mu})^{-1}) && \text{by Equation (3.4)} \\ &= \ell(\mathsf{read}_{\tau_P}(T)\mathsf{read}_{\tau_P}(T_{\lambda/\mu})^{-1}) && \text{by Definition 3.1.} \end{aligned}$$

Therefore, $\mathsf{read}_{\tau_P}(T_{\lambda/\mu}) \preceq_L \mathsf{read}_{\tau_P}(T)$. In the same manner, we can prove that $\mathsf{read}_{\tau_P}(T) \preceq_L \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})$. □

4. An equivalence relation on $\mathrm{Int}(n)$

Recall that $\mathrm{Int}(n)$ denotes the set of nonempty left weak Bruhat intervals in $\mathfrak{S}_n$; that is,

$$\mathrm{Int}(n) = \{[\sigma, \rho]_L \mid \sigma, \rho \in \mathfrak{S}_n \text{ and } \sigma \preceq_L \rho\}.$$

For $I_1, I_2 \in \mathrm{Int}(n)$, a poset isomorphism $f : (I_1, \preceq_L) \to (I_2, \preceq_L)$ is called *descent-preserving* if $\mathrm{Des}_L(\gamma) = \mathrm{Des}_L(f(\gamma))$ for all $\gamma \in I_1$. In this section, we study the classification of left weak Bruhat intervals in $\mathrm{Int}(n)$ up to descent-preserving poset isomorphism. In particular, in the case where $P \in \mathsf{RSP}_n$, we explicitly describe the isomorphism class of $\Sigma_L(P)$.

We begin by explaining the reason why we consider descent-preserving poset isomorphisms. Note that every interval $I \in \mathrm{Int}(n)$ can be represented by the colored digraph whose vertices are given by the permutations in I and $\{1, 2, \dots, n-1\}$-colored arrows are given by

$$\gamma \xrightarrow{i} \gamma' \quad \text{if and only if} \quad \gamma \preceq_L \gamma' \text{ and } s_i\gamma = \gamma'.$$

For intervals $I_1, I_2 \in \mathrm{Int}(n)$, a map $f : I_1 \to I_2$ is called a *colored digraph isomorphism* if f is bijective and satisfies that for all $\gamma, \gamma' \in I_1$ and $1 \le i \le n-1$,

$$\gamma \xrightarrow{i} \gamma' \quad \text{if and only if} \quad f(\gamma) \xrightarrow{i} f(\gamma').$$

If there exists a descent-preserving colored digraph isomorphism between two intervals I_1 and I_2, then $\mathsf{B}(I_1)$ is isomorphic to $\mathsf{B}(I_2)$. Motivated by this fact, in [19, Subsection 3.1], the authors posed the classification problem of weak Bruhat intervals up to descent-preserving colored digraph isomorphism.

A colored digraph isomorphism between I_1 and I_2 is a poset isomorphism with respect to $\preceq_L$, but a poset isomorphism $f : I_1 \to I_2$ may not be a colored digraph isomorphism. For instance, the poset isomorphism $f : [1234, 2134]_L \to [1234, 1324]_L$ defined by $f(1234) = 1234$ and $f(2134) = 1324$ is not a colored digraph isomorphism since

$$\begin{array}{ccc} 1234 & & 1234 \\ \downarrow{\scriptstyle 1} & \xrightarrow{f} & \downarrow{\scriptstyle 2} \\ 2134 & & 1324 \end{array}.$$

However, if a poset isomorphism between left weak Bruhat intervals is descent-preserving, it indeed proves to be a colored digraph isomorphism.

Proposition 4.1. *Let $I_1, I_2 \in \mathrm{Int}(n)$. Every descent-preserving poset isomorphism $f : I_1 \to I_2$ is a colored digraph isomorphism.*

Proof. Let $\gamma, \gamma' \in I_1$ with $\gamma \preceq_L^c \gamma'$. Since f is a poset isomorphism, we have $f(\gamma) \preceq_L^c f(\gamma')$. Let $i, j \in [n-1]$, satisfying $\gamma' = s_i\gamma$ and $f(\gamma') = s_j f(\gamma)$. For the assertion, it suffices to show that $i = j$. Let $D_1 := \mathrm{Des}_L(\gamma)$ and $D_2 := \mathrm{Des}_L(\gamma')$. Since $\gamma' = s_i\gamma$, we have

$$\{i\} \subseteq (D_1 \cup D_2) \setminus (D_1 \cap D_2) \subseteq \{i-1, i, i+1\}.$$

In addition, since $D_1 = \mathrm{Des}_L(f(\gamma))$, $D_2 = \mathrm{Des}_L(f(\gamma'))$ and $f(\gamma') = s_j f(\gamma)$, we have $j \in D_2 \setminus D_1$, and it follows that j is one of $i-1$, i, and $i+1$.

If $j = i-1$, then $i-1, i \in D_2$. It follows that

$$\gamma' = \cdots\ i+1\ \cdots\ i\ \cdots\ i-1\ \cdots \quad \text{in one-line notation;}$$

equivalently,

$$\gamma = \cdots\ i\ \cdots\ i+1\ \cdots\ i-1\ \cdots \quad \text{in one-line notation.}$$

This implies that $j = i-1 \in D_1$, which is a contradiction to $j \notin \mathrm{Des}_L(f(\gamma))$. Therefore, $j \neq i-1$.

In a similar manner, one can show that $j \neq i+1$. Hence, $j = i$, as required. □

Proposition 4.1 says that classifying weak Bruhat intervals up to descent-preserving colored digraph isomorphism is equivalent to classifying weak Bruhat intervals up to descent-preserving poset isomorphism. With this equivalence in mind, we introduce an equivalence relation, whose reflexivity, symmetricity and transitivity are obvious.

Definition 4.2. We define an equivalence relation $\overset{D}{\simeq}$ on $\mathrm{Int}(n)$ by $I_1 \overset{D}{\simeq} I_2$ if there is a descent-preserving (poset) isomorphism between $(I_1, \preceq_L)$ and $(I_2, \preceq_L)$.

For each equivalence class C, we define

$$\xi_C := \rho\sigma^{-1} \quad \text{for any } [\sigma, \rho]_L \in C.$$

By Proposition 4.1, ξ_C does not depend on the choice of $[\sigma,\rho]_L \in C$. We also define

$$\overline{\min}(C) := \{\sigma \mid [\sigma,\rho]_L \in C\} \quad \text{and} \quad \overline{\max}(C) := \{\rho \mid [\sigma,\rho]_L \in C\}.$$

From now on, we always regard $\overline{\min}(C)$ and $\overline{\max}(C)$ as subposets of $(\mathfrak{S}_n, \preceq_R)$.

The following lemma is the initial step in the proof of the main result of this section (Theorem 4.6).

Lemma 4.3. *Let C be an equivalence class under $\overset{D}{\simeq}$ with $\ell(\xi_C) = 1$. Then, $\overline{\min}(C)$ is a right weak Bruhat interval in $(\mathfrak{S}_n, \preceq_R)$.*

Before proving the lemma, we provide an outline of the proof for the reader's understanding. We first classify the equivalence classes under consideration according to the set X in Equation (4.2). Then, case by case, we show that $\overline{\min}(C)$ has a unique minimal element σ_0. In particular, in **Case 3**, we introduce a specific permutation $\mathbf{w}_0$ and show that it is the unique minimal element of $\overline{\min}(C)$. Using these results, we next show that $\overline{\min}(C)$ has the unique maximal element σ_1. Finally, we show that $[\sigma_0, \sigma_1]_R \subseteq \overline{\min}(C)$.

Proof. From the condition $\ell(\xi_C) = 1$, it follows that $\xi_C = s_{i_0}$ for some $i_0 \in [n-1]$.

First, let us prove that there exists a unique minimal element in $\overline{\min}(C)$. Let

$$D_1 := \mathrm{Des}_L(\sigma) \quad \text{and} \quad D_2 := \mathrm{Des}_L(s_{i_0}\sigma) \quad \text{for any } [\sigma, s_{i_0}\sigma]_L \in C \tag{4.1}$$

and

$$X := (D_1 \cup D_2) \setminus (D_1 \cap D_2). \tag{4.2}$$

One sees that $\{i_0\} \subseteq X \subseteq \{i_0-1, i_0, i_0+1\}$, and therefore, X can be one of the following:

$$\{i_0-1, i_0, i_0+1\}, \quad \{i_0\}, \quad \{i_0-1, i_0\}, \quad \text{and} \quad \{i_0, i_0+1\}.$$

Case 1: $X = \{i_0-1, i_0, i_0+1\}$. Since $i_0-1, i_0+1 \in D_1$ and $i_0 \notin D_1$,

$$w_0(D_1) = \cdots\ i_0\ i_0-1\ \cdots\ i_0+2\ i_0+1\ \cdots$$

in one-line notation. Considering this equality, one can see that $[w_0(D_1), s_{i_0}w_0(D_1)]_L \in C$. By Lemma 2.2, $w_0(D_1) \preceq_R \sigma$ for all $\sigma \in \mathfrak{S}_n$ with $\mathrm{Des}_L(\sigma) = D_1$. Thus, $w_0(D_1)$ is a unique minimal element in $\overline{\min}(C)$.

Case 2: $X = \{i_0\}$. In this case, we have

$$w_0(D_2) = \cdots\ i_0+1\ i_0 \cdots$$

in one-line notation. Considering this equality, one can see that $[s_{i_0}w_0(D_2), w_0(D_2)]_L \in C$. Again, by Lemma 2.2, $w_0(D_2) \preceq_R \sigma$ for all $\sigma \in \mathfrak{S}_n$ with $\mathrm{Des}_L(\sigma) = D_2$. Thus, $s_{i_0}w_0(D_2)$ is a unique minimal element in $\overline{\min}(C)$.

Case 3: $X = \{i_0-1, i_0\}$. When $i_0+1 \notin D_1$, following the way as in **Case 1**, one can see that $w_0(D_1)$ is a unique minimal element in $\overline{\min}(C)$.

From now on, assume that $i_0+1 \in D_1$. We begin by introducing necessary notation. Let

$$\mathsf{m}_1 := \min\{m \in [n-1] \mid [m, i_0-1] \subseteq D_1\} \quad \text{and} \quad \mathsf{m}_2 := \max\{m \in [n-1] \mid [i_0+1, m] \subseteq D_1\}.$$

And set

$$p_1 := \mathsf{m}_1 - 1, \quad p_2 := \mathsf{m}_2 - i_0, \quad p_3 := i_0 - \mathsf{m}_1, \quad \text{and} \quad p_4 := n - (\mathsf{m}_2 + 1).$$

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Let

- $\mathbf{w}^{(1)}$ be the longest element of the subgroup $\mathfrak{S}_{D_1\cap[p_1-1]}$ of $\mathfrak{S}_{p_1}$,
- $\mathbf{w}^{(2)}$ be the longest element of $\mathfrak{S}_{p_2}$,
- $\mathbf{w}^{(3)}$ be the longest element of $\mathfrak{S}_{p_3}$, and
- $\mathbf{w}^{(4)}$ be the longest element of the subgroup $\mathfrak{S}_{D_1\cap[p_4-1]}$ of $\mathfrak{S}_{p_4}$.

With this notation, we define $\mathbf{w}_0$ to be the permutation given by

$$\mathbf{w}_0(k) := \begin{cases} \mathbf{w}^{(1)}(k) & \text{if } k \in [p_1], \\ \mathbf{w}^{(2)}(k-p_1)+(i_0+1) & \text{if } k \in [p_1+1, p_1+p_2], \\ i_0 & \text{if } k = p_1+p_2+1, \\ \mathbf{w}^{(3)}(k-(p_1+p_2+1))+(\mathsf{m}_1-1) & \text{if } k \in [p_1+p_2+2, p_1+p_2+p_3+1], \\ i_0+1 & \text{if } k = p_1+p_2+p_3+2, \\ \mathbf{w}^{(4)}(k-(p_1+p_2+p_3+2))+(\mathsf{m}_2+1) & \text{if } k \in [p_1+p_2+p_3+3, n]. \end{cases}$$

It should be remarked that

$$\begin{aligned} \mathbf{w}_0([1,p_1]) &= [1, \mathsf{m}_1-1], \\ \mathbf{w}_0([p_1+1, p_1+p_2]) &= [i_0+2, \mathsf{m}_2+1], \\ \mathbf{w}_0(p_1+p_2+1) &= i_0, \\ \mathbf{w}_0([p_1+p_2+2, p_1+p_2+p_3+1]) &= [\mathsf{m}_1, i_0-1], \\ \mathbf{w}_0(p_1+p_2+p_3+2) &= i_0+1, \quad \text{and} \\ \mathbf{w}_0([p_1+p_2+p_3+3, n]) &= [\mathsf{m}_2+2, n]. \end{aligned}$$

From the definition of $\mathbf{w}_0$, it follows that $[\mathbf{w}_0, s_{i_0}\mathbf{w}_0]_L \in C$; equivalently, $\mathbf{w}_0 \in \overline{\min}(C)$. We claim that $\mathbf{w}_0$ is a unique minimal element in $\overline{\min}(C)$. This can be verified by showing that every minimal element in $\overline{\min}(C)$ is equal to $\mathbf{w}_0$. Let σ_0 be a minimal element in $\overline{\min}(C)$. Set

$$\begin{aligned} \mathcal{I}_{\mathrm{L}} &:= \{\sigma_0(k) \mid 1 \le k < \sigma_0^{-1}(i_0)\}, \\ \mathcal{I}_{\mathrm{C}} &:= \{\sigma_0(k) \mid \sigma_0^{-1}(i_0) < k < \sigma_0^{-1}(i_0+1)\}, \\ \mathcal{I}_{\mathrm{R}} &:= \{\sigma_0(k) \mid \sigma_0^{-1}(i_0+1) < k \le n\}. \end{aligned}$$

To begin with, we establish the equalities

$$\mathcal{I}_{\mathrm{L}} = [\mathsf{m}_1-1] \cup [i_0+2, \mathsf{m}_2+1], \quad \mathcal{I}_{\mathrm{C}} = [\mathsf{m}_1, i_0-1], \quad \text{and} \quad \mathcal{I}_{\mathrm{R}} = [\mathsf{m}_2+2, n]. \tag{4.3}$$

These equalities are derived by verifying the following claims.

Claim 1. $\mathcal{I}_{\mathrm{C}} = [\mathsf{m}_1, i_0-1]$. Let us show $\mathcal{I}_{\mathrm{C}} \subseteq [\mathsf{m}_1, i_0-1]$. First, to prove $\mathcal{I}_{\mathrm{C}} \subseteq [i_0-1]$, we assume that there exists $i \in \mathcal{I}_{\mathrm{C}}$ such that $i \ge i_0$. Let

$$k_1 := \max\{k \in [n] \mid \sigma_0(k) \in \mathcal{I}_{\mathrm{C}} \text{ and } \sigma_0(k) \ge i_0\}.$$

By the definition of $\mathcal{I}_{\mathrm{C}}$, $i_0, i_0+1 \notin \mathcal{I}_{\mathrm{C}}$. If $i_0+2 \in \mathcal{I}_{\mathrm{C}}$, then $i_0+1 \in D_1 \setminus D_2$, which contradicts the assumption that $i_0+1 \notin X$. Since $\sigma_0(k_1) \in \mathcal{I}_{\mathrm{C}}$, it follows that $\sigma_0(k_1) \ge i_0+3$. And, by the choice of k_1, we have $\sigma_0(k_1+1) \le i_0+1$. Putting these together yields that $[\sigma_0 s_{k_1}, s_{i_0}\sigma_0 s_{k_1}]_L \overset{D}{\simeq} [\sigma_0, s_{i_0}\sigma_0]_L$ and $\sigma_0 s_{k_1} \prec_R \sigma_0$. This contradicts the minimality of σ_0 in $\overline{\min}(C)$; therefore, $\mathcal{I}_{\mathrm{C}} \subseteq [i_0-1]$. Next, to prove $\mathcal{I}_{\mathrm{C}} \subseteq [\mathsf{m}_1, n]$, we assume that there exists $i \in \mathcal{I}_{\mathrm{C}}$ such that $i < \mathsf{m}_1$. Let

$$k_2 := \min\{k \in [n] \mid \sigma_0(k) \in \mathcal{I}_{\mathrm{C}} \text{ and } \sigma_0(k) < \mathsf{m}_1\}.$$

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By the choice of k_2, we have $\sigma_0(k_2)+1 \le \mathsf{m}_1 \le \sigma_0(k_2-1)$. In addition, if $\sigma_0(k_2)+1 = \mathsf{m}_1 = \sigma_0(k_2-1)$, then $\mathsf{m}_1-1 \in \mathrm{Des}_L(\sigma_0)$, which cannot happen by the definition of m_1. Therefore, $\sigma_0(k_2)+1 < \sigma_0(k_2-1)$, which implies that $[\sigma_0 s_{k_2-1}, s_{i_0}\sigma_0 s_{k_2-1}]_L \overset{D}{\simeq} [\sigma_0, s_{i_0}\sigma_0]_L$ and $\sigma_0 s_{k_2-1} \prec_R \sigma_0$. This contradicts the minimality of σ_0 in $\overline{\min}(C)$. Thus, $\mathcal{I}_\mathrm{C} \subseteq [\mathsf{m}_1, i_0-1]$.

Let us show $\mathcal{I}_\mathrm{C} \supseteq [\mathsf{m}_1, i_0-1]$. Assume for the sake of contradiction that there exists $i \in [\mathsf{m}_1, i_0-1]$ such that $i \notin \mathcal{I}_\mathrm{C}$. Let j be the maximal element in $[\mathsf{m}_1, i_0-1]$ such that $j \notin \mathcal{I}_\mathrm{C}$. Since $i_0-1 \in X$, we have $i_0-1 \in \mathcal{I}_\mathrm{C}$. It follows that $j < i_0-1$, so $j+1 \in [\mathsf{m}_1, i_0-1]$. Combining this with the maximality of j, we have $j+1 \in \mathcal{I}_\mathrm{C}$. And, by the definition of m_1, we have $j \in D_1$. Putting these together yields that $j \in \mathcal{I}_\mathrm{R}$. Let

$$k_3 := \min\{k \in [n] \mid \sigma_0(k) \in \mathcal{I}_\mathrm{R} \text{ and } \sigma_0(k) \le j\}.$$

If $\sigma_0(k_3-1) \le \sigma_0(k_3)+1$, then $\sigma_0(k_3-1) \le j+1 < i_0$. So, $\sigma_0(k_3-1) \ne i_0+1$, which implies $\sigma_0(k_3-1) \in \mathcal{I}_\mathrm{R}$. This, together with the minimality of k_3, yields that $j+1 \le \sigma_0(k_3-1)$. It follows that $\sigma_0(k_3-1) = j+1$, which is a contradiction because $\sigma_0(k_3-1) \in \mathcal{I}_\mathrm{R}$, but $j+1 \in \mathcal{I}_\mathrm{C}$. Therefore, we have $\sigma_0(k_3)+1 < \sigma_0(k_3-1)$. In addition, since $j < i_0-1$, we have $\sigma_0(k_3) < i_0-1$. Putting these together yields that $[\sigma_0 s_{k_3-1}, s_{i_0}\sigma_0 s_{k_3-1}]_L \overset{D}{\simeq} [\sigma_0, s_{i_0}\sigma_0]_L$ and $\sigma_0 s_{k_3-1} \prec_R \sigma_0$, which contradicts the minimality of σ_0 in $\overline{\min}(C)$. Thus, $\mathcal{I}_\mathrm{C} \supseteq [\mathsf{m}_1, i_0-1]$.

<u>*Claim 2.*</u> $[\mathsf{m}_1-1] \cup [i_0+2, \mathsf{m}_2+1] \subseteq \mathcal{I}_\mathrm{L}$. By the definition of m_2, we have $[i_0+1, \mathsf{m}_2] \subseteq \mathrm{Des}_L(\sigma_0)$. Since $i_0+2 \notin \mathcal{I}_\mathrm{C}$, we have $[i_0+2, \mathsf{m}_2+1] \subseteq \mathcal{I}_\mathrm{L}$. To prove $[\mathsf{m}_1-1] \subseteq \mathcal{I}_\mathrm{L}$, suppose that there exists $i \in [\mathsf{m}_1-1]$ such that $i \notin \mathcal{I}_\mathrm{L}$. Let

$$k_4 := \min\{k \in [n] \mid \sigma_0(k) \in [\mathsf{m}_1-1] \text{ and } \sigma_0(k) \notin \mathcal{I}_\mathrm{L}\}.$$

Since $\sigma_0(k_4) \notin \mathcal{I}_\mathrm{L}$ and $\sigma_0(k_4) < \mathsf{m}_1$, we have $\sigma_0(k_4) \in \mathcal{I}_\mathrm{R}$. This implies that $\sigma_0(k_4-1) \notin \mathcal{I}_\mathrm{L} \cup \{i_0\} \cup \mathcal{I}_\mathrm{C}$. In addition, the minimality of k_4 gives $\sigma_0(k_4-1) \ge \mathsf{m}_1$. Since $[\mathsf{m}_1, i_0-1] = \mathcal{I}_\mathrm{C}$, we have $\sigma_0(k_4-1) \ge i_0+1$. Putting the above inequalities together, we have

$$\sigma_0(k_4) < \mathsf{m}_1 \le i_0-1 < i_0+1 \le \sigma_0(k_4-1),$$

and so $\sigma_0(k_4)+2 < \sigma_0(k_4-1)$. It follows that $[\sigma_0 s_{k_4-1}, s_{i_0}\sigma_0 s_{k_4-1}]_L \overset{D}{\simeq} [\sigma_0, s_{i_0}\sigma_0]_L$ and $\sigma_0 s_{k_4-1} \prec_R \sigma_0$. This contradicts the minimality of σ_0 in $\overline{\min}(C)$; thus, $[\mathsf{m}_1-1] \subseteq \mathcal{I}_\mathrm{L}$.

<u>*Claim 3.*</u> $[\mathsf{m}_2+2, n] \subseteq \mathcal{I}_\mathrm{R}$. Suppose that there exists $i \in [\mathsf{m}_2+2, n]$ such that $i \notin \mathcal{I}_\mathrm{R}$. Let

$$k_5 := \max\{k \in [n] \mid \sigma_0(k) \notin \mathcal{I}_\mathrm{R} \;\text{ and }\; \sigma_0(k) \in [\mathsf{m}_2+2, n]\}.$$

Since $k_5 \notin \mathcal{I}_\mathrm{R}$ and $\sigma_0(k_5) > i_0+1$, we have $\sigma_0(k_5) \in \mathcal{I}_\mathrm{L}$, which implies that $\sigma_0(k_5+1) \notin \mathcal{I}_\mathrm{R}$. By the maximality of k_5, we have $\sigma_0(k_5+1) \le \mathsf{m}_2+1$. If $\sigma_0(k_5+1) < \mathsf{m}_2+1$, then

$$\sigma_0(k_5+1)+1 < \mathsf{m}_2+2 \le \sigma_0(k_5).$$

If $\sigma_0(k_5+1) = \mathsf{m}_2+1$, then $\sigma_0^{-1}(\mathsf{m}_2+2) > k_5+1$ due to the maximality of m_2, so $\sigma_0(k_5) \ne \mathsf{m}_2+2$, which implies

$$\sigma_0(k_5+1)+1 < \sigma_0(k_5).$$

Putting these together with the inequalities $\sigma_0(k_5) \ge \mathsf{m}_2+2 > i_0+2$ yields that $[\sigma_0 s_{k_5}, s_{i_0}\sigma_0 s_{k_5}]_L \overset{D}{\simeq} [\sigma_0, s_{i_0}\sigma_0]_L$ and $\sigma_0 s_{k_5} \prec_R \sigma_0$. This contradicts the minimality of σ_0 in $\overline{\min}(C)$; thus, $[\mathsf{m}_2+2, n] \subseteq \mathcal{I}_\mathrm{R}$.

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Now, we are ready to show that $\sigma_0 = \mathbf{w}_0$. Let

$$\mathcal{I}_{\mathrm{L}}^{(1)} := \{\sigma_0(k) \in \mathcal{I}_{\mathrm{L}} \mid 1 \le k \le \mathsf{m}_1 - 1\} \quad \text{and} \quad \mathcal{I}_{\mathrm{L}}^{(2)} := \{\sigma_0(k) \in \mathcal{I}_{\mathrm{L}} \mid \mathsf{m}_1 \le k < \sigma_0^{-1}(i_0)\}.$$

We claim that $\mathcal{I}_{\mathrm{L}}^{(1)} = [\mathsf{m}_1 - 1]$ and $\mathcal{I}_{\mathrm{L}}^{(2)} = [i_0 + 2, \mathsf{m}_2 + 1]$. We may assume that $\mathsf{m}_1 > 1$; otherwise, the claim is obvious. To prove our claim, suppose that there exists $i \in \mathcal{I}_{\mathrm{L}}^{(1)}$ such that $i \in [i_0 + 2, \mathsf{m}_2 + 1]$. Then, there exists $1 \le k < \sigma_0^{-1}(i_0) - 1$ such that $\sigma_0(k) \in [i_0 + 2, \mathsf{m}_2 + 1]$ and $\sigma_0(k+1) \in [\mathsf{m}_1 - 1]$. It follows that $[\sigma_0 s_k, s_{i_0}\sigma_0 s_k]_L \overset{D}{\simeq} [\sigma_0, s_{i_0}\sigma_0]_L$ and $\sigma_0 s_k \prec_R \sigma_0$. Again, this contradicts the minimality of σ_0 in $\overline{\min}(C)$, so

$$\mathcal{I}_{\mathrm{L}}^{(1)} = [\mathsf{m}_1 - 1] \quad \text{and} \quad \mathcal{I}_{\mathrm{L}}^{(2)} = [i_0 + 2, \mathsf{m} + 1]. \tag{4.4}$$

Putting Lemma 2.2, Equation (4.3), Equation (4.4) and the minimality of σ_0 together, we conclude that $\sigma_0 = \mathbf{w}_0$.

Case 4: $X = \{i_0, i_0+1\}$. Take $[\sigma, s_{i_0}\sigma]_L \in C$ and let C' be the equivalence class of $[\sigma^{w_0}, (s_{i_0}\sigma)^{w_0}]_L$. By mimicking Equation (4.1) and Equation (4.2), we define

$$D_1' := \mathrm{Des}_L(\sigma^{w_0}), \quad D_2' := \mathrm{Des}_L((s_{i_0}\sigma)^{w_0}), \quad \text{and} \quad X' := (D_1' \cup D_2') \setminus (D_1' \cap D_2').$$

Since $D_1' = \{n - i \mid i \in D_1\}$ and $D_2' = \{n - i \mid i \in D_2\}$, we have

$$X' = \{n - i_0, (n - i_0) + 1\}.$$

Following the proof of **Case 3**, we see that $\overline{\min}(C')$ has a unique minimal element $\mathbf{w}_0'$. And one can easily see that the map $f : \overline{\min}(C) \to \overline{\min}(C')$, $\gamma \mapsto \gamma^{w_0}$ is a well-defined bijection and that for $\gamma_1, \gamma_2 \in \overline{\min}(C)$, $\gamma_1 \preceq_R \gamma_2$ if and only if $f(\gamma_1) \preceq_R f(\gamma_2)$. Thus, $(\mathbf{w}_0')^{w_0}$ is a unique minimal element in $\overline{\min}(C')$.

Second, we will show that $\overline{\min}(C)$ has a unique maximal element. Recall that we take $[\sigma, s_{i_0}\sigma]_L \in C$. Let C'' be the equivalence class of $[s_{i_0}\sigma w_0, \sigma w_0]_L$. Due to the previous arguments, we know that there is a unique minimal element γ_0 in $\overline{\min}(C'')$. One can easily see that the map $g : \overline{\min}(C) \to \overline{\min}(C'')$, $\gamma \mapsto \gamma w_0$ is a well-defined bijection and that for $\gamma_1, \gamma_2 \in \overline{\min}(C)$, $\gamma_1 \preceq_R \gamma_2$ if and only if $g(\gamma_1) \succeq_R g(\gamma_2)$. Therefore, $\gamma_0 w_0$ is the unique maximal element in $\overline{\min}(C)$.

Finally, we will show that $\overline{\min}(C)$ is a right weak Bruhat interval in $(\mathfrak{S}_n, \preceq_R)$. Let σ_0 and σ_1 be the minimal and maximal elements in $\overline{\min}(C)$, respectively. Let $\gamma \in [\sigma_0, \sigma_1]_R$. Since $\mathrm{Des}_L(\sigma_0) = \mathrm{Des}_L(\sigma_1)$, we have $\mathrm{Des}_L(\gamma) = \mathrm{Des}_L(\sigma_0)$ by Lemma 2.2. Next, let us examine $\mathrm{Des}_L(s_{i_0}\gamma)$. Since $\mathrm{Des}_L(\sigma_0) = \mathrm{Des}_L(\gamma)$, it follows that $\gamma \preceq_L s_{i_0}\gamma$. By Lemma 2.1, we have that $s_{i_0}\gamma \in [s_{i_0}\sigma_0, s_{i_0}\sigma_1]_R$. Since $\mathrm{Des}_L(s_{i_0}\sigma_0) = \mathrm{Des}_L(s_{i_0}\sigma_1)$, we have $\mathrm{Des}_L(s_{i_0}\gamma) = \mathrm{Des}_L(s_{i_0}\sigma_0)$ by Lemma 2.2. Thus, $\gamma \in \overline{\min}(C)$. □

Example 4.4. Let $C \subseteq \mathrm{Int}(4)$ be the equivalence class of $[2134, 2143]_L$. One sees that

$$C = \{[2134, 2143]_L, [2314, 2413]_L, [2341, 2431]_L\}.$$

So, $\overline{\min}(C) = \{2134, 2314, 2341\}$ and $\overline{\max}(C) = \{2143, 2413, 2431\}$ which are equal to $[2134, 2341]_R$ and $[2143, 2431]_R$, respectively. For the readers' convenience, we draw the left weak Bruhat intervals in C within the left weak Bruhat graph of $\mathfrak{S}_4$ on the left-hand side of Figure 1. We also draw the right weak Bruhat intervals $\overline{\min}(C)$ and $\overline{\max}(C)$ within the right weak Bruhat graph of $\mathfrak{S}_4$ on the right-hand side of Figure 1.

Lemma 4.5. *The intersection of two right weak Bruhat intervals in $\mathfrak{S}_n$ is again a right weak Bruhat interval.*

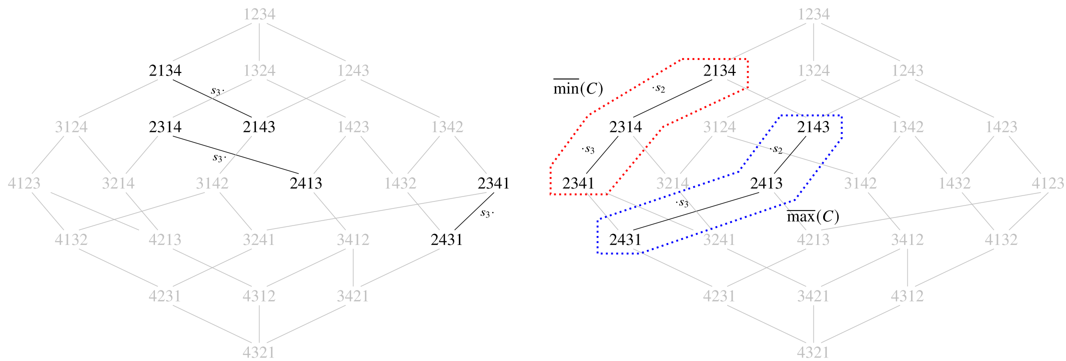

Figure 1. *The left weak Bruhat intervals in C on $(\mathfrak{S}_4, \preceq_L)$ and the right weak Bruhat intervals $\overline{\min}(C)$ and $\overline{\max}(C)$ on $(\mathfrak{S}_4, \preceq_R)$ in Example 4.4.*

Proof. It is well known that $(\mathfrak{S}_n, \preceq_R)$ is a lattice; that is, every two-element subset $\{\gamma_1, \gamma_2\} \subseteq \mathfrak{S}_n$ has the least upper bound and greatest lower bound (for example, see [6, Section 3.2]). Combining this with the fact $|\mathfrak{S}_n| < \infty$, we derive the desired result. □

The following theorem provides significant information regarding equivalence classes under $\overset{D}{\simeq}$.

Theorem 4.6. *Let C be an equivalence class under $\overset{D}{\simeq}$. Then $\overline{\min}(C)$ and $\overline{\max}(C)$ are right weak Bruhat intervals in $(\mathfrak{S}_n, \preceq_R)$.*

Proof. Note that $\sigma \preceq_L \xi_C\sigma$ for any $\sigma \in \overline{\min}(C)$ and that $\overline{\max}(C) = \xi_C \cdot \overline{\min}(C)$. If we prove that $\overline{\min}(C)$ is a right weak Bruhat interval, then Lemma 2.1 implies that $\overline{\max}(C)$ is also a right weak Bruhat interval. So we will only prove that $\overline{\min}(C)$ is a right weak Bruhat interval.

When $\ell(\xi_C) = 0$, the assertion follows from Lemma 2.2. From now on, assume that $\ell(\xi_C) \geq 1$. We will prove the assertion by using mathematical induction on $\ell(\xi_C)$. When $\ell(\xi_C) = 1$, the assertion is true by Lemma 4.3. Let k be an arbitrary positive integer and suppose that the assertion holds for every equivalence class $C \in \mathscr{C}(n)$ with $\ell(\xi_C) \leq k$. Let $C \in \mathscr{C}(n)$ with $\ell(\xi_C) = k + 1$. Set

$$\mathcal{A} := \{i \in [n-1] \mid s_i \in [\mathrm{id}, \xi_C]_L\}.$$

Given $i \in \mathcal{A}$ and $\sigma \in \overline{\min}(C)$, note that

$$[\sigma, s_i\sigma]_L \overset{D}{\simeq} [\sigma', s_i\sigma']_L \quad \text{and} \quad [s_i\sigma, \xi_C\sigma]_L \overset{D}{\simeq} [s_i\sigma', \xi_C\sigma']_L$$

for all $\sigma' \in \overline{\min}(C)$. This says that the equivalence classes of $[\sigma, s_i\sigma]_L$ and $[s_i\sigma, \xi_C\sigma]_L$ do not depend on $\sigma \in \overline{\min}(C)$. For each $i \in \mathcal{A}$, we set

$$\begin{aligned} E_i &:= \text{the equivalence class of } [\sigma, s_i\sigma]_L, \\ E_i' &:= \text{the equivalence class of } [s_i\sigma, \xi_C\sigma]_L \end{aligned}$$

for any $\sigma \in \overline{\min}(C)$. Then, we set

$$J_i := \overline{\max}(E_i) \cap \overline{\min}(E_i') \quad \text{for } i \in \mathcal{A}, \quad \text{and} \quad J := \bigcap_{i \in \mathcal{A}} s_i \cdot J_i.$$

Now, the desired assertion can be achieved by proving the following claims:

(i) $\overline{\min}(C) = J$.
(ii) J is a right weak Bruhat interval.

First, let us prove $\overline{\min}(C) = J$. By the definition of J_i, we have $s_i\sigma \in J_i$ for all $i \in \mathcal{A}$ and $\sigma \in \overline{\min}(C)$. It follows that $\overline{\min}(C) \subseteq J$. To prove the opposite inclusion $\overline{\min}(C) \supseteq J$, take $\sigma \in J$. By the definition of J, we have

$$[\sigma, s_i\sigma]_L \in E_i \quad \text{and} \quad [s_i\sigma, \xi_C\sigma]_L \in E'_i \quad \text{for all } i \in \mathcal{A}.$$

And Lemma 2.1 implies that

$$\{s_i\sigma \mid i \in \mathcal{A}\} = \{\gamma \in [\sigma, \xi_C\sigma]_L \mid \sigma \prec_L^{\mathrm{c}} \gamma\}.$$

Putting these together yields that $[\sigma, \xi_C\sigma]_L \in C$; therefore, $\sigma \in \overline{\min}(C)$.

Next, let us prove that J is a right weak Bruhat interval. Due to Lemma 4.5, it suffices to show that $s_i \cdot J_i$ is a right weak Bruhat interval for $i \in \mathcal{A}$. Let us fix $i \in \mathcal{A}$. Since $\ell(\xi_{E_i}) = 1$ and $\ell(\xi_{E'_i}) = k$, $\overline{\max}(E_i)$ and $\overline{\min}(E'_i)$ are right weak Bruhat intervals by the induction hypothesis. Combining this with Lemma 4.5 yields that J_i is a right weak Bruhat interval. In addition, we have $s_i\gamma \preceq_L \gamma$ for all $\gamma \in J_i$. Therefore, by Lemma 2.1, $s_i \cdot J_i$ is a right weak Bruhat interval. □

According to Theorem 4.6, every equivalence class C can be expressed as follows:

$$C = \{[\gamma, \xi_C\gamma]_L \mid \gamma \in [\sigma_0, \sigma_1]_R\},$$

where σ_0 and σ_1 represent the minimal and maximal elements in $\overline{\min}(C)$, respectively. In particular, when C is the equivalence class of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$, we can provide an explicit description of it.

Theorem 4.7. *Let $P \in \mathsf{RSP}_n$ and C the equivalence class of $\Sigma_L(P)$ under $\overset{D}{\simeq}$. Then*

$$C = \{\Sigma_L(Q) \mid Q \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_Q) = \mathrm{sh}(\tau_P)\}.$$

Proof. Let $\lambda/\mu = \mathrm{sh}(\tau_P)$. Combining Equation (3.3) with Theorem 3.9 yields that

$$\{\Sigma_L(Q) \mid Q \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_Q) = \lambda/\mu\} = \{[\mathsf{read}_\tau(T_{\lambda/\mu}), \mathsf{read}_\tau(T'_{\lambda/\mu})]_L \mid \tau \in \mathsf{DS}(\lambda/\mu)\}.$$

Therefore, for the assertion, we have only to show the equality

$$C = \{[\mathsf{read}_\tau(T_{\lambda/\mu}), \mathsf{read}_\tau(T'_{\lambda/\mu})]_L \mid \tau \in \mathsf{DS}(\lambda/\mu)\}.$$

First, let us show that $\{[\mathsf{read}_\tau(T_{\lambda/\mu}), \mathsf{read}_\tau(T'_{\lambda/\mu})]_L \mid \tau \in \mathsf{DS}(\lambda/\mu)\} \subseteq C$. This can be done by proving that for $\tau \in \mathsf{DS}(\lambda/\mu)$, the map

$$\begin{aligned} f_{P;\tau} : [\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})]_L &\to [\mathsf{read}_\tau(T_{\lambda/\mu}), \mathsf{read}_\tau(T'_{\lambda/\mu})]_L \\ \mathsf{read}_{\tau_P}(T) &\mapsto \mathsf{read}_\tau(T) \quad (T \in \mathrm{SYT}(\lambda/\mu)) \end{aligned}$$

is a descent-preserving isomorphism. Let us fix $\tau \in \mathsf{DS}(\lambda/\mu)$. The definition of τ-reading implies that for any $T_1, T_2 \in \mathrm{SYT}(\lambda/\mu)$,

$$\mathsf{read}_\tau(T_1) \preceq_L^{\mathrm{c}} \mathsf{read}_\tau(T_2) \quad \text{if and only if} \quad \mathsf{read}_{\tau_P}(T_1) \preceq_L^{\mathrm{c}} \mathsf{read}_{\tau_P}(T_2),$$

and therefore, $f_{P;\tau}$ is a poset isomorphism. To show that $f_{P;\tau}$ is descent-preserving, choose arbitrary $T \in \mathrm{SYT}(\lambda/\mu)$ and $i \in \mathrm{Des}_L(\mathsf{read}_{\tau_P}(T))$. Combining the conditions $T \in \mathrm{SYT}(\lambda/\mu)$ and $\tau_P \in \mathsf{DS}(\lambda/\mu)$ with $i \in \mathrm{Des}_L(\mathsf{read}_{\tau_P}(T))$ yields that $i+1$ appears weakly above and strictly right of i in T. It follows that $i \in \mathrm{Des}_L(\mathsf{read}_\tau(T))$, so $\mathrm{Des}_L(\mathsf{read}_{\tau_P}(T)) \subseteq \mathrm{Des}_L(\mathsf{read}_\tau(T))$. In the same manner, one can show that $\mathrm{Des}_L(\mathsf{read}_\tau(T)) \subseteq \mathrm{Des}_L(\mathsf{read}_{\tau_P}(T))$. Therefore, $f_{P;\tau}$ is a descent-preserving isomorphism.

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Next, let us show $C \subseteq \{[\mathsf{read}_\tau(T_{\lambda/\mu}), \mathsf{read}_\tau(T'_{\lambda/\mu})]_L \mid \tau \in \mathsf{DS}(\lambda/\mu)\}$. In the previous paragraph, we prove that $[\mathsf{read}_\tau(T_{\lambda/\mu}), \mathsf{read}_\tau(T'_{\lambda/\mu})]_L \in C$ for any $\tau \in \mathsf{DS}(\lambda/\mu)$. This implies that $\mathsf{read}_\tau(T'_{\lambda/\mu}) = \xi_C \mathsf{read}_\tau(T_{\lambda/\mu})$, and so it suffices to show that

$$\overline{\min}(C) \subseteq \{\mathsf{read}_\tau(T_{\lambda/\mu}) \mid \tau \in \mathsf{DS}(\lambda/\mu)\}.$$

Due to Lemma 3.7, this inclusion can be obtained by proving

$$\gamma \in [\mathsf{read}_{\tau_0}(T_{\lambda/\mu}), \mathsf{read}_{\tau_1}(T_{\lambda/\mu})]_R \quad \text{for any } \gamma \in \overline{\min}(C).$$

Let $\gamma \in \overline{\min}(C)$. Since $\mathsf{read}_{\tau_0}(T_{\lambda/\mu}) \in \overline{\min}(C)$, we have $\mathrm{Des}_L(\mathsf{read}_{\tau_0}(T_{\lambda/\mu})) = \mathrm{Des}_L(\gamma)$. In addition, by the definitions of τ_0 and $T_{\lambda/\mu}$, we have $\mathsf{read}_{\tau_0}(T_{\lambda/\mu}) = w_0(\alpha^{\mathrm{c}})$, where $\alpha = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_{\ell(\lambda)} - \mu_{\ell(\lambda)})$. Putting these equalities together with Lemma 2.2 yields that $\mathsf{read}_{\tau_0}(T_{\lambda/\mu}) \preceq_R \gamma$. Similarly, we have

$$\mathrm{Des}_L(\mathsf{read}_{\tau_1}(T'_{\lambda/\mu})) = \mathrm{Des}_L(\xi_C\gamma) \quad \text{and} \quad \mathsf{read}_{\tau_1}(T'_{\lambda/\mu}) = w_0(\beta^{\mathrm{c}})w_0,$$

where $\beta = (\lambda^{\mathrm{t}}_1 - \mu^{\mathrm{t}}_1, \lambda^{\mathrm{t}}_2 - \mu^{\mathrm{t}}_2, \ldots, \lambda^{\mathrm{t}}_{\ell(\lambda^{\mathrm{t}})} - \mu^{\mathrm{t}}_{\ell(\lambda^{\mathrm{t}})})$. This, together with Lemma 2.2, yields that $\xi_C\gamma \preceq_R \mathsf{read}_{\tau_1}(T'_{\lambda/\mu})$. Since $\mathsf{read}_{\tau_1}(T_{\lambda/\mu}) \preceq_L \mathsf{read}_{\tau_1}(T'_{\lambda/\mu}) = \xi_C \mathsf{read}_{\tau_1}(T_{\lambda/\mu})$, we have $\gamma \preceq_R \mathsf{read}_{\tau_1}(T_{\lambda/\mu})$. Therefore, $\gamma \in [\mathsf{read}_{\tau_0}(T_{\lambda/\mu}), \mathsf{read}_{\tau_1}(T_{\lambda/\mu})]_R$, as desired. □

Theorem 4.7 tells us that $\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\}$ is closed under $\overset{D}{\simeq}$ and the equivalence classes inside it are parametrized by the skew partitions of size n. Given a skew partition λ/μ of size n, let $C_{\lambda/\mu}$ be the equivalence class parametrized by λ/μ; that is,

$$C_{\lambda/\mu} = \{\Sigma_L(P) \mid P \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_P) = \lambda/\mu\}.$$

Corollary 4.8. *With the above notation, we have*

$$\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\} = \bigsqcup_{|\lambda/\mu|=n} C_{\lambda/\mu} \quad (\textit{disjoint union}).$$

5. The classification of M_P's for $P \in \mathsf{RSP}_n$

Let $P, Q \in \mathsf{RSP}_n$. By combining Proposition 4.1 with Theorem 4.7, we can see that if τ_P and τ_Q have the same shape, then the $H_n(0)$-modules M_P and M_Q are isomorphic. The purpose of this section is to demonstrate that the converse of this implication also holds. Let us briefly explain our strategy. First, we provide both a projective cover and an injective hull of M_P for every $P \in \mathsf{RSP}_n$. We discover that these modules are completely determined by the shape of τ_P, as demonstrated in Lemma 5.4. Then, we establish that if τ_P and τ_Q have different shapes, M_P and M_Q have either nonisomorphic projective covers or nonisomorphic injective hulls, as proven in Theorem 5.5.

To begin with, we present a brief overview of the background knowledge concerning projective modules and injective modules of the 0-Hecke algebras. In [11, Proposition 4.1], it was shown that $H_n(0)$ is a Frobenius algebra. It is well known that every Frobenius algebra is self injective, and for a finitely generated module M of a self injective algebra, M is projective if and only if it is injective (for instance, see [3, Proposition 1.6.2]). In [30], a complete list of non-isomorphic projective indecomposable $H_n(0)$-modules was provided.

In the work [19], it was shown that this list can also be expressed in terms of weak Bruhat interval modules, specifically as $\{\mathbf{P}_\alpha \mid \alpha \models n\}$, where

$$\mathbf{P}_\alpha := \mathsf{B}(w_0(\alpha^{\mathrm{c}}), w_0 w_0(\alpha)) \quad \text{for } \alpha \models n.$$

We note that $\mathbf{P}_\alpha/\mathrm{rad}\,\mathbf{P}_\alpha$ is isomorphic to $\mathbf{F}_\alpha$, where $\mathrm{rad}\,\mathbf{P}_\alpha$ is the *radical* of $\mathbf{P}_\alpha$, the intersection of maximal submodules of $\mathbf{P}_\alpha$.

In the following, we recall the definition of a projective cover and an injective hull. Let M be a finitely generated $H_n(0)$-module. A *projective cover* of M is a pair $(\boldsymbol{P}, f)$ consisting of a projective $H_n(0)$-module $\boldsymbol{P}$ and an $H_n(0)$-module epimorphism $f : \boldsymbol{P} \to M$ such that $\ker(f) \subseteq \mathrm{rad}(\boldsymbol{P})$. An *injective hull* of M is a pair $(\boldsymbol{I}, \iota)$, where $\boldsymbol{I}$ is an injective $H_n(0)$-module and $\iota : M \to \boldsymbol{I}$ is an $H_n(0)$-module monomorphism satisfying $\iota(M) \supseteq \mathrm{soc}(\boldsymbol{I})$. Here, $\mathrm{soc}(\boldsymbol{I})$ is the *socle* of $\boldsymbol{I}$, the sum of all irreducible submodules of $\boldsymbol{I}$. A projective cover and an injective hull of M always exist, and they are unique up to isomorphism. For more information, refer to [1, 23].

The projective modules introduced by Huang [18] play an important role in describing the projective cover and injective hull of M_P for $P \in \mathsf{RSP}_n$. We briefly review these projective modules from the viewpoint of weak Bruhat interval modules. A *generalized composition* $\boldsymbol{\alpha}$ of n is a formal expression $\alpha^{(1)} \star \alpha^{(2)} \star \cdots \star \alpha^{(k)}$, where $\alpha^{(i)} \models n_i$ for positive integers n_i's with $n_1 + n_2 + \cdots + n_k = n$. For compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$, let

$$\alpha \cdot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}, \beta_1, \beta_2, \ldots, \beta_{\ell(\beta)}) \quad \text{and} \quad \alpha \odot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)} + \beta_1, \beta_2, \ldots, \beta_{\ell(\beta)}).$$

For a generalized composition $\boldsymbol{\alpha} = \alpha^{(1)} \star \alpha^{(2)} \star \cdots \star \alpha^{(k)}$, let

$$\boldsymbol{\alpha}_\bullet := \alpha^{(1)} \cdot \alpha^{(2)} \cdot \cdots \cdot \alpha^{(k)}, \quad \boldsymbol{\alpha}_\odot := \alpha^{(1)} \odot \alpha^{(2)} \odot \cdots \odot \alpha^{(k)}$$

and let

$$\boldsymbol{\alpha}^{\mathrm{c}} := (\alpha^{(1)})^{\mathrm{c}} \star (\alpha^{(2)})^{\mathrm{c}} \star \cdots \star (\alpha^{(k)})^{\mathrm{c}}, \quad \boldsymbol{\alpha}^{\mathrm{r}} := (\alpha^{(k)})^{\mathrm{r}} \star (\alpha^{(k-1)})^{\mathrm{r}} \star \cdots \star (\alpha^{(1)})^{\mathrm{r}},$$

and $\boldsymbol{\alpha}^{\mathrm{c}\cdot\mathrm{r}} := (\boldsymbol{\alpha}^{\mathrm{c}})^{\mathrm{r}}$. Normally, $(\boldsymbol{\alpha}_\bullet)^{\mathrm{c}} \neq (\boldsymbol{\alpha}^{\mathrm{c}})_\bullet$ and $(\boldsymbol{\alpha}_\odot)^{\mathrm{c}} \neq (\boldsymbol{\alpha}^{\mathrm{c}})_\odot$ for a generalized composition $\boldsymbol{\alpha}$. Despite the potential for confusion, for the sake of brevity, we denote $(\boldsymbol{\alpha}_\bullet)^{\mathrm{c}}$ and $(\boldsymbol{\alpha}_\odot)^{\mathrm{c}}$ as $\boldsymbol{\alpha}^{\mathrm{c}}_\bullet$ and $\boldsymbol{\alpha}^{\mathrm{c}}_\odot$, respectively. Then, we define

$$\mathbf{P}_{\boldsymbol{\alpha}} := \mathsf{B}(w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\bullet), w_0 w_0(\boldsymbol{\alpha}_\odot)).$$

Huang decomposed $\mathbf{P}_{\boldsymbol{\alpha}}$ into projective indecomposable modules, and thus showed that it is projective. To be precise, the following lemma was shown.

Lemma 5.1 [18, Theorem 3.3]. *For a generalized composition* $\boldsymbol{\alpha} = \alpha^{(1)} \star \alpha^{(2)} \star \cdots \star \alpha^{(k)}$ *of* n*,*

$$\mathbf{P}_{\boldsymbol{\alpha}} \cong \mathbf{P}_{\alpha^{(1)}} \boxtimes \mathbf{P}_{\alpha^{(2)}} \boxtimes \cdots \boxtimes \mathbf{P}_{\alpha^{(k)}} \cong \bigoplus_{\beta \in [\boldsymbol{\alpha}]} \mathbf{P}_\beta,$$

where $[\boldsymbol{\alpha}] := \{\alpha^{(1)} \,\square\, \alpha^{(2)} \,\square\, \cdots \,\square\, \alpha^{(k)} \mid \square = \cdot \text{ or } \odot\}$.

It is clear from Lemma 5.1 that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are distinct generalized compositions of n, then $\mathbf{P}_{\boldsymbol{\alpha}}$ and $\mathbf{P}_{\boldsymbol{\beta}}$ are nonisomorphic. Let $\boldsymbol{\alpha}$ be a generalized composition of n. For $\rho \in [w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\odot), w_0 w_0(\boldsymbol{\alpha}_\odot)]_L$, let $\Upsilon_{\boldsymbol{\alpha};\rho} : \mathbf{P}_{\boldsymbol{\alpha}} \to \mathsf{B}(w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\bullet), \rho)$ be a $\mathbb{C}$-linear map given by

$$\gamma \mapsto \begin{cases} \gamma & \text{if } \gamma \in [w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\bullet), \rho]_L, \\ 0 & \text{if } \gamma \in [w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\bullet), w_0 w_0(\boldsymbol{\alpha}_\odot)]_L \setminus [w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\bullet), \rho]_L. \end{cases}$$

Clearly, $\Upsilon_{\boldsymbol{\alpha};\rho}$ is an $H_n(0)$-module epimorphism. In addition, it follows from [20, Lemma 6.2] that $\ker(\Upsilon_{\boldsymbol{\alpha};\rho}) \subseteq \mathrm{rad}(\mathbf{P}_{\boldsymbol{\alpha}})$. Consequently, we have the following lemma.

Lemma 5.2. *[(cf. [20, Lemma 6.2])] For a generalized composition* $\boldsymbol{\alpha}$ *of* n *and* $\rho \in [w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\odot), w_0 w_0(\boldsymbol{\alpha}_\odot)]_L$*, the pair* $(\mathbf{P}_{\boldsymbol{\alpha}}, \Upsilon_{\boldsymbol{\alpha};\rho})$ *is a projective cover of* $\mathsf{B}(w_0(\boldsymbol{\alpha}^{\mathrm{c}}_\bullet), \rho)$.

Let us provide notation and a lemma needed to describe a projective cover and an injective hull of M_P for $P \in \mathsf{RSP}_n$. For a connected skew partition λ/μ of size n, define

$$\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) := (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_{\ell(\lambda)} - \mu_{\ell(\lambda)}).$$

And, for a disconnected skew partition λ/μ of size n, define

$$\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) := \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(1)}/\mu^{(1)}) \star \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(2)}/\mu^{(2)}) \star \cdots \star \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(k)}/\mu^{(k)}),$$

where $\lambda/\mu = \lambda^{(1)}/\mu^{(1)} \star \lambda^{(2)}/\mu^{(2)} \star \cdots \star \lambda^{(k)}/\mu^{(k)}$ with connected skew partitions $\lambda^{(i)}/\mu^{(i)}$'s $(1 \le i \le k)$.

Lemma 5.3. *Let λ/μ be a skew partition of size n.*

(1) $\mathsf{read}_{\tau_0}(T_{\lambda/\mu}) = w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)^{\mathrm{c}}_{\bullet})$.
(2) $\mathsf{read}_{\tau_0}(T'_{\lambda/\mu}) \in [w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)^{\mathrm{c}}_{\odot}), w_0 w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)_{\odot})]_L$.

Proof. By the definition of $\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)$, the assertion (1) is clear. In addition, one can easily see that $\mathrm{Des}_R(\mathsf{read}_{\tau_0}(T'_{\lambda/\mu})) = \mathrm{set}(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)^{\mathrm{c}}_{\odot})$. So, by Lemma 2.2, the assertion (2) follows. □

Let $P \in \mathsf{RSP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. By Theorem 3.9, $\mathsf{M}_P = \mathsf{B}(\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu}))$. Furthermore, by Theorem 4.7, we have an $H_n(0)$-module isomorphism

$$f_P : \mathsf{M}_{\mathsf{poset}(\tau_0)} \to \mathsf{M}_P, \quad \mathsf{read}_{\tau_0}(T) \mapsto \mathsf{read}_{\tau_P}(T) \quad (T \in \mathrm{SYT}(\lambda/\mu)).$$

Set

$$\eta_P := f_P \circ \Upsilon_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu);\mathsf{read}_{\tau_0}(T'_{\lambda/\mu})}.$$

Combining Lemma 5.2 and Lemma 5.3 implies that the pair $\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)}, \eta_P\right)$ is a projective cover of M_P.

To find an injective hull of M_P, we note that

$$\mathsf{read}_{\tau_1}(T_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})w_0 = \mathsf{read}_{\tau_0}(T'_{\lambda/\mu}) \quad \text{and} \quad \mathsf{read}_{\tau_1}(T'_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})w_0 = \mathsf{read}_{\tau_0}(T_{\lambda/\mu}).$$

Combining these equalities with [19, Theorem 4] yields the following $H_n(0)$-module isomorphism:

$$\begin{aligned} g_1 : \mathbf{T}^{-}_{\widehat{\theta}}\left(\mathsf{M}_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}\right) &\to \mathsf{M}_{\mathsf{poset}(\tau_0^{\lambda/\mu})}, \\ \gamma^* &\mapsto (-1)^{\ell(\gamma\, \mathsf{read}_{\tau_1}(T'_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})^{-1})}\gamma w_0, \end{aligned}$$

where $\gamma \in [\mathsf{read}_{\tau_1}(T_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}}), \mathsf{read}_{\tau_1}(T'_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})]_L$ and γ^* denotes the dual of γ with respect to the basis $[\mathsf{read}_{\tau_1}(T_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}}), \mathsf{read}_{\tau_1}(T'_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})]_L$ for $\mathsf{M}_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}$. Set

$$\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu) := \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})^{\mathrm{c}\cdot\mathrm{r}}.$$

Again, by [19, Theorem 4], we have the $H_n(0)$-module isomorphism

$$\begin{aligned} g_2 : \mathbf{T}^{-}_{\widehat{\theta}}\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})}\right) &\to \mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)}, \\ \gamma^* &\mapsto (-1)^{\ell(\gamma(w_0 w_0(\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)_{\odot}))^{-1})}\gamma w_0, \end{aligned}$$

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where $\gamma \in [w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})^{\mathrm{c}}_{\bullet}), w_0 w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})_{\odot})]_L$ and γ^* denotes the dual of γ with respect to the basis $[w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})^{\mathrm{c}}_{\bullet}), w_0 w_0(\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})_{\odot})]_L$ for $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})}$. Set $\eta_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})} := f_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})} \circ \Upsilon_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}});\mathsf{read}_{\tau_0}(T'_{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}$. As above, the pair $\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})}, \eta_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}\right)$ is a projective cover of $\mathsf{M}_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}$. And since $\mathbf{T}^-_{\widehat{\theta}}$ is contravariant, $\left(\mathbf{T}^-_{\widehat{\theta}}\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})}\right), \mathbf{T}^-_{\widehat{\theta}}\left(\eta_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}\right)\right)$ is an injective hull of $\mathbf{T}^-_{\widehat{\theta}}(\mathsf{M}_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})})$ (for the definition of $\mathbf{T}^-_{\widehat{\theta}}$, see Section 2.4). Consequently, the pair $\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)}, \iota_P\right)$ is an injective hull of M_P, where

$$\iota_P = g_2 \circ \mathbf{T}^-_{\widehat{\theta}}\left(\eta_{\mathsf{poset}(\tau_1^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}})}\right) \circ g_1^{-1} \circ f_P^{-1}.$$

To summarize, we can state the following lemma.

Lemma 5.4. *Let $P \in \mathsf{RSP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$.*

(1) $\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)}, \eta_P\right)$ *is a projective cover of* M_P.
(2) $\left(\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)}, \iota_P\right)$ *is an injective hull of* M_P.

Now, we are ready to state the classification of M_P's for $P \in \mathsf{RSP}_n$ up to $H_n(0)$-module isomorphism.

Theorem 5.5. *Let $P, Q \in \mathsf{RSP}_n$. Then*

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \textit{if and only if} \quad \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q).$$

Proof. The 'if' part follows from Proposition 4.1 and Theorem 4.7. To prove the 'only if' part, suppose that $\mathsf{M}_P \cong \mathsf{M}_Q$. For simplicity, let $\lambda/\mu = \mathrm{sh}(\tau_P)$ and $\nu/\kappa = \mathrm{sh}(\tau_Q)$. By Lemma 5.4, $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)} \cong \mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\nu/\kappa)}$ and $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)} \cong \mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\nu/\kappa)}$, and therefore, $\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) = \boldsymbol{\alpha}_{\mathrm{proj}}(\nu/\kappa)$ and $\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu) = \boldsymbol{\alpha}_{\mathrm{inj}}(\nu/\kappa)$. Since $\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) = \boldsymbol{\alpha}_{\mathrm{proj}}(\nu/\kappa)$, the number of boxes in the same row of $\mathtt{yd}(\lambda/\mu)$ and $\mathtt{yd}(\nu/\kappa)$ are the same. Similarly, since $\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu) = \boldsymbol{\alpha}_{\mathrm{inj}}(\nu/\kappa)$, the number of boxes in the same column of $\mathtt{yd}(\lambda/\mu)$ and $\mathtt{yd}(\nu/\kappa)$ are same. Thus, we have $\lambda/\mu = \nu/\kappa$. □

Note that Theorem 5.5 is the classification theorem concerning the class of $H_n(0)$-modules $\{\mathsf{M}_P \mid P \in \mathsf{RSP}_n\}$. Consequently, a natural question arises: can this theorem be extended to the classes $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$ or $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$? This question appears to be highly nontrivial, as it involves the investigation of a broader set of modules. As a specific instance, let us examine the characterization of posets $Q \in \mathsf{RP}_n$ such that $\mathsf{M}_Q \cong \mathsf{M}_P$ when $P \in \mathsf{RSP}_n$. This problem can be readily addressed by assuming the validity of the following conjecture due to Stanley.

Conjecture 5.6 [34, p. 81]. *For $P \in \mathsf{P}_n$, if K_P is symmetric, then $P \in \mathsf{SP}_n$.*

In more detail, by combining Stanley's conjecture with Theorem 2.9(1), we can deduce that $\mathrm{ch}([\mathsf{M}_Q])$ is not symmetric, and as a consequence, $\mathsf{M}_Q \not\cong \mathsf{M}_P$ unless $Q \in \mathsf{SP}_n$. This observation leads to the following conclusion from Theorem 5.5:

$$\{Q \in \mathsf{RP}_n \mid \mathsf{M}_Q \cong \mathsf{M}_P\} = \{Q \in \mathsf{RSP}_n \mid \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q)\}.$$

If the shape of τ_P is non-skew, it is indeed possible to derive this conjectural identity without depending on the validity of Stanley's conjecture (see Proposition 7.1). However, tackling the general case remains beyond our current comprehension. For further discussions on classifications, refer to Section 7.1.

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6. A characterization of regular Schur labeled skew shape posets P and distinguished filtrations of M_P

In this section, we prove that a poset $P \in \mathsf{P}_n$ is a regular Schur labeled skew shape poset if and only if $\Sigma_L(P)$ is dual plactic closed (Theorem 6.4). Then, by considering the dual plactic closedness of $\Sigma_L(P)$, we construct filtrations

$$0 =: M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_l := \mathsf{M}_P$$

such that $\mathrm{ch}([M_k/M_{k-1}])$ is a Schur function for all $1 \le k \le l$ (Theorem 6.7).

6.1. A characterization of regular Schur labeled skew shape posets

Let $P \in \mathsf{P}_n$ and let $\Sigma_R(P) := \{\gamma^{-1} \mid \gamma \in \Sigma_L(P)\}$. In [27, Fact 1], it was stated that if $P \in \mathsf{SP}_n$, then $\Sigma_R(P)$ is plactic-closed. This, however, is not true. For instance, considering the case where $\lambda/\mu = (3,2)/(2)$ and $\tau = \begin{array}{ccc} & & \boxed{2} \\ \boxed{3} & \boxed{1} & \end{array} \in \mathsf{S}(\lambda/\mu)$, we have

$$\Sigma_R(\mathsf{poset}(\tau)) = \{312, 231, 321\},$$

which is not plactic-closed.

The purpose of this subsection is to prove that $P \in \mathsf{RSP}_n$ if and only if $\Sigma_R(P)$ is plactic-closed. We begin by providing background knowledge relevant to the plactic congruence. For instance, see [6, 14, 31, 35].

For $\sigma \in \mathfrak{S}_n$ and $1 < i < n$, we write $\sigma \overset{1}{\cong} \sigma s_i$ if

$$\sigma(i) < \sigma(i-1) < \sigma(i+1) \quad \text{or} \quad \sigma(i+1) < \sigma(i-1) < \sigma(i).$$

And we write $\sigma \overset{2}{\cong} \sigma s_{i-1}$ if

$$\sigma(i-1) < \sigma(i+1) < \sigma(i) \quad \text{or} \quad \sigma(i) < \sigma(i+1) < \sigma(i-1).$$

The *Knuth equivalence* (or *plactic congruence*) is an equivalence relation $\overset{K}{\cong}$ on $\mathfrak{S}_n$ defined by $\sigma \overset{K}{\cong} \rho$ if and only if there are $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathfrak{S}_n$ such that

$$\sigma = \gamma_1 \overset{a_1}{\cong} \gamma_2 \overset{a_2}{\cong} \cdots \overset{a_{k-1}}{\cong} \gamma_k = \rho,$$

where $a_1, a_2, \ldots a_{k-1} \in \{1, 2\}$. A subset S of $\mathfrak{S}_n$ is called *plactic-closed* if for any $\sigma \in S$, every $\rho \in \mathfrak{S}_n$ with $\rho \overset{K}{\cong} \sigma$ is also an element of S; in other words, S is a union of equivalence classes under $\overset{K}{\cong}$.

The *dual Knuth equivalence* (or *dual plactic congruence*) is an equivalence relation $\overset{K^*}{\cong}$ on $\mathfrak{S}_n$ defined by

$$\sigma \overset{K^*}{\cong} \rho \quad \text{if and only if} \quad \sigma^{-1} \overset{K}{\cong} \rho^{-1}.$$

A subset S of $\mathfrak{S}_n$ is called *dual plactic-closed* if for any $\sigma \in S$, every $\rho \in \mathfrak{S}_n$ with $\rho \overset{K^*}{\cong} \sigma$ is also an element of S; in other words, S is a union of equivalence classes under $\overset{K^*}{\cong}$.

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The Knuth and dual Knuth equivalences are closely related to the *Robinson–Schensted correspondence*, which is a one-to-one correspondence between $\mathfrak{S}_n$ and $\bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$. For $\sigma \in \mathfrak{S}_n$, we use the notation $(\mathtt{ins}(\sigma), \mathtt{rec}(\sigma))$ to represent the image of σ under this bijection. We call $\mathtt{ins}(\sigma)$ and $\mathtt{rec}(\sigma)$ as the *insertion tableau* and *recording tableau* of σ, respectively. It is well known that $\mathtt{ins}(\sigma) = \mathtt{rec}(\sigma^{-1})$ and

$$\sigma \overset{K}{\cong} \rho \quad \text{if and only if} \quad \mathtt{ins}(\sigma) = \mathtt{ins}(\rho) \quad \text{for } \sigma, \rho \in \mathfrak{S}_n.$$

Putting these together, one can easily derive that

$$\sigma \overset{K^*}{\cong} \rho \quad \text{if and only if} \quad \mathtt{rec}(\sigma) = \mathtt{rec}(\rho) \quad \text{for } \sigma, \rho \in \mathfrak{S}_n.$$

For a subset S of $\mathfrak{S}_n$, S is plactic-closed if and only if $\{\gamma^{-1} \mid \gamma \in S\}$ is dual plactic-closed. Based on this fact, we will consider the claim that $\Sigma_R(P)$ is plactic-closed and the claim that $\Sigma_L(P)$ is dual plactic-closed to be identical.

Let us collect the terminologies and lemmas necessary for the proof of the main result of this subsection. Let T be a standard Young tableau of skew shape. Denote by $\mathrm{Rect}(T)$ the *rectification of T* – that is, the unique standard Young tableau of partition shape obtained by applying jeu de taquin slides to T (see [14, Section 1.2]). Then

$$\mathrm{Rect}(T) = \mathtt{ins}(\mathsf{read}_{\tau_0}(T) w_0) \quad \text{for any } T \in \mathrm{SYT}(\lambda/\mu). \tag{6.1}$$

Define T^{t} to be the tableau obtained from T by flipping it along its main diagonal.

Lemma 6.1. *Let λ/μ be a skew partition and $T \in \mathrm{SYT}(\lambda/\mu)$. Then*

$$\mathtt{ins}(\mathsf{read}_\tau(T)) = \mathrm{Rect}(T)^{\mathrm{t}} \quad \textit{for any } \tau \in \mathsf{DS}(\lambda/\mu).$$

Proof. It is well known that

$$\mathtt{ins}(\sigma w_0) = \mathtt{ins}(\sigma)^{\mathrm{t}} \quad \text{for any } \sigma \in \mathfrak{S}_n \tag{6.2}$$

(for instance, see [31, Theorems 3.2.3]). Therefore, due to Equation (6.1), the assertion can be verified by showing that

$$\mathtt{ins}(\mathsf{read}_\tau(T)) = \mathtt{ins}(\mathsf{read}_{\tau_0^{\lambda/\mu}}(T)) \quad \text{for any } \tau \in \mathsf{DS}(\lambda/\mu). \tag{6.3}$$

Applying Taşkin's result [36, Proposition 3.2.5] to the weak order[3] on SYT_n given in [36, Definition 3.1.3], we derive that for $\sigma, \rho \in \mathfrak{S}_n$ with $\sigma \preceq_R \rho$,

$$\mathtt{ins}(\sigma) = \mathtt{ins}(\rho) \quad \text{or} \quad \mathrm{sh}(\mathtt{ins}(\rho)) \triangleleft \mathrm{sh}(\mathtt{ins}(\sigma)). \tag{6.4}$$

Here, $\trianglelefteq$ denotes the dominance order on the set of partitions of n. And, Lemma 3.7 says that

$$\mathsf{read}_{\tau_0^{\lambda/\mu}}(T) \preceq_R \mathsf{read}_\tau(T) \preceq_R \mathsf{read}_{\tau_1^{\lambda/\mu}}(T) \quad \text{for } \tau \in \mathsf{DS}(\lambda/\mu). \tag{6.5}$$

Note that

$$\mathtt{ins}(\mathsf{read}_{\tau_0^{\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}}}(T^{\mathrm{t}}) w_0) \underset{Eq.(6.1)}{=} \mathrm{Rect}(T^{\mathrm{t}}) = \mathrm{Rect}(T)^{\mathrm{t}} \underset{Eq.(6.1)}{=} \mathtt{ins}(\mathsf{read}_{\tau_0^{\lambda/\mu}}(T) w_0)^{\mathrm{t}}.$$

[3]This order was originally defined in [29, 2.5.1], where it is called the *induced Duflo order*.

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Since $\mathsf{read}_{\tau_1^{\lambda/\mu}}(T) = \mathsf{read}_{\tau_0^{\lambda^t/\mu^t}}(T^t)w_0$, it follows from Equation (6.2) that

$$\mathtt{ins}(\mathsf{read}_{\tau_1^{\lambda/\mu}}(T)) = \mathtt{ins}(\mathsf{read}_{\tau_0^{\lambda/\mu}}(T)). \tag{6.6}$$

Now, the equality in Equation (6.3) is obtained by combining Equation (6.4), Equation (6.5) and Equation (6.6). □

We introduce two important results due to Malvenuto [27].

Lemma 6.2 [27, Theorem 1]. *For $P \in \mathsf{P}_n$, if $\Sigma_R(P)$ is plactic-closed, then P is a Schur labeled skew shape poset.*

For $P \in \mathsf{P}_n$, we say a subposet Q of P is *convex* if Q satisfies the property that for any $x \in P$ if there exist $y_1, y_2 \in Q$ such that $y_1 \preceq_P x \preceq_P y_2$, then $x \in Q$. For a subposet $Q = \{i_1 < i_2 < \cdots < i_{|Q|}\}$ of $P \in \mathsf{P}_n$, the *standardization of Q*, denoted by $\mathsf{st}(Q)$, is the poset obtained from Q by replacing i_j with j for $1 \le j \le |Q|$.

Lemma 6.3 [27, Corollary 1]. *Let $P \in \mathsf{P}_n$ such that $\Sigma_R(P)$ is plactic-closed. For any convex subposet Q of P, $\Sigma_R(\mathsf{st}(Q))$ is plactic-closed.*

Now, we are ready to prove the main result of this subsection.

Theorem 6.4. *For $P \in \mathsf{P}_n$, P is a regular Schur labeled skew shape poset if and only if $\Sigma_L(P)$ is dual plactic-closed.*

Proof. To establish the 'only if' part, let $P \in \mathsf{RSP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. Due to Lemma 3.2, we have that

$$\Sigma_L(P) = \mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu)).$$

We claim that $\mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$ is dual plactic-closed.

As mentioned in [15, Property A], one can easily see that $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))$ is dual plactic-closed.[4] In addition, by Lemma 6.1, we have

$$\mathtt{ins}(\mathsf{read}_{\tau_P}(T)) = \mathtt{ins}(\mathsf{read}_{\tau_0}(T)) \quad \text{for all } T \in \mathrm{SYT}(\lambda/\mu). \tag{6.7}$$

Therefore, given $T \in \mathrm{SYT}(\lambda/\mu)$, if we show that

$$\mathsf{read}_{\tau_P}(T) \overset{K^*}{\cong} \mathsf{read}_{\tau_P}(U) \qquad \text{for all } U \in \mathrm{SYT}(\lambda/\mu) \text{ with } \mathsf{read}_{\tau_0}(T) \overset{K^*}{\cong} \mathsf{read}_{\tau_0}(U),$$

then we have

$$\{\gamma \in \mathfrak{S}_n \mid \gamma \overset{K^*}{\cong} \mathsf{read}_{\tau_P}(T)\} \subseteq \mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu)).$$

Let $T, U \in \mathrm{SYT}(\lambda/\mu)$ with $\mathsf{read}_{\tau_0}(T) \overset{K^*}{\cong} \mathsf{read}_{\tau_0}(U)$. Since $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))$ is dual plactic-closed, there exist standard Young tableaux $T_0 := T, T_1, \ldots, T_l := U$ of shape λ/μ such that for any $1 \le k \le l$,

$$\mathsf{read}_{\tau_0}(T_k) \overset{K^*}{\cong} \mathsf{read}_{\tau_0}(T) \;\text{ and }\; \mathsf{read}_{\tau_0}(T_k) = s_{i_k}\mathsf{read}_{\tau_0}(T_{k-1}) \;\text{ for some } i_k \in [n-1].$$

Combining Equation (6.7) with the equality $\mathrm{sh}(\mathtt{ins}(\mathsf{read}_{\tau_0}(T_k))) = \mathrm{sh}(\mathtt{ins}(\mathsf{read}_{\tau_0}(T)))$, we have

$$\mathrm{sh}(\mathtt{ins}(\mathsf{read}_{\tau_P}(T_k))) = \mathrm{sh}(\mathtt{ins}(\mathsf{read}_{\tau_P}(T))) \quad \text{for all } 1 \le k \le l. \tag{6.8}$$

[4] [15, Property A] is stated as 'For any skew diagram D the collection $W^{-1}(D)$ is a union of Knuth equivalence classes'. Following the notation of this paper, $W^{-1}(\mathsf{yd}(\lambda/\mu)) = \{(\mathsf{read}_{\tau_0}(T)w_0)^{-1} \mid T \in \mathrm{SYT}(\lambda/\mu)\}$. So, [15, Property A] says that the set $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))w_0 := \{\mathsf{read}_{\tau_0}(T)w_0 \mid T \in \mathrm{SYT}(\lambda/\mu)\}$ is dual plactic-closed. Although $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))$ is different from $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))w_0$, the dual plactic closedness of $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))$ can be proved in the same way as that of $\mathsf{read}_{\tau_0}(\mathrm{SYT}(\lambda/\mu))w_0$.

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Note that Equation (6.4) is equivalent to the statement that for $\sigma, \rho \in \mathfrak{S}_n$ with $\sigma \preceq_L \rho$,

$$\sigma \overset{K^*}{\cong} \rho \quad \text{or} \quad \mathrm{sh}(\mathtt{rec}(\rho)) \triangleleft \mathrm{sh}(\mathtt{rec}(\sigma)). \tag{6.9}$$

Putting Equation (6.8) together with Equation (6.9), we have $\mathsf{read}_{\tau_P}(T) \overset{K^*}{\cong} \mathsf{read}_{\tau_P}(U)$. Since we chose arbitrary $T, U \in \mathrm{SYT}(\lambda/\mu)$, we conclude that $\mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$ is dual plactic-closed.

To establish the 'if' part of the assertion, we prove the contraposition; that is, if P is not a regular Schur labeled skew shape poset, then $\Sigma_L(P)$ is not dual plactic-closed. If P is not a Schur labeled skew shape poset, then Lemma 6.2 says that $\Sigma_L(P)$ is not dual plactic-closed. So, we assume that $P \in \mathsf{P}_n$ is a non-regular Schur labeled skew shape poset.

One can easily check that if $n = 1, 2, 3$, then $\Sigma_L(P)$ is not dual plactic-closed. Suppose $n > 3$. Then, by Lemma 3.6, τ_P is a non-distinguished Schur labeling. This implies that there exists $k \in \mathbb{Z}_{>0}$ such that $\mathsf{cnt}_k(\tau_P)$ is not filled with consecutive integers. Let k_0 be the minimum among these integers and let m_0 be the minimum element among $m \in \mathsf{cnt}_{k_0}(\tau_P)$ such that $m + 1 \notin \mathsf{cnt}_{k_0}(\tau_P)$. Since $\mathsf{cnt}_{k_0}(\tau_P)$ is not filled with consecutive integers, we can choose

$$m_1 = \min\{m \in \mathsf{cnt}_{k_0}(\tau_P) \mid m > m_0\}.$$

Since m_0 and m_1 are in the same connected component of the Schur labeling τ_P and $m_0 < m_1$, we can take $m_{-1} \in \mathsf{cnt}_{k_0}(\tau_P)$ such that $m_{-1} < m_1$ and m_{-1} is adjacent to m_1. Here, the sentence 'm_{-1} is adjacent to m_1' means that the box containing m_{-1} and that containing m_1 share an edge. We note that m_{-1} can be m_0. Because of the choice of m_1, we have $m_{-1} < m_0 + 1 < m_1$. Let Q be the subposet of P whose underlying set is $\{m_{-1}, m_0 + 1, m_1\}$. In P, m_1 covers m_{-1} and $m_0 + 1$ is incomparable with both m_1 and m_{-1}. This implies that Q is a convex subposet of P. In addition, since $m_{-1} < m_0 + 1 < m_1$, we have $\Sigma_L(\mathsf{st}(Q)) = \{123, 132, 213\}$ or $\Sigma_L(\mathsf{st}(Q)) = \{312, 231, 321\}$. Thus, $\Sigma_L(\mathsf{st}(Q))$ is not dual plactic closed. Combining this with Lemma 6.3 yields that $\Sigma_L(P)$ is not dual plactic closed, as desired. □

6.2. *Distinguished filtrations of* M_P *for* $P \in \mathsf{RSP}_n$

We begin by introducing the definition of distinguished filtrations.

Definition 6.5. Let $\mathcal{B} = \{\mathcal{B}_\alpha \mid \alpha \in I\}$ be a linearly independent subset of QSym_n with the property that $\mathcal{B}_\alpha$ is F-positive for all $\alpha \in I$, where I is an index set. Given a finite dimensional $H_n(0)$-module M, a *distinguished filtration of M with respect to* $\mathcal{B}$ is an $H_n(0)$-submodule series of M

$$0 =: M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l := M$$

such that for all $1 \le k \le l$, $\mathrm{ch}([M_k/M_{k-1}]) = \mathcal{B}_\alpha$ for some $\alpha \in I$.

As seen in Example 6.6, a distinguished filtration of M with respect to $\mathcal{B}$ may not exist even if $\mathrm{ch}([M])$ expands positively in $\mathcal{B}$. This is because the category $H_n(0)$-**mod** is neither semisimple nor representation-finite when $n > 3$ ([10, 11]).

Example 6.6. Let $\mathcal{B} = \{s_\lambda \mid \lambda \vdash 4\}$. For $B = \{2314, 1423, 3214, 2413, 1432, 3412\}$, let M be the $H_4(0)$-module with underlying space $\mathbb{C}B$ and with the $H_4(0)$-action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma), \\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin B, \\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in B. \end{cases}$$

The $H_4(0)$-action on $B \cup \{0\}$ is illustrated in the following figure:

$$\begin{tikzcd}
& 2314 \arrow[loop right, "\pi_1"] \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_3"] & & 1423 \arrow[loop right, "\pi_3"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
3214 \arrow[loop right, "\pi_1, \pi_2"] \arrow[d, "\pi_3"] & & 2413 \arrow[loop right, "\pi_1, \pi_3"] \arrow[d, "\pi_2"] & & 1432 \arrow[loop right, "\pi_2, \pi_3"] \arrow[d, "\pi_1"] \\
0 & & 3412 \arrow[loop right, "\pi_2"] \arrow[d, "\pi_1, \pi_3"] & & 0 \\
& & 0 & &
\end{tikzcd}$$

One sees that

$$\mathrm{ch}([M]) = s_{(3,1)} + s_{(2,1,1)} = (F_{(3,1)} + F_{(2,2)} + F_{(1,3)}) + (F_{(2,1,1)} + F_{(1,2,1)} + F_{(1,1,2)}).$$

So, if there exists a distinguished filtration of M with respect to $\mathcal{B}$, then there exists a three-dimensional $H_4(0)$-submodule N of M such that $\mathrm{ch}([N])$ is equal to either $s_{(3,1)}$ or $s_{(2,1,1)}$. We claim that such a submodule N does not exist.

Note that

$$M = \mathbb{C}\{2314 - 3214, 1423 - 1432, 2413, 3412\} \oplus \mathbb{C}\{3214\} \oplus \mathbb{C}\{1432\}. \tag{6.10}$$

Here, $\mathbb{C}\{3214\}$ and $\mathbb{C}\{1432\}$ are irreducible. And $\mathbb{C}\{2314 - 3214, 1423 - 1432, 2413, 3412\}$ is indecomposable since it is isomorphic to a submodule of the injective indecomposable module $\mathbf{P}_{(1,2,1)}$. Therefore, Equation (6.10) is a decomposition of M into indecomposables. The $H_4(0)$-action on $\{2314 - 3214, 1423 - 1432, 2413, 3412, 3214, 1432\} \cup \{0\}$ is illustrated in the following figure:

$$\begin{tikzcd}
& & 2314 - 3214 \arrow[loop right, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & & 1423 - 1432 \arrow[loop right, "\pi_3"] \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] & & \\
3214 \arrow[loop right, "\pi_1, \pi_2"] \arrow[d, "\pi_3"] & \oplus & 0 & 2413 \arrow[loop right, "\pi_1, \pi_3"] \arrow[d, "\pi_2"] & 0 & \oplus & 1432 \arrow[loop right, "\pi_2, \pi_3"] \arrow[d, "\pi_1"] \\
0 & & & 3412 \arrow[loop right, "\pi_2"] \arrow[d, "\pi_1, \pi_3"] & & & 0 \\
& & & 0 & & &
\end{tikzcd}$$

The injective hulls of $\mathbb{C}\{2314-3214, 1423-1432, 2413, 3412\}$, $\mathbb{C}\{3214\}$ and $\mathbb{C}\{1432\}$ are $\mathbf{P}_{(1,2,1)}$, $\mathbf{P}_{(1,3)}$ and $\mathbf{P}_{(3,1)}$, respectively. This implies that the socle of M is $\mathbb{C}\{3412\} \oplus \mathbb{C}\{3214\} \oplus \mathbb{C}\{1432\}$. It follows that for every three-dimensional submodule N of M, $1 \leq \dim \mathrm{soc}(N) \leq 3$. We list all three-dimensional submodules N of M in Table 1. Based on this, we conclude that there are no $H_4(0)$-submodules N of M such that $\mathrm{ch}([N]) = s_{(3,1)}$ or $s_{(2,1,1)}$.

Let $f \in \mathrm{QSym}_n$ and $\mathcal{B} = \{\mathcal{B}_\alpha \mid \alpha \in I\}$ be the linearly independent set given in Definition 6.5. When f expands positively in $\mathcal{B}$, that is,

$$f = \sum_{\alpha \in I} c_\alpha \mathcal{B}_\alpha \quad (c_\alpha \in \mathbb{Z}_{\geq 0}), \tag{6.11}$$

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Table 1. *The complete list of three-dimensional submodules of M in Example 6.6.*

Three-dimensional submodules N of M	$\dim \mathrm{soc}(N)$	$\mathrm{ch}([N])$
$\mathbb{C}\{3214, 1432, 3412\}$	3	$F_{(3,1)} + F_{(1,3)} + F_{(1,2,1)}$
$\mathbb{C}\{3214, 1423 - 1432 - 2413, 3412\}$	2	$F_{(3,1)} + F_{(1,1,2)} + F_{(1,2,1)}$
$\mathbb{C}\{3214, 2314 - 3214 - 2413, 3412\}$	2	$F_{(3,1)} + F_{(2,1,1)} + F_{(1,2,1)}$
$\mathbb{C}\{3214, 2413, 3412\}$	2	$F_{(3,1)} + F_{(2,2)} + F_{(1,2,1)}$
$\mathbb{C}\{1432, 1423 - 1432 - 2413, 3412\}$	2	$F_{(1,3)} + F_{(1,1,2)} + F_{(1,2,1)}$
$\mathbb{C}\{1432, 2314 - 3214 - 2413, 3412\}$	2	$F_{(1,3)} + F_{(2,1,1)} + F_{(1,2,1)}$
$\mathbb{C}\{1432, 2413, 3412\}$	2	$F_{(1,3)} + F_{(2,2)} + F_{(1,2,1)}$
$\mathbb{C}\{2314 - 3214, 2413, 3412\}$	1	$F_{(2,1,1)} + F_{(2,2)} + F_{(1,2,1)}$
$\mathbb{C}\{1423 - 1432, 2413, 3412\}$	1	$F_{(1,1,2)} + F_{(2,2)} + F_{(1,2,1)}$

finding an $H_n(0)$-module M such that

(C1) $\mathrm{ch}([M]) = f$,
(C2) it is not a direct sum of irreducible modules, yet it possesses a combinatorial model that can be effectively handled, and
(C3) it has a distinguished filtration with respect to $\mathcal{B}$

is a very important problem in the sense that this filtration can be considered as a nice representation theoretic interpretation of Equation (6.11).

In this subsection, we focus on the above problem in the case where $\mathcal{B}$ is $\mathcal{S} := \{s_\lambda \mid \lambda \vdash n\}$ and $f = s_{\lambda/\mu}$ for a skew partition λ/μ of size n. Note that for all $P \in \mathsf{RSP}_n$ with $\mathrm{sh}(\tau_P) = \lambda/\mu$, M_P satisfies (C1) and (C2) because $\mathrm{ch}([\mathsf{M}_P]) = s_{\lambda/\mu}$ by Theorem 2.9(2) and it has a combinatorial model $\Sigma_L(P)$. In the following, we show that M_P satisfies (C3).

Theorem 6.7. *For every $P \in \mathsf{RSP}_n$, M_P has a distinguished filtration with respect to $\mathcal{S}$.*

Proof. To begin with, we choose any total order $\ll$ on SYT_n subject to the condition that

$$T \ll S \quad \text{whenever } \mathrm{sh}(T) \lhd \mathrm{sh}(S). \tag{6.12}$$

Write $\{\mathtt{rec}(\gamma) \mid \gamma \in \Sigma_L(P)\}$ as

$$\{T_1 \ll T_2 \ll \cdots \ll T_l\}.$$

For $0 \le k \le l$, set

$$B_k := \{\gamma \in \mathfrak{S}_n \mid \mathtt{rec}(\gamma) = T_i \text{ for some } 1 \le i \le k\}.$$

It is clear that $\emptyset = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_l$. And, by Theorem 6.4, we have $B_l = \Sigma_L(P)$. We claim that

$$0 = \mathbb{C}B_0 \subset \mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \cdots \subset \mathbb{C}B_l = \mathsf{M}_P \tag{6.13}$$

is a distinguished filtration of M_P with respect to $\mathcal{S}$.

First, we show that for $1 \le k \le l$,

$$\pi_i \cdot \gamma \in B_k \cup \{0\} \quad \text{for all } i \in [n-1] \text{ and } \gamma \in B_k.$$

Take any $i \in [n-1]$ and $\gamma \in B_k$. If $\pi_i \cdot \gamma = 0$ or γ, then there is nothing to prove. Assume that $\pi_i \cdot \gamma = s_i\gamma$. Then, by the definition of $H_n(0)$-action on M_P, we have $\gamma \preceq_L s_i\gamma$. Combining this inequality with Equation (6.9) yields that

$$\gamma \overset{K^*}{\cong} s_i\gamma \quad \text{or} \quad \mathrm{sh}(\mathtt{rec}(s_i\gamma)) \lhd \mathrm{sh}(\mathtt{rec}(\gamma)).$$

This implies that $s_i\gamma \in B_k$, as desired.

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Next, we show that the filtration given in Equation (6.13) is distinguished with respect to $\mathcal{S}$. For $1 \le k \le l$, $\{\gamma + M_{k-1} \mid \gamma \in B_k \setminus B_{k-1}\}$ is a basis for M_k/M_{k-1} and $B_k \setminus B_{k-1}$ is an equivalence class under $\overset{K^*}{\cong}$. It follows that $\mathrm{ch}([M_k/M_{k-1}])$ is a Schur function; more precisely, $\mathrm{ch}([M_k/M_{k-1}]) = s_{\mathrm{sh}(T_k)^{\mathrm{t}}}$. □

Example 6.8. Let $P = \mathsf{poset}(\tau_0^{(4,2,1)/(2,1)})$. Following the method presented in the proof of Theorem 6.7, we will construct two distinguished filtrations of M_P with respect to $\{s_\lambda \mid \lambda \vdash 4\}$ by choosing two distinct total orders on SYT_4.

Note that $\{\mathtt{rec}(\gamma) \mid \gamma \in \Sigma_L(P)\}$ is given by

$$\left\{ Q_1 := \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array},\ Q_2 := \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array},\ Q_3 := \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array},\ Q_4 := \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array},\ Q_5 := \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array} \right\}$$

and $\mathrm{sh}(Q_1) \lessdot \mathrm{sh}(Q_2) = \mathrm{sh}(Q_3) \lessdot \mathrm{sh}(Q_4) \lessdot \mathrm{sh}(Q_5)$. Choose a total order $\ll_1$(resp. $\ll_2$) on SYT_4 satisfying both Equation (6.12) and $Q_2 \ll_1 Q_3$ (resp. $Q_3 \ll_2 Q_2$). For $1 \le k \le 5$, let

$$B'_k := \{\gamma \in \mathfrak{S}_4 \mid \mathtt{rec}(\gamma) = Q_k\}.$$

When we use $\ll_1$, we let

$$B_k := \bigsqcup_{1 \le l \le k} B'_l \quad \text{for } 0 \le k \le 5,$$

and when we use $\ll_2$, we let

$$B_k := \bigsqcup_{1 \le l \le k} B'_l \quad \text{for } k = 0,1,3,4,5 \quad \text{and} \quad B_2 := B'_1 \sqcup B'_3.$$

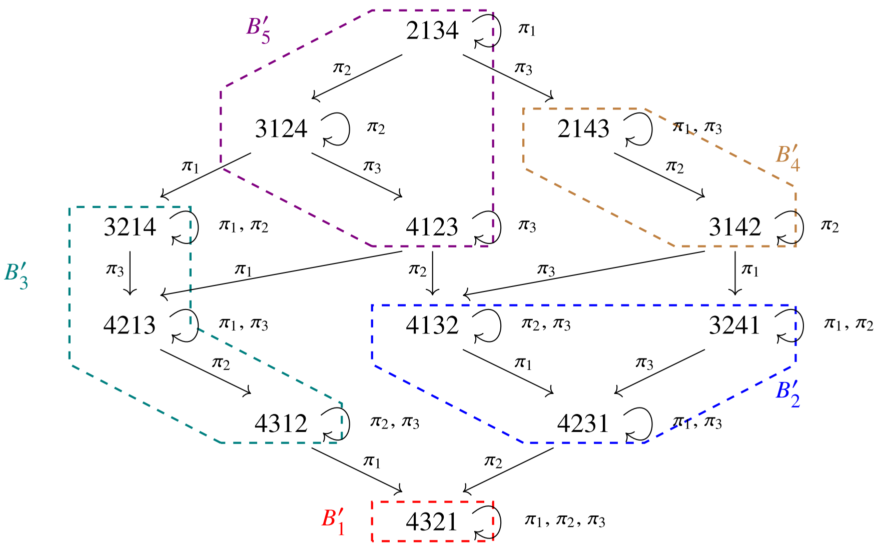

Figure 2. *The $H_4(0)$-action on the basis $\Sigma_L(P) = [2134, 4321]_L$ for M_P and the sets $B'_k (1 \le k \le 5)$ in Example 6.8.*

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Then

$$0 = \mathbb{C}B_0 \subset \mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \mathbb{C}B_3 \subset \mathbb{C}B_4 \subset \mathbb{C}B_5 = \mathsf{M}_P$$

is the desired distinguished filtration of M_P with respect to $\{s_\lambda \mid \lambda \vdash 4\}$.

For the readers' convenience, we draw the $H_4(0)$-action on the basis $\Sigma_L(P) = [2134, 4321]_L$ for M_P and the sets $B'_k (1 \le k \le 5)$ in Figure 2.

7. Further avenues

In this section, we discuss future directions regarding the classification problem, the decomposition problem, and how to recover M_P for $P \in \mathsf{RSP}_n$ from a module of the generic Hecke algebra $H_n(q)$ by specializing q to 0.

7.1. *The classification problem*

In Theorem 5.5, we successfully classify M_P for $P \in \mathsf{RSP}_n$. To be precise, we show that for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q). \tag{7.1}$$

Recall that $\mathsf{RSP}_n = \mathsf{RP}_n \cap \mathsf{SP}_n$. Hence, it would be natural to consider the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$ and $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$.

7.1.1. A remark on the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$

Since the notion 'the shape of τ_P' is available for $P \in \mathsf{SP}_n$, one may expect that the classification given in Equation (7.1) can be extended to $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$. Unfortunately, this expectation turns out to be false.

Let

$$\tau_1 := \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 2 & 1 \\ \hline 4 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \tau_2 := \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 4 & 2 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \text{and} \quad \tau_3 := \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 4 & 1 \\ \hline 3 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}.$$

Then $\mathsf{M}_{\mathsf{poset}(\tau_i)} (i = 1, 2, 3)$ is decomposed into indecomposables as follows:

$$\begin{aligned} \mathsf{M}_{\mathsf{poset}(\tau_1)} &\cong \mathbf{P}_{(4)} \oplus \mathbf{P}_{(2,2)}, \\ \mathsf{M}_{\mathsf{poset}(\tau_2)} &\cong \mathbf{F}_{(1,2,1)} \oplus \mathsf{B}(4213, 4312) \oplus \mathbf{F}_{(3,1)} \oplus \mathbf{F}_{(2,2)} \oplus \mathbf{F}_{(4)}, \\ \mathsf{M}_{\mathsf{poset}(\tau_3)} &\cong \mathbf{F}_{(1,2,1)} \oplus \mathsf{B}(4213, 4312) \oplus \mathsf{B}(2431, 3421) \oplus \mathbf{F}_{(4)}, \end{aligned}$$

where $\mathsf{B}(4213, 4312)$ and $\mathsf{B}(2431, 3421)$ are 2-dimensional indecomposable modules. These decompositions show that $\mathsf{M}_{\mathsf{poset}(\tau_1)}, \mathsf{M}_{\mathsf{poset}(\tau_2)}$ and $\mathsf{M}_{\mathsf{poset}(\tau_3)}$ are pairwise non-isomorphic although all $\tau_{\mathsf{poset}(\tau_i)}$'s have the same shape.

7.1.2. A conjecture on the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$

Note that for $P \in \mathsf{RP}_n$, the notion 'the shape of τ_P' has not been defined. This leads us to introduce a classification of $\{\mathsf{M}_P \mid P \in \mathsf{RSP}_n\}$ without using this notion. To be precise, by combining Theorem 4.7 and Theorem 5.5, we derive that for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q). \tag{7.2}$$

We expect that this classification can be extended to RP_n in its current form. The validity of this expectation has been checked for values of n up to 6 with the aid of the computer program SageMath.

Table 2. *Seven pairs* $(I_1^{(k)}, I_2^{(k)})$ *in* $\mathfrak{A}_6$.

k	$I_1^{(k)}$	$I_2^{(k)}$
1	$[123456, 426351]_L$	$[123456, 624153]_L$
2	$[123456, 354612]_L$	$[123456, 561324]_L$
3	$[123456, 356412]_L$	$[123456, 561342]_L$
4	$[123456, 563124]_L$	$[123456, 534612]_L$
5	$[123456, 536412]_L$	$[123456, 563142]_L$
6	$[123456, 465312]_L$	$[123456, 645132]_L$
7	$[123456, 564213]_L$	$[123456, 546231]_L$

Let us provide an overview of our verification process. We first classify all left weak Bruhat intervals in $\mathfrak{S}_n$ ($n \le 6$) up to descent-preserving isomorphism and choose a complete list $\mathfrak{I}_n$ of inequivalent representatives. Next, we let $\mathfrak{A}_n$ be the set of all unordered pairs $([\sigma_1, \rho_1]_L, [\sigma_2, \rho_2]_L)$ of intervals in $\mathfrak{I}_n$ satisfying that $[\sigma_1, \rho_1]_L \neq [\sigma_2, \rho_2]_L$ and

$$\mathrm{ch}([\mathsf{B}(\sigma_1, \rho_1)]) = \mathrm{ch}([\mathsf{B}(\sigma_2, \rho_2)]), \ \ \mathrm{Des}_L(\sigma_1) = \mathrm{Des}_L(\sigma_2), \ \ \mathrm{Des}_L(\rho_1) = \mathrm{Des}_L(\rho_2). \tag{7.3}$$

Note that Equation (7.3) is a necessary condition for $\mathsf{B}(\sigma_1, \rho_1) \cong \mathsf{B}(\sigma_2, \rho_2)$. Finally, we show that for all $(I_1, I_2) \in \mathfrak{A}_n$, $\mathsf{B}(I_1) \ncong \mathsf{B}(I_2)$. When $n \le 5$, there is nothing to prove because $\mathfrak{A}_n = \emptyset$. When $n = 6$, $\mathfrak{A}_6$ has fourteen pairs. Note that if $(I_1, I_2) \in \mathfrak{A}_6$, then $(w_0 \cdot I_1 \cdot w_0, w_0 \cdot I_2 \cdot w_0) \in \mathfrak{A}_6$ and

$$\mathsf{B}(I_1) \cong \mathsf{B}(I_2) \quad \underset{[19,\ \text{Theorem 4}]}{\Longleftrightarrow} \quad \mathsf{B}(w_0 \cdot I_1 \cdot w_0) \cong \mathsf{B}(w_0 \cdot I_2 \cdot w_0).$$

Therefore, it suffices to examine seven pairs $(I_1^{(k)}, I_2^{(k)})$ listed in Table 2. For $3 \le k \le 7$, using Lemma 5.2, one can see that the projective covers of $\mathsf{B}(I_1^{(k)})$ and $\mathsf{B}(I_2^{(k)})$ are not isomorphic. Therefore, $\mathsf{B}(I_1^{(k)})$ and $\mathsf{B}(I_2^{(k)})$ are not isomorphic. For $k = 1, 2$, one can see that $\mathsf{B}(I_1^{(k)})$ and $\mathsf{B}(I_2^{(k)})$ are not isomorphic in a brute force manner.

Let us give another evidence for our expectation. Specifically, we show that Equation (7.2) holds when $P \in \mathsf{RSP}_n$, $Q \in \mathsf{RP}_n$, and $\mathrm{ch}([\mathsf{M}_P])$ is a Schur function. This can be derived from the proposition presented below.

Proposition 7.1. *Let P be a poset in RSP_n such that $\mathrm{ch}([\mathsf{M}_P])$ is a Schur function.*

(1) *If $Q \in \mathsf{P}_n$ satisfies that $\mathsf{M}_Q \cong \mathsf{M}_P$, then $Q \in \mathsf{RSP}_n$.*
(2) *The isomorphism class of M_P within $\{\mathsf{M}_Q \mid Q \in \mathsf{P}_n\}$ is equal to the isomorphism class of M_P within $\{\mathsf{M}_Q \mid Q \in \mathsf{RSP}_n\}$ as sets.*

Proof. (1) Suppose that $\mathrm{ch}([\mathsf{M}_P]) = s_\lambda$ for some $\lambda \vdash n$. By [39, Theorem 2.2], $\mathrm{sh}(\tau_P)$ is either λ or λ°, where λ° denotes the skew partition whose Young diagram is obtained by rotating $\mathsf{yd}(\lambda)$ by 180°.

First, we consider the case where $\mathrm{sh}(\tau_P) = \lambda$. Let $f : \mathsf{M}_P \to \mathsf{M}_Q$ be an $H_n(0)$-module isomorphism. By Theorem 3.9, we see that $\Sigma_L(P) = [\mathsf{read}_{\tau_P}(T_\lambda), \mathsf{read}_{\tau_P}(T'_\lambda)]_L$, and therefore, $\mathsf{read}_{\tau_P}(T_\lambda)$ is a cyclic generator of M_P. In addition, in view of [32, Lemma 3.12], we have that

$$\mathrm{Des}_L(\mathsf{read}_{\tau_P}(T)) \not\supseteq \mathrm{Des}_L(\mathsf{read}_{\tau_P}(T_\lambda)) \text{ for all } T \in \mathrm{SYT}(\lambda) \setminus \{T_\lambda\}. \tag{7.4}$$

Combining (7.4) with the equality $\mathrm{ch}([\mathsf{M}_P]) = \mathrm{ch}([\mathsf{M}_Q])$, we can deduce that there exists a unique $\sigma \in \Sigma_L(Q)$ such that $\mathrm{Des}_L(\sigma) \supseteq \mathrm{Des}_L(\mathsf{read}_{\tau_P}(T_\lambda))$. This fact implies that $f(\mathsf{read}_{\tau_P}(T_\lambda)) = c\sigma$ for some nonzero $c \in \mathbb{C}$. We may assume that $c = 1$ by considering the isomorphism $\frac{1}{c}f$ instead of f. Since f is an $H_n(0)$-module isomorphism, $\Sigma_L(Q)$ is equal to $f(\Sigma_L(P))$ and therefore is a left weak Bruhat interval. Furthermore, it holds that

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$$\mathrm{Des}_L(f(\gamma)) = \mathrm{Des}_L(\gamma) \quad \text{for all } \gamma \in \Sigma_L(P).$$

As a consequence, we obtain a descent-preserving isomorphism $f|_{\Sigma_L(P)} : \Sigma_L(P) \to \Sigma_L(Q)$. Now the assertion follows from Theorem 4.7.

Next, consider the case where $\mathrm{sh}(\tau_P) = \lambda^\circ$. Let $\overline{P}^*$ and $\overline{Q}^*$ be the posets in P_n whose orders are defined by

$$u \preceq_{\overline{P}^*} v \iff n+1-v \preceq_P n+1-u \quad \text{and} \quad u \preceq_{\overline{Q}^*} v \iff n+1-v \preceq_Q n+1-u,$$

respectively. Since P is a poset in RSP_n with $\mathrm{sh}(\tau_P) = \lambda^\circ$, $\overline{P}^*$ is a poset in RSP_n with $\mathrm{sh}(\tau_{\overline{P}^*}) = \lambda$. By [9, Theorem 3.6(a)], we have $\mathsf{M}_{\overline{P}^*} \cong \mathbf{T}^+_\phi(\mathsf{M}_P)$ and $\mathsf{M}_{\overline{Q}^*} \cong \mathbf{T}^+_\phi(\mathsf{M}_Q)$, which implies that $\mathsf{M}_{\overline{Q}^*} \cong \mathsf{M}_{\overline{P}^*}$. It follows from the first case that $\overline{Q}^* \in \mathsf{RSP}_n$, thus $Q \in \mathsf{RSP}_n$.

(2) It follows from (1). □

Based on these evidences, we propose the following conjecture.

Conjecture 7.2. *Let $P, Q \in \mathsf{RP}_n$. If $\mathsf{M}_P \cong \mathsf{M}_Q$, then $\Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q)$.*

We remark that the converse of Conjecture 7.2 holds due to Proposition 4.1.

7.2. *The decomposition problem of* M_P *for* $P \in \mathsf{RSP}_n$

A Young diagram of skew shape is called a *ribbon* if it does not contain any 2×2 square. For simplicity, we call a skew partition a *ribbon* if the corresponding Young diagram is a ribbon. Note that our ribbons are not necessarily connected. Consider a skew partition

$$\lambda/\mu = \lambda^{(1)}/\mu^{(1)} \star \lambda^{(2)}/\mu^{(2)} \star \cdots \star \lambda^{(k)}/\mu^{(k)}$$

such that $\lambda^{(i)}/\mu^{(i)}$ is connected for all $1 \le i \le k$. We say that λ/μ *contains a disconnected ribbon* if there exists an index $1 \le j \le k-1$ such that both $\lambda^{(j)}/\mu^{(j)}$ and $\lambda^{(j+1)}/\mu^{(j+1)}$ are ribbons. With this notation, we state the following proposition.

Proposition 7.3. *Let $P \in \mathsf{RSP}_n$.*

(1) *If $\mathrm{sh}(\tau_P)$ is connected, then M_P is indecomposable.*
(2) *If $\mathrm{sh}(\tau_P)$ contains a disconnected ribbon, then M_P is not indecomposable.*

Proof. (1) It follows from Lemma 5.4.

(2) Suppose that $\mathrm{sh}(\tau_P)$ contains a disconnected ribbon. Let $\lambda/\mu = \mathrm{sh}(\tau_P)$. Write λ/μ as $\lambda^{(1)}/\mu^{(1)} \star \lambda^{(2)}/\mu^{(2)} \star \cdots \star \lambda^{(k)}/\mu^{(k)}$, where $\lambda^{(i)}/\mu^{(i)}$ is connected for all $1 \le i \le k$ and both $\lambda^{(j)}/\mu^{(j)}$ and $\lambda^{(j+1)}/\mu^{(j+1)}$ are ribbons for some $1 \le j \le k-1$.

In Appendix A, we constructed an $H_n(0)$-module $X_{\lambda/\mu}$ satisfying that $X_{\lambda/\mu} \cong \mathsf{M}_P$. From now on, we will prove the assertion for $X_{\lambda/\mu}$ instead of M_P. By Proposition A.2(1), we have the $H_n(0)$-module isomorphism

$$X_{\lambda/\mu} \cong X_{\lambda^{(1)}/\mu^{(1)}} \boxtimes \cdots \boxtimes X_{\lambda^{(k)}/\mu^{(k)}}.$$

Set $X^{(1)} := X_{\lambda^{(1)}/\mu^{(1)}} \boxtimes \cdots \boxtimes X_{\lambda^{(j-1)}/\mu^{(j-1)}}$ and $X^{(2)} := X_{\lambda^{(j+2)}/\mu^{(j+2)}} \boxtimes \cdots \boxtimes X_{\lambda^{(k)}/\mu^{(k)}}$. Since $\lambda^{(j)}/\mu^{(j)}$ and $\lambda^{(j+1)}/\mu^{(j+1)}$ are ribbons, $X_{\lambda^{(j)}/\mu^{(j)}} \cong \mathbf{P}_\alpha$ and $X_{\lambda^{(j+1)}/\mu^{(j+1)}} \cong \mathbf{P}_\beta$, where $\alpha = \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(j)}/\mu^{(j)})$ and $\beta = \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(j+1)}/\mu^{(j+1)})$. Therefore,

$$X_{\lambda/\mu} \cong X^{(1)} \boxtimes \mathbf{P}_\alpha \boxtimes \mathbf{P}_\beta \boxtimes X^{(2)}.$$

Combining Lemma 5.1 with the fact that $\boxtimes$ is distributive over $\oplus$, we derive the $H_n(0)$-module isomorphism

$$X_{\lambda/\mu} \cong (X^{(1)} \boxtimes \mathbf{P}_{\alpha\cdot\beta} \boxtimes X^{(2)}) \oplus (X^{(1)} \boxtimes \mathbf{P}_{\alpha\odot\beta} \boxtimes X^{(2)}).$$

This shows $X_{\lambda/\mu}$ is not indecomposable. □

The contraposition of Proposition 7.3(2) says that if M_P is indecomposable, then $\mathrm{sh}(\tau_P)$ does not contain any disconnected ribbon. We ask if the converse is true. In the case where $\mathrm{sh}(\tau_P)$ is connected, it is true by Proposition 7.3(1). In the case where $\mathrm{sh}(\tau_P)$ is disconnected, we verified its validity when $|P| \le 6$. Indeed, this was done by showing that $\mathrm{End}(\mathsf{M}_P)$ has no idempotent except for 0 and id. Refer to the following example.

Example 7.4. Let $\lambda/\mu = (3,3,1)/(1,1)$ and $P = \mathsf{poset}(\tau_0^{\lambda/\mu})$. Then, $\Sigma_L(P) = [21435, 42531]_L$ is a basis for M_P. Let $f \in \mathrm{End}(\mathsf{M}_P)$ be an idempotent and let

$$f(21435) = \sum_{\gamma\in[21435,42531]_L} c_\gamma \gamma \quad (c_\gamma \in \mathbb{C}).$$

Note that

$$\{\gamma \in [21435, 42531]_L \mid \mathrm{Des}_L(21435) \subseteq \mathrm{Des}_L(\gamma)\} = \{21435, 21543, 42531\}.$$

Since f is an $H_5(0)$-module homomorphism, this equality implies that $c_\gamma = 0$ for all $\gamma \in [21435, 42531]_L \setminus \{21435, 21543, 42531\}$. In addition, $c_{21543} = 0$ since

$$\pi_1\pi_2 \cdot 21435 = 0 \quad \text{and} \quad \pi_1\pi_2 \cdot f(21435) = c_{21543}\, 32541.$$

Hence, $f - c_{21435}\,\mathrm{id}$ is an $H_5(0)$-module homomorphism such that

$$(f - c_{21435}\mathrm{id})(\gamma) = \begin{cases} c_{42531} 42531 & \text{if } \gamma = 21435, \\ 0 & \text{if } \gamma \in [21435, 42531]_L \setminus \{21435\}, \end{cases}$$

and therefore, $(f - c_{21435}\mathrm{id})^2 = 0$. Since f is an idempotent, the possible values for c_{21435} are 0 or 1. Using the fact that f is an idempotent again, we have that $c_{42531} = 0$. As a consequence, f is 0 or id.

Based on the above discussion, we propose the following conjecture.

Conjecture 7.5. *Let $P \in \mathsf{RSP}_n$. Suppose that $\mathrm{sh}(\tau_P)$ is disconnected and does not contain any disconnected ribbon. Then, M_P is indecomposable.*

7.3. *Recovering M_P for $P \in \mathsf{RSP}_n$ from an $H_n(q)$-module by specializing q to 0*

Let $q \in \mathbb{C}$. The Hecke algebra $H_n(q)$ is the associative $\mathbb{C}$-algebra with 1 generated by $T_1, T_2, \ldots, T_{n-1}$ subject to the following relations:

$$\begin{aligned} T_i^2 &= (q-1)T_i + q \quad \text{for } 1 \le i \le n-1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for } 1 \le i \le n-2, \\ T_iT_j &= T_jT_i \quad \text{if } |i-j| \ge 2. \end{aligned}$$

Let $q \in \mathbb{C}$ be generic; that is, q is neither zero nor a root of unity. It is well known that $H_n(q)$ is isomorphic to the group algebra $\mathbb{C}[\mathfrak{S}_n]$, and thus, the category of left finite dimensional $H_n(q)$-modules is semisimple and there exists a ring isomorphism ([22, Section 3.2])

$$\mathbf{ch}_q : \bigoplus_{n\geq 0} \mathcal{G}_0(H_n(q)) \to \mathrm{Sym}, \quad [V^\lambda(q)] \mapsto s_\lambda.$$

Here, $\bigoplus_{n\geq 0} \mathcal{G}_0(H_n(q))$ is the Grothendieck ring of the tower of generic Hecke algebras equipped with addition and multiplication from direct sum and induction product, Sym is the ring of symmetric functions, and $V^\lambda(q)$ is the irreducible $H_n(q)$-module attached to a partition λ of size n. The explicit description of $V^\lambda(q)$ can be found in [21, p.7].

Let $P \in \mathsf{RSP}_n$. Viewing q as an indeterminate, one may ask if M_P can be obtained from an $H_n(q)$-module by specializing q to 0. However, it should be noted that the process of 'specializing q to 0' depends on the choice of bases for the $H_n(q)$-module under consideration, as illustrated in the example below.

Example 7.6. The irreducible $H_3(q)$-module $V^{(2,1)}(q)$ has the underlying space $\mathbb{C}\{v_1, v_2\}$, and the $H_3(q)$-action defined by

$$\begin{cases} T_1 \cdot v_1 = -v_1, \\ T_2 \cdot v_1 = v_2, \end{cases} \quad \text{and} \quad \begin{cases} T_1 \cdot v_2 = -q^2 v_1 + q v_2, \\ T_2 \cdot v_2 = q v_1 + (q-1) v_2. \end{cases}$$

By the specialization $q = 0$, we have the $H_3(0)$-action on $\mathbb{C}\{v_1, v_2\}$ defined by

$$\begin{cases} \overline{\pi}_1 \cdot v_1 = -v_1, \\ \overline{\pi}_2 \cdot v_1 = v_2, \end{cases} \quad \text{and} \quad \begin{cases} \overline{\pi}_1 \cdot v_2 = 0, \\ \overline{\pi}_2 \cdot v_2 = -v_2. \end{cases}$$

The resulting module is isomorphic to $\mathbf{T}^+_\theta(\mathsf{M}_{P_1})$, where $P_1 = \mathsf{poset}(\tau_0^{(2,1)}) \in \mathsf{RSP}_3$.

However, if we choose the basis $\{w_1 := q v_1 - v_2, w_2 := (q^2 - q) v_1 - q v_2\}$ for $V^{(2,1)}(q)$, then we have

$$\begin{cases} T_1 \cdot w_1 = w_2, \\ T_2 \cdot w_1 = -w_1, \end{cases} \quad \text{and} \quad \begin{cases} T_1 \cdot w_2 = q w_1 + (q-1) w_2, \\ T_2 \cdot w_2 = -q^2 w_1 + q w_2. \end{cases}$$

By the specialization $q = 0$, we have the $H_3(0)$-action on $\mathbb{C}\{w_1, w_2\}$ defined by

$$\begin{cases} \overline{\pi}_1 \cdot w_1 = w_2, \\ \overline{\pi}_2 \cdot w_1 = -w_1, \end{cases} \quad \text{and} \quad \begin{cases} \overline{\pi}_1 \cdot w_2 = -w_2, \\ \overline{\pi}_2 \cdot w_2 = 0. \end{cases}$$

The resulting module is isomorphic to $\mathbf{T}^+_\theta(\mathsf{M}_{P_2})$, where $P_2 = \mathsf{poset}(\tau_0^{(2,2)/(1)}) \in \mathsf{RSP}_3$. It is worthwhile to remark that while $\mathbf{T}^+_\theta(\mathsf{M}_{P_1})$ and $\mathbf{T}^+_\theta(\mathsf{M}_{P_2})$ have the same quasisymmetric characteristic $\mathbf{ch}_q([V^{(2,1)}(q)])$, they are not isomorphic.

We expect that for $P \in \mathsf{RSP}_n$, $\mathbf{T}^+_\theta(\mathsf{M}_P)$ can be obtained from an $H_n(q)$-module, whose image under $\mathbf{ch}_q$ equals K_P, by applying the specialization $q = 0$ to a suitable basis.

A. A tableau description of M_P for $P \in \mathsf{RSP}_n$

Let $P \in \mathsf{RSP}_n$. Note that $\Sigma_L(P)$ is a basis of M_P consisting of permutations. Here, we construct an $H_n(0)$-module that is isomorphic to M_P and has a tableau basis.

For a skew partition λ/μ of size n, consider the bijection

$$f : \mathrm{SYT}(\lambda/\mu) \to \Sigma_L(\mathsf{poset}(\tau_0^{\lambda/\mu})), \quad T \mapsto \mathsf{read}_{\tau_0}(T).$$

Let $\widetilde{f} : \mathbb{C}\mathrm{SYT}(\lambda/\mu) \to \mathsf{M}_{\mathrm{poset}(\tau_0^{\lambda/\mu})}$ be the $\mathbb{C}$-linear isomorphism obtained by extending f by linearity. We endow $\mathbb{C}\mathrm{SYT}(\lambda/\mu)$ with an $H_n(0)$-module structure by letting

$$h \cdot x := \widetilde{f}^{-1}(h \cdot \widetilde{f}(x)) \quad \text{for } h \in H_n(0) \text{ and } x \in \mathbb{C}\mathrm{SYT}(\lambda/\mu).$$

One can see that for $T \in \mathrm{SYT}(\lambda/\mu)$ and $1 \le i \le n-1$,

$$\pi_i \cdot T = \begin{cases} T & \text{if } i \text{ is strictly left of } i+1 \text{ in } T, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same column of } T, \\ s_i \cdot T & \text{if } i \text{ is strictly right of } i+1 \text{ in } T. \end{cases}$$

Here, $s_i \cdot T$ is the tableau obtained from T by swapping i and $i+1$. We denote the resulting module by $X_{\lambda/\mu}$. By Theorem 5.5, we have

- $\mathsf{M}_P \cong X_{\mathrm{sh}(\tau_P)}$ for $P \in \mathsf{RSP}_n$, and
- $X_{\lambda/\mu} \ncong X_{\nu/\kappa}$ for distinct skew partitions λ/μ, ν/κ of size n.

Therefore, $X_{\mathrm{sh}(\tau_P)}$ can be viewed as a representative of the isomorphism class of M_P in the category $H_n(0)$-**mod**.

Remark A.1. (1) For a composition α, Searles [32] constructed an indecomposable 0-Hecke module X_α whose image under the quasisymmetric characteristic is an extended Schur function. In particular, when α is a partition, our X_α is identical to X_α.

(2) For a generalized composition $\boldsymbol{\alpha}$, let λ/μ be a unique skew partition satisfying the conditions that $\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) = \boldsymbol{\alpha}$ and λ/μ is a ribbon. Then, $X_{\lambda/\mu} \cong \mathbf{P}_{\boldsymbol{\alpha}}$.

The following proposition shows how $X_{\lambda/\mu}$'s behave with respect to induction product, restrictions and (anti-)automorphism twists of ϕ and $\widehat{\theta}$.

Proposition A.2. *We have the following isomorphisms.*

(1) *For skew partitions λ/μ of size n and ν/κ of size m,*

$$X_{\lambda/\mu} \boxtimes X_{\nu/\kappa} \cong X_{\lambda/\mu \star \nu/\kappa} \quad \text{as } H_{n+m}(0)\text{-modules.}$$

(2) *For a skew partition λ/μ of size n and $1 \le k \le n-1$,*

$$X_{\lambda/\mu} \downarrow_{H_k(0)\otimes H_{n-k}(0)} \cong \bigoplus_{\substack{|\nu/\mu|=k \\ \mu\subset\nu\subset\lambda}} X_{\overline{\nu/\mu}} \otimes X_{\overline{\lambda/\nu}} \quad \text{as } H_k(0) \otimes H_{n-k}(0)\text{-modules.}$$

Here, $\overline{\nu/\mu}$ and $\overline{\lambda/\nu}$ denote the basic skew partitions whose Young diagrams are obtained from $\mathtt{yd}(\nu/\mu)$ and $\mathtt{yd}(\lambda/\nu)$, respectively, by removing empty rows and empty columns.

(3) *For a skew partition λ/μ of size n,*

$$\mathbf{T}^+_\phi(X_{\lambda/\mu}) \cong X_{(\lambda/\mu)^\circ} \quad \text{and} \quad \mathbf{T}^-_{\widehat{\theta}}(X_{\lambda/\mu}) \cong X_{\lambda^t/\mu^t}.$$

Here, $(\lambda/\mu)^\circ$ is the skew partition whose Young diagram is obtained by rotating $\mathtt{yd}(\lambda/\mu)$ by $180°$.

Proof. The first assertion follows from [19, Lemma 4], the second from [19, Theorem 2] and the third from [19, Theorem 4]. □

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References

[1] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras* (Cambridge Studies in Advanced Mathematics) vol. 36 (Cambridge University Press, Cambridge, 1995).

[2] J. Bardwell and D. Searles, '0-Hecke modules for Young row-strict quasisymmetric Schur functions', *European J. Combin.* **102** (2022), 103494, 18. https://doi.org/10.1016/j.ejc.2021.103494.

[3] D. J. Benson, *Representations and Cohomology. I* (Cambridge Studies in Advanced Mathematics) vol. 30 (Cambridge University Press, Cambridge, 1991). Basic representation theory of finite groups and associative algebras.

[4] C. Berg, N. Bergeron, F. Saliola, L. Serrano and M. Zabrocki, 'Indecomposable modules for the dual immaculate basis of quasi-symmetric functions', *Proc. Amer. Math. Soc.* **143**(3) (2015), 991–1000. http://doi.org/10.1090/S0002-9939-2014-12298-2

[5] N. Bergeron and H. Li, 'Algebraic structures on Grothendieck groups of a tower of algebras', *J. Algebra* **321**(8) (2009), 2068–2084. http://doi.org/10.1016/j.jalgebra.2008.12.005

[6] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups* (Graduate Texts in Mathematics) vol. 231 (Springer, New York, 2005).

[7] A. Björner and M. L. Wachs, 'Generalized quotients in Coxeter groups', *Trans. Amer. Math. Soc.* **308**(1) (1988), 1–37. http://doi.org/10.2307/2000946.

[8] A. Björner and M. L. Wachs, 'Permutation statistics and linear extensions of posets', *J. Combin. Theory Ser. A* **58**(1) (1991), 85–114. http://doi.org/10.1016/0097-3165(91)90075-R.

[9] S.-I. Choi, Y.-H. Kim and Y.-T. Oh, 'Poset modules of the 0-Hecke algebras and related quasisymmetric power sum expansions', *European J. Combin.* **120** (2024), Paper No. 103965, 34. https://doi.org/10.1016/j.ejc.2024.103965.

[10] B. Deng and G. Yang, 'Representation type of 0-Hecke algebras', *Sci. China Math.* **54**(3) (2011), 411–420. http://doi.org/10.1007/s11425-010-4145-x

[11] G. Duchamp, F. Hivert and J.-Y. Thibon, 'Noncommutative symmetric functions. VI. Free quasi-symmetric functions and related algebras', *Internat. J. Algebra Comput.* **12**(5) (2002), 671–717. http://doi.org/10.1142/S0218196702001139.

[12] G. Duchamp, D. Krob, B. Leclerc and J.-Y. Thibon, 'Fonctions quasi-symétriques, fonctions symétriques non commutatives et algèbres de Hecke à $q = 0$', *C. R. Acad. Sci. Paris Sér. I Math.* **322**(2) (1996), 107–112.

[13] M. Fayers, '0-Hecke algebras of finite Coxeter groups', *J. Pure Appl. Algebra* **199**(1–3) (2005), 27–41. http://doi.org/10.1016/j.jpaa.2004.12.001.

[14] W. Fulton, *Young tableaux* (London Mathematical Society Student Texts) vol. 35 (Cambridge University Press, Cambridge, 1997). With applications to representation theory and geometry.

[15] A. M. Garsia and J. Remmel, 'Shuffles of permutations and the Kronecker product', *Graphs Combin.* **1**(3) (1985), 217–263. http://doi.org/10.1007/BF02582950.

[16] I. M. Gessel, 'Multipartite P-partitions and inner products of skew Schur functions', in *Combinatorics and Algebra (Boulder, Colo., 1983)* (Contemp. Math.) vol. 34 (Amer. Math. Soc., Providence, RI, 1984), 289–317. https://doi.org/10.1090/conm/034/777705.

[17] F. Hivert, J.-C. Novelli and J.-Y. Thibon, 'Yang-Baxter bases of 0-Hecke algebras and representation theory of 0-Ariki–Koike–Shoji algebras', *Adv. Math.* **205**(2) (2006), 504–548. http://doi.org/10.1016/j.aim.2005.07.016.

[18] J. Huang, 'A tableau approach to the representation theory of 0-Hecke algebras', *Ann. Comb.* **20**(4) (2016), 831–868. http://doi.org/10.1007/s00026-016-0338-5.

[19] W.-S. Jung, Y.-H. Kim, S.-Y. Lee and Y.-T. Oh, 'Weak Bruhat interval modules of the 0-Hecke algebra', *Math. Z.* **301**(4) (2022), 3755–3786. http://doi.org/10.1007/s00209-022-03025-4.

[20] Y.-H. Kim and S. Yoo, 'Weak Bruhat interval modules of the 0-Hecke algebra for genomic Schur functions', Preprint, 2022, arXiv:2211.06575 [math.RT].

[21] R. C. King and B. G. Wybourne, 'Representations and traces of the Hecke algebras $H_n(q)$ of type A_{n-1}', *J. Math. Phys.* **33**(1) (1992), 4–14. http://doi.org/10.1063/1.529925

[22] D. Krob and J.-Y. Thibon, 'Noncommutative symmetric functions. IV. Quantum linear groups and Hecke algebras at $q = 0$', *J. Algebraic Combin.* **6**(4) (1997), 339–376. http://doi.org/10.1023/A:1008673127310.

[23] T. Y. Lam, *Lectures on Modules and Rings* (Graduate Texts in Mathematics) vol. 189 (Springer-Verlag, New York, 1999).

[24] A. Lascoux, B. Leclerc and J.-Y. Thibon, 'Flag varieties and the Yang-Baxter equation,' *Lett. Math. Phys.* **40**(1) (1997), 75–90. http://doi.org/10.1023/A:1007307826670.
[25] A. Lascoux and M. P. Schützenberger, 'Symmetrization operators on polynomial rings', *Funct. Anal. Appl.* **21**(4) (1987), 324–326. http://doi.org/10.1007/BF01077811.
[26] K. Luoto, S. Mykytiuk and S. van Willigenburg, *An Introduction to Quasisymmetric Schur Functions* (SpringerBriefs in Mathematics) (Springer, New York, 2013).
[27] C. Malvenuto, '*P*-partitions and the plactic congruence', *Graphs Combin.* **9**(1) (1993), 63–73. http://doi.org/10.1007/BF01195328
[28] P. McNamara, 'Cylindric skew Schur functions', *Adv. Math.* **205**(1) (2006), 275–312. http://doi.org/10.1016/j.aim.2005.07.011.
[29] A. Melnikov, 'On orbital variety closures in sl_n. I. Induced Duflo order', *J. Algebra* **271**(1) (2004), 179–233. http://doi.org/10.1016/j.jalgebra.2003.09.012.
[30] P. Norton, '0-Hecke algebras', *J. Austral. Math. Soc. Ser. A* **27**(3) (1979), 337–357. http://doi.org/10.1017/S1446788700012453.
[31] B. E. Sagan, *The Symmetric Group - Representations, Combinatorial Algorithms, and Symmetric Functions* (Wadsworth & Brooks/Cole Mathematics Series) (Wadsworth, 1991).
[32] D. Searles, 'Indecomposable 0-Hecke modules for extended Schur functions', *Proc. Amer. Math. Soc.* **148**(5) (2020), 1933–1943. http://doi.org/10.1090/proc/14879.
[33] D. Searles, 'Diagram supermodules for 0-Hecke-Clifford algebras', Preprint, 2022, arXiv:2202.12022 [math.RT].
[34] R. Stanley, *Ordered Structures and Partitions* (Memoirs of the American Mathematical Society) no. 119 (American Mathematical Society, Providence, RI, 1972).
[35] R. Stanley, *Enumerative Combinatorics. Vol. 2* (Cambridge Studies in Advanced Mathematics) vol. 62 (Cambridge University Press, Cambridge, 1999).
[36] M. Taskin, 'Properties of four partial orders on standard Young tableaux', ProQuest LLC, Ann Arbor, MI, 2006, PhD dissertation, University of Minnesota.
[37] V. Tewari and S. van Willigenburg, 'Modules of the 0-Hecke algebra and quasisymmetric Schur functions', *Adv. Math.* **285** (2015), 1025–1065. http://doi.org/10.1016/j.aim.2015.08.012.
[38] V. Tewari and S. van Willigenburg, 'Permuted composition tableaux, 0-Hecke algebra and labeled binary trees', *J. Combin. Theory Ser. A* **161** (2019), 420–452. http://doi.org/10.1016/j.jcta.2018.09.003.
[39] S. van Willigenburg, 'Equality of Schur and skew Schur functions', *Ann. Comb.* **9**(3) (2005), 355–362. http://doi.org/10.1007/s00026-005-0263-5.