

GEOMETRIC PROOFS OF SOME CLASSICAL RESULTS ON BOUNDARY VALUES FOR ANALYTIC FUNCTIONS

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ABSTRACT. In this note we are going to give a geometric proof, using the method of the extremal metric, of the following result of Beurling. For any analytic function $f(z)$ in the unit disc Δ of the plane with a bounded Dirichlet integral, the set E on the boundary of the unit disc where the nontangential limits of $f(z)$ do not exist has logarithmic capacity zero. Also, using an unpublished result of Beurling, we will prove different results on boundary values for different classes of functions.

1. Introduction. Our aim in this paper is to present simpler geometric proofs of some known results on boundary values for different classes of analytic functions. In this introduction we shall introduce the classes of functions that we are going to study, and will state the results. Let us start with the definitions.

DEFINITION A. Let $f(z)$ be a regular function in the unit disc. If

$$\iint_{\Delta} |f'(z)|^2 dx dy < \infty,$$

we say that $f(z)$ is in the Dirichlet class D .

DEFINITION B. Let $f(z)$ be a regular non-constant function in the unit disc. Define

$$n(w) = n(w, \Delta, f)$$

to be the number of roots of the equation $f(z) = w$ in Δ and write

$$p(R) = p(R, \Delta, f) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\theta}) d\theta.$$

Then, if there exists a positive number p such that,

$$\frac{1}{\pi} \iint_{\{z: |f(z)| \leq R\}} |f'(z)|^2 dx dy = \int_0^R p(\rho) d\rho^2 \leq pR^2$$

for all positive R , we say that the function $f(z)$ is an areally mean p -valent function.

Let us denote by AMP the class of regular areally mean p -valent functions in the unit disc. It is not difficult to see that neither AMP is a subset of D nor D is a subset of AMP.

In this work we shall use the concepts of nontangential or angular limits and capacity of a set. For their corresponding definitions we refer the reader to [CL].

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We are going to present simpler geometric proofs of the Fatou theorem and the M. and F. Riesz theorem for functions in the class D . The original proofs of these theorems for regular univalent functions appeared in [BEU], where Beurling made the striking observation that for some classes of functions the right tool to measure the sets on the boundary of the unit disc on which radial limits do not exist and the sets of uniqueness is not the Lebesgue measure but the logarithmic capacity.

Tsuji in [TSU], gave a proof of both the Fatou theorem and the M. and F. Riesz theorem for functions in the class D . Carleson [CAR] later showed that for the M. and F. Riesz theorem for functions in the class D some condition on the limiting value α must be imposed if we want the uniqueness set E to have zero logarithmic capacity. That condition had been defined by Beurling, and was that the limiting value α had to be an ordinary value. Namely, let α be a complex number in the plane, we denote by $\Delta_\rho(\alpha) = \{z : |f(z) - \alpha| < \rho\}$, where $f(z)$ is in the class D . Consider the following area integral $\iint_{\Delta_\rho(\alpha)} |f'(z)|^2 dx dy = A_\rho(\alpha)$, then we have the following definition.

DEFINITION C. We say that α is an ordinary value of $f(z)$ if,

$$\limsup_{\rho \rightarrow 0} \frac{A_\rho(\alpha)}{\rho^2} < \infty.$$

Now we pass to state Tsuji's theorems.

THEOREM 1. Suppose that $f(z)$ is in the class D and that $\lim_{z \rightarrow e^{i\theta}} f(z) = \alpha$ nontangentially whenever $e^{i\theta}$ is in E , where E is a subset of $\partial\Delta$. If α is an ordinary value of $f(z)$ and E has positive logarithmic capacity, then $f(z)$ is a constant function.

THEOREM 2. Any function $f(z) \in D$ has nontangential limits for all $e^{i\theta} \in \partial\Delta$ except possibly for a set E in $\partial\Delta$ of zero logarithmic capacity.

Theorem 1 has been improved first by Villamor [VI2], and more recently by Koskela [KOS]. Our proof will rely on an unpublished result of Beurling [MAR], which we shall adapt to our context, and the fact that up to a linear transformation functions in the class D are the same than functions in AMP. We will make the last assertion clearer later.

The fact that for a zero free unbounded AMP function $f(z)$ the harmonic function $u(z) = \log |f(z)|$ is slowly increasing unbounded as defined in [JO] and [VI1], is also essential in our argument for the proof of Theorem 1.

As for Theorem 2 our proof is a simple and straightforward application of the method of the extremal metric. In any case our proofs are simpler than the ones presented in Tsuji's paper and of a more geometric nature, which in our opinion make them more interesting.

Before we get into the proof of our main results, we shall need some preliminary results and definitions.

2. **Preliminaries.** Let $u(z)$ be a harmonic function in a plane domain Ω , consider the level curves of $u(z)$, $l(c) = \{z \in \Omega : u(z) = c\}$ for $-\infty < c < \infty$, and let $\Theta(c) = \int_{l(c)} |*du|$ for $-\infty < c < \infty$, where $*du$ is the differential of the harmonic conjugate function of $u(z)$. With the agreement that $\Theta(c) = 0$ if $l(c) = \emptyset$. We have the following definition.

DEFINITION A. Let $u(z)$ be a harmonic function in a domain Ω . If there exists $a \in u(\Omega)$ such that, $\int_a^b \frac{dc}{\Theta(c)} < \infty$ for every $b > a$, and $\lim_{b \rightarrow \infty} \int_a^b \frac{dc}{\Theta(c)} = \infty$, then we say that the function $u(z)$ is a slowly increasing unbounded harmonic function.

Consider the family of curves $\Gamma(a, b) = \{l(c) : a < c < b\}$ for $-\infty \leq a \leq b < \infty$. We denote its module by the symbol $\mu(a, b)$, then it is known that if $\Gamma(a, b) \neq \emptyset$ and $(a, b) \subset u(\Omega)$ then

$$\mu(a, b) = \int_a^b \frac{dc}{\Theta(c)}.$$

Jenkins and Oikawa [JO] observed that if the function $f(z)$ is zero free, areally mean p -valent and unbounded, then $u(z) = \log |f(z)|$ is a slowly increasing unbounded harmonic function satisfying the following inequality

$$[u(z) - a] \leq 2\pi p \mu[a, u(z)] + \tau,$$

where τ is a universal constant.

For more information on the class of slowly increasing unbounded harmonic functions we refer the reader to the papers [JO] and [VI1].

We pass to state and comment Beurling's unpublished result which we will use in the proofs of the Theorems 1 and 2.

BEURLING'S THEOREM. Suppose f is analytic in a neighborhood of the closed unit disc $\bar{\Delta}$ and suppose that $|f(z)| \leq M$ whenever $|z| \leq r$. Then

$$\text{cap}\{e^{i\theta} : |f(e^{i\theta})| > x\} \leq r^{-1/2} \exp\left\{-\pi \int_M^x \frac{dt}{|f(\gamma_t)|}\right\},$$

where $|f(\gamma_t)| = f_{\gamma_t} |f'(z)| |dz|$, $\gamma_t = \{z : |f(z)| = t\}$.

Marshall in [MAR] used this result, combined with the method of the extremal metric, to prove the following sharp inequality concerning the Dirichlet integral.

MARSHALL'S THEOREM. There is a constant $C < \infty$ such that if f is analytic on Δ , $f(0) = 0$, and $\left(\frac{1}{\pi} \int_{\Delta} |f'(z)|^2 dx dy\right)^{1/2} \leq 1$ then

$$\int_0^{2\pi} e^{|f(e^{i\theta})|^2} d\theta \leq C.$$

3. Intermediate result. In this section we prove a lemma which is the key in the proofs of the main results of the next section. The proof of the lemma uses Beurling’s theorem combined with the theory of slowly increasing unbounded harmonic functions.

LEMMA. *Let f be an unbounded areally mean p -valent function in a neighborhood of the closed unit disc Δ . If M denotes the radius of the largest disc contained in $f(\Delta)$, then*

$$\text{cap}\{e^{i\theta} : |f(e^{i\theta})| > x\} \leq 4 \exp\{-\pi\mu(\log M, \log x)\},$$

where the function u in the definition of $\mu(a, b)$ is $u(z) = \log |f(z)|$.

PROOF OF THE LEMMA. As it can be seen in [VI3], there is no loss of generality to assume that f is zero free. Then let f be a zero free unbounded areally mean p -valent function in Δ , and $M > 0$ be the radius of the largest disc inside $f(\Delta)$. By Beurling’s theorem we have that

$$(2.1) \quad \text{cap}\{e^{i\theta} : |f(e^{i\theta})| > x\} \leq r^{-1/2} \exp\left\{-\pi \int_M^x \frac{dt}{|f(\gamma_t)|}\right\},$$

with r as in the theorem.

Köbe’s $\frac{1}{4}$ -theorem for zero free areally mean p -valent functions asserts that $r^{-1/2} \leq 4$, therefore we have that

$$\text{cap}\{e^{i\theta} : |f(e^{i\theta})| > x\} \leq 4 \exp\left\{-\pi \int_M^x \frac{dt}{|f(\gamma_t)|}\right\}.$$

Let $u(z) = \log |f(z)|$, by [JO], $u(z)$ is a slowly increasing unbounded harmonic function, and therefore

$$|\log |f(z)| - a| \leq 2\pi p\mu(a, \log |f(z)|) + \tau,$$

where,

$$\mu(a, \log |f(z)|) = \int_a^{\log |f(z)|} \frac{dc}{\Theta(c)}.$$

Since,

$$\Theta(c) = \int_{l(c)=\{z:u(z)=c\}} |* du| = \int_{\gamma_{e^c}} \frac{|f'(z)|}{|f(z)|} |dz| = \frac{|f(\gamma_{e^c})|}{e^c},$$

we obtain that

$$\mu(a, \log |f(z)|) = \int_a^{\log |f(z)|} \frac{e^c}{|f(\gamma_{e^c})|} dc.$$

In the last integral we make the change of variable $e^c = s$ to obtain that

$$\mu(a, \log |f(z)|) = \int_{e^a}^{|f(z)|} \frac{ds}{|f(\gamma_s)|}.$$

Choose $a = \log M$, then we have that

$$\mu(\log M, \log |f(z)|) = \int_M^{|f(z)|} \frac{ds}{|f(\gamma_s)|}.$$

In the above equality we replace $|f(z)|$ by x , to obtain that

$$(2.2) \quad \mu(\log M, \log x) = \int_M^x \frac{ds}{|f(\gamma_s)|}.$$

Thus combining (2.1) with (2.2) we obtain

$$\text{cap}\{e^{i\theta} : |f(e^{i\theta})| > x\} \leq 4 \exp\{-\pi\mu(\log M, \log x)\},$$

and this proves the Lemma.

4. Proof of the main results. We shall start with the proof of Theorem 2. We would like to remark that the following proof as well as the proof of Theorem 1 works for functions in the class D_S of meromorphic functions in the unit disc with bounded spherical Dirichlet integral.

PROOF OF THEOREM 2. Let $f \in D$, and consider the set $E = \{e^{i\theta} : \lim_{z \rightarrow e^{i\theta}} f(z)$ non-tangentially does not exist $\}$, by Pflüger’s theorem [PF], $\text{cap}(E) = 0$ if and only if $d_\Delta(E, C_{r_0}) = \infty$. Where $d_\Delta(E, C_{r_0})$ is the extremal length in Δ between E and $C_{r_0} = \{z : |z| = r_0\}$, for some $r_0 < 1$. Therefore it is enough to prove that $d_\Delta(E, C_{r_0}) = \infty$. It is well known that, see [PO], that functions in the class D are normal, therefore by [LV], they satisfy Lindelöf’s theorem. Hence, taking the metric $\rho(z) = |f'(z)|$ for any $z \in \Delta$, we have that for any curve γ in Δ joining E with C_{r_0} ,

$$\int_\gamma \rho(z)|dz| = \int_\gamma |f'(z)| |dz| = \infty,$$

by the definition of the set E . Thus,

$$d_\Delta(E, C_{r_0}) = \sup_\rho \frac{(\inf_\gamma \int_\gamma \rho(z)|dz|)^2}{\int_\Delta \rho^2(z) dz} \geq \frac{(\inf_\gamma \int_\gamma |f'(z)| |dz|)^2}{\int_\Delta |f'(z)|^2 dz},$$

since $\int_\Delta |f'(z)|^2 dz \leq M < \infty$, we have that

$$d_\Delta(E, C_{r_0}) \geq \frac{(\inf_\gamma \int_\gamma |f'(z)| |dz|)^2}{M} \geq \frac{\infty}{M} = \infty,$$

and this proves the theorem.

PROOF OF THEOREM 1. Without loss of generality we can assume that $\alpha = 0$. Since α is an ordinary value by assumption, we can construct, as in [VI3], a zero free areally mean p -valent function $F(z)$ for some large p such that the non-tangential limit $\lim_{z \rightarrow e^{i\theta}} F(z) = \infty$ for any $e^{i\theta} \in E$.

Consider now $F_r(z) = F(rz)$ for $r < 1$. It is clearly an areally mean p -valent function in a neighborhood of the closed unit disc. Therefore by the Lemma in Section 2 we have

that

$$\text{cap}\{e^{i\theta} : |F_r(e^{i\theta})| > x\} \leq 4 \exp\{-\pi\mu(\log M_r, \log x)\},$$

where M_r is the radius of the largest disc contained in $F_r(\Delta)$. It is clear that $\lim_{r \rightarrow 1} M_r = M$, where M is the radius of the largest disc contained in $F(\Delta)$. Since we know that for zero free unbounded areally mean p -valent functions,

$$[u(z) - a] \leq 2\pi p\mu[a, u(z)] + \tau,$$

where $u(z) = \log |F_r(z)|$, and τ is a universal constant. We have that

$$\begin{aligned} \text{cap}\{e^{i\theta} : |F_r(e^{i\theta})| > x\} &\leq 4 \exp\left\{-\pi \frac{1}{2\pi p} [\log x - \log M_r - \tau]\right\} \\ &= 4 \exp\left\{-\frac{1}{2p} \left[\log \frac{x}{\bar{M}_r}\right]\right\} \\ &= 4 \left(\frac{\bar{M}_r}{x}\right)^{1/2p}, \end{aligned}$$

where \bar{M}_r is a constant that depends only on τ and r . Since $\lim_{z \rightarrow e^{i\theta}} F(z) = \infty$ for any $e^{i\theta} \in E$ non-tangentially, we have that $\lim_{r \rightarrow 1} F(re^{i\theta}) = \lim_{r \rightarrow 1} F_r(e^{i\theta}) = \infty$ for any $e^{i\theta} \in E$. By Egoroff's theorem we may assume that E is closed and $\lim_{r \rightarrow 1} F_r(e^{i\theta}) = \infty$ uniformly on E . Thus, for any large value x , there exists $r_0 < 1$, such that for any $r_0 < r < 1$, $|F_r(e^{i\theta})| > x$ for any $e^{i\theta} \in E$. That is, the set $E \subset \{e^{i\theta} : |F_r(e^{i\theta})| > x\}$ for any $r_0 < r < 1$. Hence,

$$\begin{aligned} \text{cap}(E) &\leq \text{cap}\{e^{i\theta} : |F_r(e^{i\theta})| > x\} \leq 4 \exp\left\{-\frac{1}{2p} \left[\log \frac{x}{\bar{M}_r}\right]\right\} \\ &= 4 \left(\frac{\bar{M}_r}{x}\right)^{1/2p}, \end{aligned}$$

for any $r_0 < r < 1$. Therefore, letting $r \rightarrow 1$ we get that

$$\text{cap}(E) \leq 4 \exp\left\{-\frac{1}{2p} \left[\log \frac{x}{M}\right]\right\} = 4 \left(\frac{M}{x}\right)^{1/2p}.$$

Now letting $x \rightarrow \infty$ we obtain that

$$\text{cap}(E) = 0,$$

and the theorem is proved.

To finish this note we shall state an immediate corollary to the Lemma in Section 2. Since the proof is similar to the one in the above theorem we omit it here.

COROLLARY 1. *Let $f(z)$ be an areally mean p -valent function in the unit disc Δ , then*

$$\text{cap}\{e^{i\theta} : \lim_{z \rightarrow e^{i\theta}} f(z) = \infty\} = 0.$$

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