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ON AN EXTENSION OF GENERALIZED INCOMPLETE GAMMA FUNCTIONS WITH APPLICATIONS

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Abstract

In this paper we have introduced extensions $\gamma_\nu(\alpha, x; b)$ and $\Gamma_\nu(\alpha, x; b)$ of the generalized Gamma functions $\gamma(\alpha, x; b)$ and $\Gamma(\alpha, x; b)$ considered recently by Chaudhry and Zubair. These extensions are found useful in the representations of the Laplace and K-transforms of a class of functions. We have also defined a generalization of the inverse Gaussian distribution. The cumulative and the reliability functions of the generalized inverse Gaussian distribution are expressed in terms of these functions. Some useful properties of the functions are also discussed.

1. Introduction

The inverse Gaussian density function

$$g(t) = \left(\nu/2\pi t^3\right)^{1/2} e^{-\nu[(t-\mu)^2/2\mu t]}, \qquad t,\ \mu,\ \nu > 0, \tag{1}$$

arises as the density of the first passage time of the Brownian motion with positive drift (see, [23], [24], [36], [44], [45], [50], [53], [58], [59], [71], [74]). The model has been used in the reliability theory and in the theory of demographic rates (see, [38], [44], [54], [68], [70], [73]). Good [33] proposed the generalized inverse Gaussian distribution

$$h(t) = \frac{1}{I(\alpha; a, b)} t^{\alpha-1} e^{-at-bt^{-1}} \qquad (t > 0,\ a > 0,\ b > 0,\ -\infty < \alpha < \infty), \tag{2}$$

which was used by Sichel ([68], [69]) to construct mixture of Poisson distributions. Barndorff-Nielsen ([10], [11]) used the generalized inverse Gaussian distribution (2) as a mixing distribution to obtain the generalized hyperbolic distribution as a

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mixture of normal distributions. Wise [74] used the model (2) in biomedicine and Marcus [44] used it as a unified stochastic model for the power laws in compartment analysis. Blaesild [14] discussed some probabilistic properties of (2). He computed the moments and cumulants and considered the shape of the density. Barndorff-Nielsen ([10], [11]) and Halgreen [36] showed that the density (2) is infinitely divisible and Halgreen [36] proved that (2) is self-decomposable (see also [40], [41]). Chhikara and Folks ([23], [24]) used the inverse Gaussian distribution (1) as a lifetime model and discussed its statistical applications.

Jorgensen [44] studied the distribution (2) systematically and discussed its applications in different fields like fractures of airconditioning equipment, traffic data, fracture toughness of MIG welds and repair time data (see also, [9], [11], [23], [38], [42], [54], [66]).

A natural generalization of the generalized inverse Gaussian model (2) is

$$f(t) = C(\alpha; a, b) t^{\alpha-1} e^{-at} W_{0,\nu+\frac{1}{2}}(2b/t) \qquad (a > 0,\ b \geq 0,\ t > 0,\ -\infty < \alpha < \infty), \tag{3}$$

where

$$C = C(\alpha; a, b) = \left(\int_0^\infty t^{\alpha-1} e^{-at}\, W_{0,\nu+\frac{1}{2}}(2b/t)\, dt \right)^{-1} \tag{4}$$

is the normalizing constant and $W_{k,\mu}$ is one of the Whittaker functions [47]. It should be noted that for $\nu = 0$ in (3) we get the generalized inverse Gaussian distribution (2). Several classical densities like Weibull, Gamma, Erlang, Exponential, Raleigh, and Chi-square can be derived from (2) by simple transformation of the variable t or by specialization of the parameters α, a and b. Therefore all of these classical densities arising in diverse fields of applications are special cases of the model proposed in (3).

The study of the model (3) will provide a unified approach to the systematical analysis of the probability densities encountered in forestry, reliability theory and in demographic rates (see [19], [20], [54], [73]). The cumulative density function of the density (3) is

$$F(x) = C \int_0^x t^{\alpha-1}\, e^{-at}\, W_{0,\nu+\frac{1}{2}}(2b/t)\, dt, \qquad (x > 0,\ b > 0), \tag{5}$$

and its reliability function is given by

$$R(x) = 1 - F(x). \tag{6}$$

The study of the functions $F(x)$ and $R(x)$ is important in statistics and in reliability theory. In particular, the systematic study of these functions will extend the usefulness of the generalized inverse Gaussian distributions in reliability and lifetesting situations with censored data.

It should be noted that according to ([47], page 279) the Whittaker function $W_{0,\nu}$ can be expressed in terms of the Macdonald function to give

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, W_{0,\nu}(2z); \tag{7}$$

therefore, equations (3), (4) and (5) can be simplified in terms of the Macdonald function K_ν by using (7).

In this paper we introduce a pair of new functions

$$\gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}(b/t)\, dt \tag{8}$$

and

$$\Gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}(b/t)\, dt, \quad (\operatorname{Re} x > 0,\ \operatorname{Re} b > 0,\ -\infty < \alpha < \infty). \tag{9}$$

It follows from (5) – (9) that

$$F(x) = Ca^{-\alpha}\gamma_\nu(\alpha, ax; ab) \tag{10}$$

and

$$R(x) = 1 - Ca^{-\alpha}\Gamma_\nu(\alpha, ax; ab). \tag{11}$$

It should be noted that the closed form solutions to a considerable number of problems in applied mathematics, astrophysics, nuclear physics, statistics and engineering can be expressed in terms of incomplete Gamma functions ([18], [21], [30], [35], [37], [43], [42], [47], [52], [57], [62], [59], [65], [75 – 77])

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt, \qquad (\operatorname{Re} \alpha > 0) \tag{12}$$

and

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt. \tag{13}$$

These functions were investigated for real x by Legendre ([48], [49]). The functional behavior of these functions and the decomposition formula

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha) \tag{14}$$

were studied by Prym ([27], page 152). The older theory of the incomplete Gamma functions (12) – (13) and references to the literature are given by Nielsen ([55], [56]) and Böhmer [15]. Recently, Chaudhry and Zubair [22] have introduced generalized Gamma functions

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \tag{15}$$

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \tag{16}$$

found useful in the analytic study of a considerable number of heat conduction problems in a semi-infinite solid with time dependent boundary conditions ([22], [21], [75 – 77]) and in probability theory ([19], [20]). It should be noted that the substitution $\nu = 0$ in (8) – (9) leads to

$$\gamma_0(\alpha, x; b) = \gamma(\alpha, x; b) \tag{17}$$

and

$$\Gamma_0(\alpha, x; b) = \Gamma(\alpha, x; b). \tag{18}$$

Therefore the functions $\gamma_\nu(\alpha, x; b)$ and $\Gamma_\nu(\alpha, x; b)$ can be regarded as extensions of the generalized Gamma functions (15) – (16) found useful in statistics, applied mathematics and engineering.

In this paper we have studied important properties of these functions such as decomposition formula, special cases, differentiation formula, recurrence relations and Laplace transform representations. It is anticipated that the work presented in this paper will inspire scientists and engineers to find wide applications of these functions in several physical problems. It should be noted that for the most part the expressions used are analytic and hence retain their validity for the complex case because of the principle of analytic continuation. The proofs of some of the identities follow from the simple manipulations of the definitions (8) – (9); we have stated these identities as theorems for completeness.

2. Main Results and Applications

THEOREM 1. (Decomposition theorem)

$$\begin{aligned}&\gamma_\nu(\alpha, x; b) + \Gamma_\nu(\alpha, x; b)\\ &= 2^{\alpha-2}\pi^{-1} b^{1/2} G_{04}^{40}\left(\frac{b^2}{16}\middle| \frac{1}{2}\left(\nu+\frac{1}{2}\right), -\frac{1}{2}\left(\nu+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+\frac{1}{2}\right), \frac{1}{2}\left(\alpha-\frac{1}{2}\right)\right),\\ &\qquad (-\infty < \alpha < \infty,\ \operatorname{Re} b > 0).\end{aligned} \tag{19}$$

PROOF. Substituting $\mu = \frac{1}{2}$ in ([29], page 375(25)) and using the fact ([47], page 112)

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}, \tag{20}$$

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we get

$$\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty x^{\alpha-\frac{3}{2}}e^{-x}K_\nu(b/x)\,dx$$
$$=2^{\alpha-3}G_{04}^{40}\left(\frac{b^2}{16}\middle|\frac{\nu}{2},-\frac{\nu}{2},\frac{1}{2}\left(\alpha+\frac{1}{2}\right),\frac{1}{2}\left(\alpha-\frac{1}{2}\right)\right). \tag{21}$$

Replacing ν by $\nu+\frac{1}{2}$ in (21) and simplifying we get

$$\left(\frac{2b}{\pi}\right)^{1/2}\int_0^\infty x^{\alpha-\frac{3}{2}}e^{-x}K_{\nu+\frac{1}{2}}(b/x)\,dx$$
$$=2^{\alpha-2}\pi^{-1}b^{1/2}G_{04}^{40}\left(\frac{b^2}{16}\middle|\frac{1}{2}\left(\nu+\frac{1}{2}\right),-\frac{1}{2}\left(\nu+\frac{1}{2}\right),\frac{1}{2}\left(\alpha+\frac{1}{2}\right),\frac{1}{2}\left(\alpha-\frac{1}{2}\right)\right). \tag{22}$$

From (8) – (9) and (22) we get the proof.

COROLLARY.

$$\gamma(\alpha,x;b)+\Gamma(\alpha,x;b)=2b^{\alpha/2}K_\alpha(2\sqrt{b}). \tag{23}$$

PROOF. This follows from (18) when we substitute $\nu=0$ and use the fact ([28], page 217(18))

$$G_{04}^{40}\left(z\middle|-\frac{1}{4},\frac{1}{4},\left(\frac{\alpha}{2}-\frac{1}{4}\right),\left(\frac{\alpha}{2}+\frac{1}{4}\right)\right)=4\pi z^{(\alpha-1)/4}K_\alpha\left(4z^{1/4}\right). \tag{24}$$

COROLLARY.

$$\gamma(\alpha,x)+\Gamma(\alpha,x)=\Gamma(\alpha),\qquad(\mathrm{Re}\,\alpha>0). \tag{25}$$

PROOF. This follows from (23) when we let $b\to0^+$ and use the fact ([47], page 136)

$$\lim_{b\to0}b^\alpha K_\alpha(b)=2^{\alpha-1}\Gamma(\alpha),\qquad(\mathrm{Re}\,\alpha>0). \tag{26}$$

REMARK.

$$\Gamma_{-\nu-1}(\alpha,x;b)=\Gamma_\nu(\alpha,x;b). \tag{27}$$

PROOF. This follows from the fact ([47], page 110) that $K_\nu(z)=K_{-\nu}(z)$. In particular, for $\nu=0$ in (27) we get

$$\Gamma_{-1}(\alpha,x;b)=\Gamma_0(\alpha,x;b)=\Gamma(\alpha,x;b). \tag{28}$$

THEOREM 2. (Recurrence relation)

$$\Gamma_\nu(\alpha+1,x;b)$$
$$= (\alpha+\nu)\Gamma_\nu(\alpha,x;b) + b\Gamma_{\nu-1}(\alpha-1,x;b) + \left(\frac{2b}{\pi}\right)^{1/2} e^{-x}x^{\alpha-\frac{1}{2}}K_{\nu+\frac{1}{2}}(b/x),$$
$$(\operatorname{Re} b>0,\ -\infty<\alpha<\infty). \tag{29}$$

PROOF. According to ([34], page 970(8.486)(12)), we have

$$\frac{d}{dt}\left[K_{\nu+\frac{1}{2}}(b/t)\right] = \frac{b}{t^2}K_{\nu-\frac{1}{2}}(b/t) + (\nu+\frac{1}{2})t^{-1}K_{\nu+\frac{1}{2}}(b/t). \tag{30}$$

Differentiating $t^{\alpha-\frac{1}{2}}e^{-t}K_{\nu+\frac{1}{2}}(b/t)$ with respect to t and using (30), we get

$$\frac{d}{dt}\left[t^{\alpha-\frac{1}{2}}e^{-t}K_{\nu+\frac{1}{2}}(b/t)\right]$$
$$= (\alpha+\nu)t^{\alpha-\frac{3}{2}}e^{-t}K_{\nu+\frac{1}{2}}(b/t) + bt^{\alpha-\frac{5}{2}}e^{-t}K_{\nu-\frac{1}{2}}(b/t) - t^{\alpha-\frac{1}{2}}e^{-t}K_{\nu+\frac{1}{2}}(b/t). \tag{31}$$

Multiplying both sides in (31) by $\left(\frac{2b}{\pi}\right)^{1/2}$ and integrating from x to ∞ and using (9), we get

$$0 - \left(\frac{2b}{\pi}\right)^{1/2} x^{\alpha-\frac{1}{2}}e^{-x}K_{\nu+\frac{1}{2}}(b/x)$$
$$= (\alpha+\nu)\Gamma_\nu(\alpha,x;b) + b\Gamma_{\nu-1}(\alpha-1,x;b) - \Gamma_\nu(\alpha+1,x;b). \tag{32}$$

A rearrangement of the terms in (32) yields the proof of the theorem.

COROLLARY. (See [22])

$$\Gamma(\alpha+1,x;b) = \alpha\Gamma(\alpha,x;b) + b\Gamma(\alpha-1,x;b) + x^\alpha e^{-x-bx^{-1}}. \tag{33}$$

PROOF. This follows from (29) when we substitute $\nu=0$ and use equations (20) and (28). In particular substituting $b=0$ in (33) we get the recurrence relation ([34], page 942(8.356))

$$\Gamma(\alpha+1,x) = \alpha\Gamma(\alpha,x) + x^\alpha e^{-x} \tag{34}$$

for the classical incomplete Gamma function.

THEOREM 3. (Laplace transform representations) *Let*

$$H(\tau) = \begin{cases} 1, \tau>0 \\ 0, \tau<0 \end{cases},$$

be the Heaviside unit step function and L be the Laplace transform operator. Then

$$L\left\{t^{\alpha-\frac{3}{2}}K_{\nu+\frac{1}{2}}(b/t)H(t-x);s\right\}=\left(\frac{\pi}{2b}\right)^{1/2}s^{-\alpha}\Gamma_\nu(\alpha,sx;sb) \tag{35}$$

and

$$L\left\{t^{\alpha-\frac{3}{2}}K_{\nu+\frac{1}{2}}(b/t)H(x-t)H(t);s\right\}=\left(\frac{\pi}{2b}\right)^{1/2}s^{-\alpha}\gamma_\nu(\alpha,sx;sb),$$
$$(x\ge 0,\ \mathrm{Re}\, b>0,\ -\infty<\alpha<\infty). \tag{36}$$

PROOF. By definition of the Laplace transformation we have

$$\begin{aligned}&L\left\{t^{\alpha-\frac{3}{2}}K_{\nu+\frac{1}{2}}(b/t)H(t-x);s\right\}\\&=\int_0^\infty t^{\alpha-\frac{3}{2}}K_{\nu+\frac{1}{2}}(b/t)e^{-st}H(t-x)\,dt,\\&=\int_x^\infty t^{\alpha-\frac{3}{2}}K_{\nu+\frac{1}{2}}(b/t)e^{-st}\,dt.\end{aligned} \tag{37}$$

Substituting $t=\tau/s$, $dt=d\tau/s$, we get

$$\int_x^\infty t^{\alpha-\frac{3}{2}}K_{\nu+\frac{1}{2}}(b/t)e^{-st}\,dt=s^{-\alpha+\frac{1}{2}}\int_{sx}^\infty \tau^{\alpha-\frac{3}{2}}e^{-\tau}K_{\nu+\frac{1}{2}}(bs/\tau)\,d\tau. \tag{38}$$

From (9), (37) and (38), we get the proof of (35). The proof of (36) follows similarly.

COROLLARY.

$$\begin{aligned}&L\left\{t^{-1/2}e^{-b/t}H(t-x);s\right\}\\&=\frac{\sqrt{\pi}}{2\sqrt{s}}\left[e^{-2\sqrt{bs}}\,\mathrm{Erfc}\left(\sqrt{sx}-\sqrt{\frac{b}{x}}\right)+e^{2\sqrt{bs}}\,\mathrm{Erfc}\left(\sqrt{sx}+\sqrt{\frac{b}{x}}\right)\right],\end{aligned} \tag{39}$$

$$\begin{aligned}&L\left\{t^{-3/2}e^{-b/t}H(t-x);s\right\}\\&=\frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{-2\sqrt{bs}}\,\mathrm{Erfc}\left(\sqrt{sx}-\sqrt{\frac{b}{x}}\right)-e^{2\sqrt{bs}}\,\mathrm{Erfc}\left(\sqrt{sx}+\sqrt{\frac{b}{x}}\right)\right].\end{aligned} \tag{40}$$

PROOF. These identities follow from (35) when we substitute $\nu=0$, $\alpha=\pm 1/2$ and use the fact [22]

$$\Gamma\left(\frac{1}{2},x;b\right)=\frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{x}-\sqrt{\frac{b}{x}}\right)+e^{2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{x}+\sqrt{\frac{b}{x}}\right)\right] \tag{41}$$

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and

$$\sqrt{b}\Gamma\left(-\frac{1}{2}, x; b\right) = \frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\operatorname{Erfc}\left(\sqrt{x}-\sqrt{\frac{b}{x}}\right) - e^{2\sqrt{b}}\operatorname{Erfc}\left(\sqrt{x}+\sqrt{\frac{b}{x}}\right)\right]. \tag{42}$$

It should be noted that the identities (39) and (40) are sharp and do not seem to be known in the literature. In particular, when we take $b = 0$ in (39) we get (see [28], page 135(15))

$$L\left\{t^{-1/2}H(t-x); s\right\} = \sqrt{\pi}s^{-1/2}\operatorname{Erfc}\left(\sqrt{sx}\right), \qquad x > 0,\ s > 0. \tag{43}$$

THEOREM 4. (Evaluation of $\Gamma_\nu(\alpha, x; b)$ for integral values of ν) *Let* $\Gamma(\alpha, x; b)$ *be the generalized gamma function as defined by (9). Then*

$$\Gamma_n(\alpha, x; b) = \sum_{m=0}^{n}\frac{(2b)^{-m}}{m!}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\Gamma(\alpha+m, x; b) \qquad (n \in \{0, 1, 2, 3, \ldots\}). \tag{44}$$

PROOF. This follows from the representation (9) and from the fact ([27], page 10) that

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{m=0}^{n}\frac{(2z)^{-m}}{m!}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}.$$

REMARK. Since the generalized gamma function $\Gamma\left(\frac{1}{2}+m, x; b\right)$, $m \in \{0, \pm1, \pm2, \pm3, \ldots\}$, can be simplified in terms of error functions [22], it follows from (44) that for $\alpha = r + \frac{1}{2}$, $r \in \{0, \pm1, \pm2, \pm3, \ldots\}$, the function $\Gamma_n\left(r+\frac{1}{2}, x; b\right)$ is expressible in terms of error functions. Moreover, according to (27), $\Gamma_{-n-1}(\alpha, x; b) = \Gamma_n(\alpha, x; b)$, therefore, $\Gamma_n\left(r+\frac{1}{2}, x; b\right)$ should be simplified in terms of error functions for all $r, n \in \{0, \pm1, \pm2, \pm3, \ldots\}$.

THEOREM 5. (Parametric differentiation)

$$\frac{\partial}{\partial b}\left(\Gamma_\nu(\alpha, x; b)\right) = -\frac{1}{b}\left[\nu\Gamma_\nu(\alpha, x; b) + b\Gamma_{\nu-1}(\alpha-1, x; b)\right]. \tag{45}$$

PROOF. Differentiating both sides of (9) with respect to the parameter b we get

$$\begin{aligned}\frac{\partial}{\partial b}\left(\Gamma_\nu(\alpha, x; b)\right) &= \frac{1}{2b}\left(\frac{2b}{\pi}\right)^{1/2}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}K_{\nu+\frac{1}{2}}(b/t)\,dt \\ &\quad + \left(\frac{2b}{\pi}\right)^{1/2}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}\frac{\partial}{\partial b}\left[K_{\nu+\frac{1}{2}}(b/t)\right]dt.\end{aligned} \tag{46}$$

It should be noted that the process of differentiation under the integral sign in (46) is justified ([16], pages 427–448). Using the differentiation formula ([32, page 970(8.486)(12)) for the Macdonald function, we get

$$\begin{aligned}
&\frac{\partial}{\partial b}(\Gamma_\nu(\alpha, x; b)) \\
&= \frac{1}{2b}\left(\frac{2b}{\pi}\right)^{1/2}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}K_{\nu+\frac{1}{2}}(b/t)\,dt - \frac{1}{b}\left(\frac{2b}{\pi}\right)^{1/2} \\
&\quad - \int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}\left\{\left(\nu+\frac{1}{2}\right)K_{\nu+\frac{1}{2}} + \frac{b}{t}K_{\nu-\frac{1}{2}}(b/t)\right\}dt \\
&= \frac{1}{2b}\Gamma_\nu(\alpha, x; b) - \frac{1}{b}\left\{\left(\nu+\frac{1}{2}\right)\Gamma_\nu(\alpha, x; b) + b\Gamma_{\nu-1}(\alpha-1, x; b)\right\}. \qquad (47)
\end{aligned}$$

The simplification in (47) yields the proof of (45). In particular for $\nu = 0$ in (45), we get the differentiation formula [22]

$$\frac{\partial}{\partial b}(\Gamma(\alpha, x; b) = -\Gamma(\alpha - 1, x; b)$$

for the generalized incomplete gamma function.

THEOREM 6. (K-transform Representation) *Let R_ν be the K-transform operator as defined by ([23], page 125)*

$$R_\nu[f(t); b] = \int_0^\infty f(t)K_\nu(bt)(bt)^{1/2}\,dt.$$

Then

$$R_\nu\left[t^{-\alpha-1}e^{-1/t}H(x-t)H(t); b\right] = \left(\frac{\pi}{2}\right)^{1/2}\Gamma_{\nu-\frac{1}{2}}\left(\alpha, \frac{1}{x}; b\right) \qquad (48)$$

and

$$R_\nu\left[t^{-\alpha-1}e^{-1/t}H(t-x); b\right] = \left(\frac{\pi}{2}\right)^{1/2}\gamma_{\nu-\frac{1}{2}}\left(\alpha, \frac{1}{x}; b\right). \qquad (49)$$

PROOF. Replacing x by $1/x$ and ν by $\nu - \frac{1}{2}$ in (9), we get

$$\Gamma_{\nu-\frac{1}{2}}\left(\alpha, \frac{1}{x}; b\right) = \left(\frac{2b}{\pi}\right)^{1/2}\int_{1/x}^\infty \xi^{\alpha-\frac{3}{2}}e^{-\xi}K_\nu(b/\xi)\,d\xi. \qquad (50)$$

Substituting $\xi = 1/t$, $d\xi = -dt/t^2$ in (50), we get

$$\int_{1/x}^\infty \xi^{\alpha-\frac{3}{2}}e^{-\xi}K_\nu(b/\xi)\,d\xi = \int_0^x t^{-\alpha-1}e^{-1/t}K_\nu(bt)t^{1/2}\,dt. \qquad (51)$$

From (50) and (51), we get

$$\Gamma_{\nu-\frac{1}{2}}\left(\alpha, \frac{1}{x}; b\right) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x t^{-\alpha-1} e^{-1/t} \left[K_\nu(bt)(bt)^{1/2}\right] dt$$
$$= \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty t^{-\alpha-1} e^{-1/t} H(x-t) H(t) \left[K_\nu(bt)(bt)^{1/2}\right] dt. \quad (52)$$

From (51) and (52), we get the proof of (48).

The proof of (49) follows similarly. It should be noted that several special cases of (48) – (49) can be listed.

REMARK. The equation (44) gives the relationship between $\Gamma_\nu(\alpha, x; b)$ and $\Gamma(\alpha, x; b)$ for integral values of ν. For non-integral values of ν we have not been able to express $\Gamma_\nu(\alpha, x; b)$ in terms of other tabulated classical functions and this remains an open problem. However, we have the following important result.

THEOREM 7. (Integration with respect to the parameter ν)

$$\Gamma\left(\alpha + \frac{1}{2}, \frac{1}{x}; b\right) = \left(\frac{2}{\pi}\right)^{1/2} b^\alpha \int_0^\infty \gamma_{-\frac{1}{2}+i\nu}(-\alpha, x; b)\, d\nu, \quad (53)$$

$$\gamma\left(\alpha + \frac{1}{2}, \frac{1}{x}; b\right) = \left(\frac{2}{\pi}\right)^{1/2} b^\alpha \int_0^\infty \Gamma_{-\frac{1}{2}+i\nu}(-\alpha, x; b)\, d\nu. \quad (54)$$

PROOF. According to (8) we have

$$\gamma_{-\frac{1}{2}+i\nu}(-\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{-\alpha-\frac{3}{2}} e^{-t} K_{i\nu}(b/t)\, dt. \quad (55)$$

Substituting $t = 1/\tau,\ dt = -d\tau/\tau^2$ we get

$$\gamma_{-\frac{1}{2}+i\nu}(-\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_{1/x}^\infty \tau^{\alpha-\frac{1}{2}} e^{-1/\tau} K_{i\nu}(b\tau)\, d\tau. \quad (56)$$

Integrating both sides in (56) with respect to ν from $\nu = 0$ to $\nu = \infty$ using the fact ([47], page 153(6.5.13)) we get

$$\int_0^\infty \gamma_{-\frac{1}{2}+i\nu}(-\alpha, x; b)\, d\nu = \left(\frac{\pi b}{2}\right)^{1/2} \int_{1/x}^\infty \tau^{\alpha-\frac{1}{2}} e^{-b\tau-\tau^{-1}}\, d\tau$$
$$= \left(\frac{\pi}{2}\right)^{1/2} b^{-\alpha} \Gamma\left(\alpha + \frac{1}{2}, \frac{b}{x}; b\right). \quad (57)$$

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It should be noted that the process of integration under the integral sign in (57) is justified ([16], pages 427–448). Multiplying both sides in (57) by $\left(\frac{2}{\pi}\right)^{1/2} b^{\alpha}$ we get the proof of (53). The proof of (54) follows similarly. In particular substituting $\alpha = 0$ in (53) and using (41) we get

$$\int_0^{\infty} \gamma_{-\frac{1}{2}+i\nu}(0, x; b)\, d\nu = \frac{\pi}{2\sqrt{2}} \left[e^{-2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{x}} - \sqrt{bx} \right) + e^{2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{x}} + \sqrt{bx} \right) \right]. \qquad (58)$$

3. Tabular and Graphical Representations

For numerical and scientific computations, the extension functions $\gamma_{\nu}(\alpha, x; b)$ and $\Gamma_{\nu}(\alpha, x; b)$ can be tabulated by using IMSL FORTRAN subroutines for mathematical applications [39]. The values of the function can be calculated by using the numerical integration subroutine QDAGI. The subroutine uses a globally adaptive scheme in an attempt to reduce the absolute error. It should be noted that QDAGI is an implementation of the subroutine QAGI, which is fully documented by Piessens *et al.* [61]. The modified Bessel functions of the third kind (equations (8) – (9)) are computed by the IMSL subroutine BSKS which is based on the work of Cody [25]. On the other hand, subroutine GAMIC which is based on the computational procedure of Gautschi [32], is used for the incomplete Gamma function. It should be added that $\Gamma(\alpha, x; 0)$, calculated by using the numerical integration QDAGI provides exactly the same results as that of the incomplete Gamma function calculated by the subroutine GAMIC.

The normalized representation of the function $\Gamma_{\nu}(\alpha, x; b)$ for $\nu = 0$ and various values of the parameters α and b is given in [22]. The representation for $\nu \neq 0$ can be found similarly.

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