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Examples of Half-Factorial Domains

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Abstract. In this paper, we determine some sufficient conditions for an $A + XB[X]$ domain to be an HFD. As a consequence we give new examples of HFDs of the type $A + XB[X]$.

Introduction

We first recall various factorization properties for an integral domain. Following P. M. Cohn [13], we say that an integral domain R is *atomic* if each nonzero nonunit of R is a finite product of irreducible elements (atoms) of R. An integral domain R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. It is well known that ACCP implies atomic, but the converse is not true; however examples are hard to come by. The first such example is due to A. Grams [17].

For an atomic domain R, a nonzero nonunit of R may have several factorizations into irreducible elements of R and two factorizations may have different lengths. Thus, following A. Zaks [21], we define R to be a *half-factorial domain* (HFD) if R is atomic and any two factorizations of a nonzero nonunit of R as products of irreducible elements have the same length. Examples of HFDs include UFDs, and more generally any Krull domain R with $|\operatorname{Cl}(R)| \leq 2$ [22, Theorem 1.4]. Moreover, A. Zaks showed that if R is a Krull domain, then $R[X]$ is an HFD if and only if $|\operatorname{Cl}(R)| \leq 2$ [22, Theorem 2.4]. If $R[X]$ is an HFD, then R is an HFD, but the converse does not hold in general (also, see [5, Example 5.4]). As it concerns the Noetherian domain R such that $R[X]$ is an HFD, a characterization has been given recently by J. Coykendall [14, Corollary 2.3]. In order to measure how far an atomic domain R is from being an HFD, we define the *elasticity* of R as $\rho(R) = sup\{\frac{m}{n} \mid x_1 \cdots x_m = y_1 \cdots y_n$, where each $x_i, y_j \in R$ is irreducible$\}$. This concept was introduced by R. Valenza [20]. Thus $1 \leq \rho(R) \leq \infty$, and $\rho(R) = 1$ if and only if R is an HFD.

In this paper, we determine some sufficient conditions for an $A + XB[X]$ domain to be an HFD. As a consequence we give new examples of HFDs of the type $A + XB[X]$.

General references for any undefined terminology or notation are [1], [4], [8], and [15]. For an integral domain R, R^* is the set of nonzero elements of R and $U(R)$ is its group of units.

Main Results

In [5], [10], [12], and [16], integral domains of the type $A + XB[X]$, where $A \subseteq B$ is an extension of integral domains are studied. In particular, some sufficient conditions

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for $A + XB[X]$ to be an HFD are given. In [5, Theorem 5.3], they showed that if B is a field, then $A + XB[X]$ is an HFD if and only if A is a field. In [12, Theorem 3.4], they showed if A is a field and $B[X]$ is a UFD, then $A + XB[X]$ is an HFD. Also, they asked a question in [12, Question, p. 75]: If $A \subset B$ is an extension of integral domains such that $U(A) = U(B)$, each irreducible element of A is irreducible in B, and B is a UFD, is $A + XB[X]$ an HFD? This question was solved positively in [10, Proposition 5.4] and with weaker sufficient conditions in [16, Proposition 1.8]. In [12, Remark 3.7] and [10, Question, end of Section 5], they ask the following question: Is $A = K + XB[X]$ an HFD when K is a field and $B[X]$ is an HFD? Next, we give a positive answer for this question. To do so, we need the following four lemmas.

Lemma 1 *Let R be an integral domain with quotient field K. Let $f, g \in R[X]$. Assume that either the leading coefficient or the coefficient of the term of lowest degree of g is a unit of R and g divides f in $K[X]$. Then g divides f in $R[X]$.*

Proof This result follows by comparing coefficients. ■

The following lemma is well known (see [18, p. 69, Exercise 9.6]).

Lemma 2 *Let R be an integrally closed integral domain with quotient field K and let $f \in R[X]$ be a nonconstant. Assume that either the leading coefficient or the constant term of f is a unit of R. Then f is irreducible in $R[X]$ if and only if f is irreducible in $K[X]$.*

To prove the following lemma, we need the fact that if $B[X]$ is an HFD, then B is integrally closed [14, Theorem 2.2]. However, in [10, Example 5.5 (b)] they gave an example where $B[X]$ is an HFD, but B is not completely integrally closed.

Lemma 3 *Let $R = K + XB[X]$, where $K \subseteq B$, K is a field, and $B[X]$ is an HFD. If $f(X) = X\big(b_1 + Xh_1(X)\big)\cdots\big(b_n + Xh_n(X)\big)$, where for each $i = 1, \ldots, n$, $0 \neq b_i \in B - U(B)$, $h_i(X) \in B[X]$, and $b_i + Xh_i(X)$ is an irreducible element of $B[X]$, then f is an irreducible element of R.*

Proof Suppose that f is not irreducible in R. Then, since K is a field, $f = \big(1 + Xg(X)\big)Xm(X)$, where $g(X)$ and $m(X)$ are nonzero elements of $B[X]$. Thus, among irreducible factors of f in $B[X]$, there is an irreducible factor of type $1 + Xg'(X)$, where $g'(X) \in B[X]$. By Lemma 2, $1 + Xg'(X)$ is irreducible (and so prime) in $L[X]$, where L is the quotient field of B. Since $1 + Xg'(X) \nmid X$, $1 + Xg'(X) \mid b_i + Xh_i(X)$ in $L[X]$ for some $1 \leq i \leq n$. It follows from Lemma 1 that $1 + Xg'(X) \mid b_i + Xh_i(X)$ in $B[X]$. Thus $b_i \in U(B)$ since $b_i + Xh_i(X)$ is irreducible in $B[X]$, a contradiction. Thus f is irreducible in R. ■

Lemma 4 **([12, Lemma 3.3])** *Let $R = A + XB[X]$, where $A \subseteq B$ is an extension of integral domains such that $U(B) \cap A = U(A)$ and let $f \in R$. If f is irreducible in $B[X]$, then it is irreducible in R.*

Now we are ready to answer a question raised in [12, Remark 3.7] and [10, Question, end of Section 5].

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Theorem 5 *Let $R = K + XB[X]$, where $K \subseteq B$, K is a field, and $B[X]$ is an HFD. Then R is an HFD.*

Proof If $0 \neq f \in R$, then $f(X) = X^r\big(b + Xh(X)\big)$, where $r \geq 0$, $0 \neq b \in B$ and $h(X) \in B[X]$.

Case 1 If $r = 0$, then $b \in K$. Since K is a field, $b \in U(R)$ and so $f(X)$ is an associate of an element of R of type $1 + Xh'(X)$, where $h'(X) \in B[X]$. In this case, the factorization of f as a product of irreducible elements in $B[X]$ is also a factorization of f as a product of irreducible elements in R. Indeed, any irreducible factor of f is of type $1 + Xh_i(X)$, and so an irreducible element of R (*cf.* Lemma 4).

Case 2 If $r \neq 0$ and $b \in U(B)$, then $f = (bX)X^{r-1}\big(1 + Xh'(X)\big)$ for some $h'(X) \in B[X]$. Since bX and X are irreducible elements of R, decomposing the factor $1 + Xh'(X)$ in $B[X]$, we get also, in this case, that f is a product of irreducible elements of R.

Case 3 If $r \neq 0$ and $b \in B - U(B)$, then consider the factorization of f into irreducible elements of $B[X]$: $f(X) = X^r\big(b + Xh(X)\big) = uX^r\big(b_1 + Xh_1(X)\big)\cdots\big(b_n + Xh_n(X)\big)$, where $u \in U(B)$, each $b_i \in B$ (at least one b_j is a nonunit) and each $h_i(X) \in B[X]$. Since the factors $b_i + Xh_i(X)$ with $b_i \in U(B)$ are associates (in $B[X]$) of elements of type $1 + Xh_i'(X)$ for some $h_i'(X) \in B[X]$, we get $f(X) = vX^r\big(b_1+Xh_1(X)\big)\cdots\big(b_k+Xh_k(X)\big)\big(1+Xh_1'(X)\big)\cdots\big(1 + Xh_s'(X)\big)$, where $v \in U(B)$, $b_1, \ldots, b_k \in B - U(B)$ and all the factors are irreducible elements in $B[X]$. It follows from Lemma 3 that $X\big(b_1 + Xh_1(X)\big)\cdots\big(b_k + Xh_k(X)\big)$ is an irreducible element of R. Thus f is a product of $r + s$ irreducible elements of R.

Thus the three cases show that R is atomic (*cf.* [12, Proposition 2.1]). To prove that R is an HFD, we have to show that if $f \in R$ has the following factorization into irreducible elements of $B[X]$: $f(X) = uX^r\big(b_1+Xf_1(X)\big)\cdots\big(b_k+Xf_k(X)\big)\big(1+Xg_1(X)\big)\cdots\big(1+Xg_s(X)\big)$, where $u \in U(B)$, each $b_i \in B - U(B)$, and each $f_i(X), g_i(X) \in B[X]$, then any factorization of f into irreducible elements of R has $s + r$ factors.

Indeed, let $f(X) = \big(a_1 + Xh_1(X)\big)\cdots\big(a_n + Xh_n(X)\big)$ be another factorization of f into irreducible elements of R. Note that if $a_i = 0$, then $Xh_i(X)$ is not divisible (in $B[X]$) by any factor $1 + Xg_j(X)$.

Claim Each factor $1 + Xg_i(X)$, where $1 \leq i \leq s$, is an associate (in R) of an element $a_j + Xh_j(X)$ with $a_j \neq 0$.

Proof of claim Since each $1+Xg_i(X)$ is irreducible in $B[X]$, it is irreducible (and so prime) in $L[X]$ by Lemma 2, where L is the quotient field of B. Since $1+Xg_i(X) \nmid Xh_k(X)$, $1+Xg_i(X)$ divides (in $L[X]$) an element $a_j + Xh_j(X)$ with $a_j \neq 0$. Thus $1 + Xg_i(X)$ divides in $B[X]$ the element $a_j + Xh_j(X)$ by Lemma 1. Write $a_j + Xh_j(X) = \big(1 + Xg_i(X)\big)h(X)$ for some $h(X) \in B[X]$. Note that $h(X) \in R$. The fact that $a_j + Xh_j(X)$ is an irreducible element of R forces $h(X) \in U(R)$.

Thus the number of indices such that $a_i \neq 0$ is exactly s. So the factorization of f into irreducible elements of R is: $f(X) = a\big(1 + Xh_1'(X)\big)\cdots\big(1 + Xh_s'(X)\big)\big(Xh_1(X)\big)\cdots\big(Xh_{n-s}(X)\big)$, where $a \in K^*$ and each $h_i'(X), h_j(X) \in B[X]$. Furthermore, since the factors

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$Xh_j(X)$ are irreducible elements of R, the polynomials $h_j(X)$ are not divisible by X (in $B[X]$). Since X^r divides f in $B[X]$ and X^{r+1} does not divide f in $B[X]$, we get that $n - s = r$, and so $n = r + s$, as we desired. ■

Example 6 Let $B = K[Y^2, YZ, Z^2]$, where K is a field. Then B is a two-dimensional Noetherian Krull domain with $|\operatorname{Cl}(B)| = 2$ [1, Example 3.4]. Thus B (and hence $B[X]$) is not a UFD. Note that, for a Krull domain D, $D[X]$ is an HFD if and only if $|\operatorname{Cl}(D)| \leq 2$ [21, Theorem 2.4]. Thus $B[X]$ is an HFD which is not a UFD. Let $R = K + XB[X]$. Then R is an HFD by Theorem 5.

Corollary 7 *Let A be an integral domain with quotient field K and let B be an extension of A such that $K \subseteq B$ and $B[X]$ is an HFD. Then $A + XB[X]$ is an HFD if and only if A is a field.*

Proof This follows from Theorem 5 and the remarks after [5, Theorem 5.3]. ■

Question 8 Let $R = K + XB[X]$, where $K \subseteq B$, and K is a field. Is $B[X]$ an HFD if R is an HFD?

Remark 9 In [12, Example 3.7], they observed that even if A and $B[X]$ are HFDs, the domain $A + XB[X]$ need not be an HFD. For example, let $A = \mathbf{Z}$ and $B = \mathbf{Z}[\sqrt{-5}]$. Then A is a UFD, $B[X]$ is an HFD, but $A + XB[X]$ is not an HFD. While, in [16, Example 3.7 (b)], there is an example such that A and B are HFDs, but not UFDs, and $A + XB[X]$ is also an HFD. For example, take $A = \mathbf{Z}[\sqrt{85}]$ and $B = \mathbf{Z}[\frac{1+\sqrt{85}}{2}]$.

In [6, Definition 2.1], they defined a splitting multiplicatively closed set as follows: Let R be an integral domain. A saturated multiplicatively closed subset S of R is said to be a *splitting set* if for each $0 \neq r \in R$, we can write r as the product $r = sa$ for some $s \in S$ and $a \in R$ with $s'R \cap aR = s'aR$ for all $s' \in S$.

For S any multiplicatively closed subset of R, let $T = \{0 \neq t \in R \mid sR \cap tR = stR$ for all $s \in S\}$. It is easily proved that T is a saturated multiplicatively closed subset of R. Thus S is a splitting set if and only if $ST = R - \{0\}$. Hence if S is a splitting set of R, each nonzero element $r \in R$ may be written in the form $r = st$ for some $s \in S$ and $t \in T$, and this factorization is unique up to unit factors. The set T is called the *complementary multiplicatively closed set for S* (or *m-complement for S*). Note that T is also a splitting set with S for its m-complement. Several conditions equivalent to S being a splitting set are given in [6, Theorem 2.2].

Note that the following theorem generalizes [10, Proposition 5.4], but its proof is essentially the same as in [10, Proposition 5.4]. For completeness we will give a proof.

Theorem 10 *Let $A \subseteq B$ be an extension of integral domains such that $U(A) = U(B)$, A is a UFD, $B[X]$ is an HFD, and each prime element in A is a prime of B. Then $R = A + XB[X]$ is an HFD.*

Proof Let p be a prime of A. By hypothesis, p is prime in B, and hence also in $B[X]$. Note that $pB \cap A = pA$. For if $pb = a \in A$, then some prime factor of a in A must be an associate of p in B, and hence in A since $U(A) = U(B)$. Thus $pR = pB[X] \cap R$, and so p is also prime in R. Hence by [7, Corollary 1.7], $S = A - \{0\}$ is a splitting multiplicative set of R (resp., $B[X]$) generated by principal primes since R satisfies ACCP by [10, Corollary 1.3] (resp., $B[X]$ is an HFD). Thus $R_S = qf(A) + XB_S[X]$ is an HFD by Theorem 5 since $B_S[X] = (B[X])_S$ is an HFD [7, Corollary 2.5], and hence R is an HFD by [7, Theorem 3.3]. ■

Example 11 Let A be a UFD, and let X, Y, Z be indeterminates. Then $B = A[Y^2, YZ, Z^2]$ is a Krull domain with $\mathrm{Cl}(B) \cong \mathbf{Z}/2\mathbf{Z}$, and hence an HFD by [21, Theorem 1.4]. Thus $R = A + XA[Y^2, YZ, Z^2][X]$ is an HFD by Theorem 10. In particular, $\mathbf{Z} + X\mathbf{Z}[Y^2, YZ, Z^2][X]$ is an HFD.

The following definition is due to D. D. Anderson *et al.* in [6, Example 4.8]. Note that the condition (1) of the following definition was used (with the name C_2^*) in [16] to study elasticity of $A + XB[X]$ domains.

Definition 12 Let $A \subseteq B$ be an extension of integral domains. We say that this extension satisfies $(*)$ if for $0 \neq b \in B$ (1) $b = au$, where $a \in A$ and $u \in U(B)$, and (2) $b = au = a'u'$ ($a, a' \in A, u, u' \in U(B)$) implies that $\frac{u}{u'} \in U(A)$.

Note that the extension $A \subseteq B$ satisfies $(*)$ precisely when the map $P_+(A) \to P_+(B)$ given by $Ax \mapsto Bx$ is an isomorphism or, equivalently, when $P(A) \to P(B)$ is an order-isomorphism. Also note that if the extension $A \subseteq B$ satisfies $(*)$, then $U(B) \cap A = U(A)$ and A is an HFD if and only if B is an HFD (*cf.* [16, Proposition 2.7]).

The following extensions of integral domains satisfy $(*)$ [6, Example 4.8].

(1) $A \subseteq A$.

(2) $K \subseteq L$, where K and L are fields. (Note that if $A \subseteq B$ satisfies $(*)$ and A or B is a field, then so is the other.)

(3) $A \subseteq \hat{A}$, where $\hat{A}$ is the completion of A for A a quasi-complete local integral domain (that is, the map $J \mapsto J\hat{A}$ is a lattice isomorphism).

(4) $A \subseteq A(\{Y_\alpha\}) = \{\frac{f}{g} \mid f, g \in A[\{Y_\alpha\}], C(g) = A\}$, where A is a Bézout domain.

Our final result is a special case of [16, Proposition 2.7]. However, the proofs are very different.

Theorem 13 *Let $A \subseteq B$ be an extension of integral domains satisfying $(*)$. Let $R = A + XB[X]$. Then R is an HFD if and only if $B[X]$ is an HFD.*

Proof Suppose that $B[X]$ is an HFD. Then clearly B satisfies ACCP. Since $U(B) \cap A = U(A)$, R satisfies ACCP [12, Proposition 1.2], and hence R is atomic. Now we will show that $\rho(R) = 1$. Let $S = \{uX^n \mid u \in U(A) \text{ for } n = 0 \text{ and } u \in U(B) \text{ for } n \geq 1\}$. Then S is a saturated multiplicatively closed subset of R. In fact, S is a saturation of $\{X^n\}_{n=0}^{\infty}$. Let $T = \{f(X) \in R \mid f(0) \neq 0\}$. Clearly T is a saturated multiplicatively closed set of R. Now by the condition (1) of Definition 12, $ST = R^*$. By the condition (2) of Definition 12, this representation is unique up to a unit factor. Hence S is a splitting multiplicatively closed set with m-complement T. Then $R_S = B[X, X^{-1}] = B[X]_X$ is an HFD [7, Corollary 2.5] since $B[X]$ is an HFD. Note that $R_T = (K + XL[X])_{T'}$ by (1) of Definition 12, where K (resp., L) is quotient field of A (resp., B) and $T' = \{f(X) \in K + XL[X] \mid f(0) \neq 0\}$. Thus R_T is atomic [7, Theorem 2.1] since $K + XL[X]$ is atomic [4, Proposition 3.1] and

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T' is a splitting set. Now we show that R_T is an HFD. Note that $D = K + XL[X]$ is an HFD [5, Theorem 5.3]. Set $S' = \{uX^n \mid u \in K^* \text{ for } n = 0 \text{ and } u \in L^* \text{ for } n \geq 1\}$. Since the pair $K \subseteq L$ satisfies $(*)$, applying the same argument as above, S' is a splitting multiplicatively closed subset of D with m-complement T'. Since $1 = \rho(D) \geq \rho(D_{T'}) \geq 1$ by [11, Theorem 2.3(1)], $\rho(R_T) = \rho(D_{T'}) = 1$. Thus $\rho(R) = \max\{\rho(R_S), \rho(R_T)\} = 1$ by [11, Theorem 2.3 (2)]. Hence R is an HFD. Conversely, suppose that R is an HFD. Let S be as above. Then $R_S = B[X, X^{-1}] = B[X]_X$. Thus R_S is atomic by [7, Theorem 2.1] and so $B[X]$ is atomic by [7, Theorem 3.1]. Now we have $1 = \rho(R) \geq \rho(R_S) = \rho(B[X]_X) \geq 1$ by [11, Theorem 2.3 (1)]. Thus $\rho(B[X]_X) = 1$, and hence $B[X]_X$ is an HFD since it is atomic [7, Theorem 2.1]. Thus by [7, Theorem 3.3] $B[X]$ is an HFD. ■

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