TAKING REINHARDT'S POWER AWAY

RICHARD MATTHEWS

Abstract. We study the notion of non-trivial elementary embeddings $j\colon V \to V$ under the assumption that V satisfies ZFC without Power Set but with the Collection Scheme. We show that no such embedding can exist under the additional assumption that it is cofinal and either $V_{\mathrm{crit}(j)}$ is a set or that the scheme of Dependent Choices of arbitrary length holds. We then study failures of instances of Collection in symmetric submodels of class forcings.

§1. Introduction. A vital tool in many set-theoretic arguments is the assumption that various large cardinal notions are consistent. That is, it is possible to have a cardinal which exhibits certain additional properties that cannot provably exist arguing from ZFC alone. Often these properties can be expressed using the first ordinal moved, or the *critical point*, of an elementary embedding

$$j\colon V \to M,$$

where M is some transitive class, with the general principle being that the closer M is to V, the stronger the resulting large cardinal assumption. For example, the critical point is said to be λ-strong if $V_\lambda \subseteq M$ or λ-supercompact if M is closed under arbitrary sequences of length λ.

There is a natural limit to these large cardinals, originally proposed by Reinhardt [21], which is for κ to be the critical point of a non-trivial elementary embedding from V to itself. However, as shown by Kunen [17] there is no such non-trivial embedding when V is a model of ZFC along with a predicate for j such that V satisfies all instances of Replacement and Separation in the language expanded to include this predicate. In fact, Kunen's proof shows that there is no non-trivial elementary embedding

$$j\colon V_{\lambda+2} \to V_{\lambda+2}$$

for any ordinal λ under the assumption that V satisfies ZFC.

Since the announcement of this result it has been a long standing and much studied question as to whether or not the Axiom of Choice is necessary for this result. Namely, if it is consistent for there to be a non-trivial elementary embedding

$j \colon V \to V$ under the assumption that V is a model of ZF. In this paper we take a different approach to generalising Kunen's inconsistency which is to study such embeddings in the theory ZFC *without Power Set*. The motivation for this is the following result which shows that such embeddings are consistent under the assumption that I_1 is consistent, which is an axiom just short of the Kunen inconsistency. In particular, I_1 will give a non-trivial embedding from H_{λ^+} to itself, which is one of the standard structures which models ZFC without Power Set.

THEOREM 2.2. *There exists a non-trivial elementary embedding* $k \colon V_{\lambda+1} \to V_{\lambda+1}$ *if and only if there exists a non-trivial elementary embedding* $j \colon H_{\lambda^+} \to H_{\lambda^+}$.

It is a well-known result that without Power Set many of the usual equivalent ways to formulate the axioms of ZFC break down. In particular, the Replacement Scheme no longer implies the stronger Collection Scheme and the Axiom of Choice does not imply that every set can be well-ordered. Without the Collection Scheme many of the basic facts that one assumes no longer hold, for example we can consistently have that ω_1 exists and is singular or that the Łoś ultrapower theorem can fail. One can find these and other similar results in [7]. So, using the notation found in that paper, we shall define *the theory* ZFC *without Power Set* as follows:

DEFINITION 1.1. Let ZF^- denote the theory consisting of the following axioms: Empty set, Extensionality, Pairing, Unions, Infinity, the Foundation Scheme, the Separation Scheme and the Replacement Scheme.
ZF^- denotes the theory ZF^- plus the Collection Scheme.
ZFC^- denotes the theory ZF^- plus the Well-Ordering Principle.
ZFC^-_{Ref} denotes the theory ZFC^- plus the Reflection Principle.
We will also use the corresponding notation for their second order versions GB^- and GB^-.

The main result of this paper is that, under mild assumptions, Kunen's inconsistency does still hold in the theory ZFC^-_j (the theory ZFC^- with a predicate for j) and therefore that, while H_{λ^+} has more structure than $V_{\lambda+1}$, the embedding given by I_1 cannot have one of the most useful properties an embedding can have, cofinality.

THEOREM 5.4. *There is no non-trivial, cofinal, Σ_0-elementary embedding* $j \colon V \to V$ *such that* $V \models ZFC^-_j$ *and* $V_{crit(j)} \in V$.

The proof of the above theorem makes essential use of the fact that the initial segment of the universe up to the critical point of j, $V_{crit(j)}$ is a set. However, by strengthening the underlying theory to also satisfy $DC_{<ORD}$, which is the DC_μ-Scheme for every cardinal μ, this can be removed.

COROLLARY 6.10. *There is no non-trivial, cofinal, Σ_0-elementary embedding* $j \colon V \to V$ *such that* $V \models (ZFC^- + DC_{<ORD})_j$.

The second half of the paper deals with a curious property of the theory $ZFC^- + DC_{<ORD}$ which is needed for the above result. This is the property that if V is a model of $ZFC^- + DC_{<ORD}$ and $\mathcal{C}$ is a definable proper class over V then for any non-zero ordinal γ there is a definable surjection of $\mathcal{C}$ onto γ. A reasonable suggestion for a

counter-example to this property without Choice is to construct a model of ZF^- with an amorphous class, where

DEFINITION 7.3. A class A is said to be *amorphous* if it is infinite but not the disjoint union of two infinite subclasses. Namely, if $A = B \cup C$ and $B \cap C = \emptyset$ then either B or C must be a finite set.

Note that such a class could not surject onto ω otherwise it could be partitioned into the class mapped to even numbers and the class mapped to odd numbers.

However, we show that, without Choice, many structures which would otherwise satisfy Collection only satisfy the Replacement Scheme. This is done by proving the following result which shows that the existence of an amorphous proper class implies the failure of Collection and thus a failure of our hoped-for counter-example. Moreover, using this we can show that a symmetric submodel of a pretame class forcing need not model the Collection Scheme.

THEOREM 7.13. *Suppose that* $\langle M, A \rangle$ *satisfies*;

1. $M \models (ZF^-)_A$,
2. $A \subseteq M$ *and* $\langle M, A \rangle \models$ "*A is a proper class*", *and*
3. $\langle M, A \rangle \models$ "*if* $B \subseteq A$ *is infinite then* B *is a proper class*".

Then the Collection Scheme fails in $\langle M, A \rangle$. *Moreover,* $\langle M, A \rangle$ *does not have a cumulative hierarchy and therefore Power Set also fails.*

§2. Embeddings of H_{λ^+}.

DEFINITION 2.1. I_1 is the assertion that for some ordinal λ there exists a non-trivial elementary embedding $k \colon V_{\lambda+1} \to V_{\lambda+1}$.

I_1 is considered one of the strongest large cardinal axioms that is not known to be inconsistent. The following result is adapted from the folklore result which gives an alternate characterisation of 1-extendible cardinals, a proof of which can be found in [1]. This theorem shows an equivalent way of considering I_1 embeddings as embeddings of H_{λ^+}, a set with much more structure than $V_{\lambda+1}$.

For convenience, we note here that the Kunen inconsistency can be proved in $H_{\lambda^{++}}$, so there is no non-trivial elementary embedding from $H_{\lambda^{++}}$ to itself. This can be seen by carefully analysing the second proof of Theorem 23.12 in [13] or equivalently by using Theorem 5.2.

THEOREM 2.2. *There exists a non-trivial elementary embedding* $k \colon V_{\lambda+1} \to V_{\lambda+1}$ *if and only if there exists a non-trivial elementary embedding* $j \colon H_{\lambda^+} \to H_{\lambda^+}$.

PROOF. ($\Leftarrow$) : By the Kunen inconsistency, λ must be the supremum of the critical sequence $\langle \kappa_n \colon n \in \omega \rangle$ of j, where κ_0 is the critical point and $\kappa_{n+1} = j(\kappa_n)$. Then each κ_n is an inaccessible cardinal and thus $2^{<\lambda} = \lambda = \beth_\lambda$. Therefore $V_\lambda = H_\lambda$ and $|V_\lambda| = \lambda$ so $V_\lambda \in H_{\lambda^+}$. This means that $V_{\lambda+1} = \{x \in H_{\lambda^+} \colon x \subseteq V_\lambda\}$, so $V_{\lambda+1}$ is a definable class in H_{λ^+}. Moreover, working in H_{λ^+}, any formula φ can be relativised to $V_{\lambda+1}$ so $j \upharpoonright V_{\lambda+1} \colon V_{\lambda+1} \to V_{\lambda+1}$ is elementary.

($\Rightarrow$) : We begin by defining a standard way to code elements of H_{λ^+} by elements of $V_{\lambda+1}$. This will be done by coding $\mathrm{trcl}(\{x\})$ by some subset of $\lambda \times \lambda$ whose

Mostowski collapse is again $\mathrm{trcl}(\{x\})$. However, since we will be working with $\mathrm{trcl}(\{x\})$ rather than x itself, it is necessary to do a simple, preliminary coding.

So, let $\hat{\mathrm{H}} := \{\mathrm{trcl}(\{x\}) \colon x \in \mathrm{H}_{\lambda^+}\}$. For $\mathrm{trcl}(\{x\}), \mathrm{trcl}(\{y\}) \in \hat{\mathrm{H}}$ define the relation $\hat{\in}$ by $\mathrm{trcl}(\{x\}) \hat{\in} \mathrm{trcl}(\{y\})$ if and only if $x \in y$ and similarly for $\hat{=}$. It is then clear that any first order statement φ about H_{λ^+} is equivalent to a formula $\hat{\varphi}$ over $\hat{\mathrm{H}}$ by the obvious coding.

Now note that $\mathrm{rank}(\lambda \times \lambda) = \lambda$ and so any subset of $\lambda \times \lambda$ has rank at most λ. So, for any $x \in \mathrm{H}_{\lambda^+}$ and bijection $f \colon |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$ let

$$C_{x,f} := \{\langle \alpha, \beta \rangle \in \lambda \times \lambda \colon f(\alpha) \in f(\beta)\} \in \mathrm{V}_{\lambda+1}.$$

Then the Mostowski collapse of $C_{x,f}$, $\mathrm{coll}(C_{x,f})$, is $\mathrm{trcl}(\{x\})$. Let $\tilde{H}$ denote the definable class in $\mathrm{V}_{\lambda+1}$ of all subsets of $\lambda \times \lambda$ which code an element of $\hat{\mathrm{H}}$ in this way. That is $X \in \tilde{H}$ iff X is a well-founded, extensional, binary relation on λ with a single maximal element and $\mathrm{dom}(X) \cup \mathrm{ran}(X)$ is a cardinal which is at most λ.

For $Z \in \tilde{H}$, let $\mathrm{fld}(Z)$ be $\mathrm{dom}(Z) \cup \mathrm{ran}(Z)$ and define $\max(Z)$ to be the unique element of $\mathrm{fld}(Z)$ which is maximal with respect to the relation on Z. Now, for $X, Y \in \tilde{H}$ define relations $\tilde{=}$ and $\tilde{\in}$ by:

$$X \tilde{=} Y \iff \exists g \colon \lambda \to \lambda \left(g \text{ is a bijection} \wedge \forall \alpha, \beta \in \lambda \right.$$

$$\left. \left(\langle \alpha, \beta \rangle \in X \leftrightarrow \langle g(\alpha), g(\beta) \rangle \in Y \right) \right),$$

$$X \tilde{\in} Y \iff \exists g \colon \lambda \to \lambda \left(g \text{ is injective} \wedge \langle g(\max(X)), \max(Y) \rangle \in Y \right.$$

$$\wedge \, \forall \alpha, \beta \in \mathrm{fld}(X) \left(\langle \alpha, \beta \rangle \in X \leftrightarrow \langle g(\alpha), g(\beta) \rangle \in Y \right)$$

$$\wedge \, \forall \gamma \in \mathrm{fld}(Y) \, \forall \delta \in \mathrm{fld}(X)$$

$$\left. \left(\langle \gamma, g(\delta) \rangle \in Y \to \exists \sigma \in \mathrm{fld}(X) \, \sigma = g(\gamma) \right) \right).[1]$$

Then $\tilde{=}$ and $\tilde{\in}$ are definable in $\mathrm{V}_{\lambda+1}$, $X \tilde{=} Y \Leftrightarrow \mathrm{coll}(X) = \mathrm{coll}(Y)$ and $X \tilde{\in} Y \Leftrightarrow \mathrm{coll}(X) \in \mathrm{coll}(Y)$. Now we have that any first order statement $\hat{\varphi}$ about $\hat{\mathrm{H}}$ is equivalent to a formula $\tilde{\varphi}$ over $\mathrm{V}_{\lambda+1}$ which is defined by the following coding:

- Replace any parameter $\mathrm{trcl}(\{x\})$ occurring in $\hat{\varphi}$ with $C_{x,f}$ for some (any) bijection $f \colon |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$.
- Replace any instance of $\hat{=}$ with $\tilde{=}$ and $\hat{\in}$ with $\tilde{\in}$.
- Replace any unbounded quantification by the same quantifier taken over $\tilde{H}$.

Then, by the elementarity of k,

$$X \tilde{=} Y \iff k(X) \tilde{=} k(Y)$$
$$\text{and}$$
$$X \tilde{\in} Y \iff k(X) \tilde{\in} k(Y).$$

---

[1] That is, g "$\mathrm{fld}(X)$ is an initial segment of $\mathrm{fld}(Y)$ with respect to the relation on Y.

Also, since $\tilde{H}$ is a definable class in $V_{\lambda+1}$ the restriction of the embedding $k \restriction \tilde{H} \colon \tilde{H} \to \tilde{H}$ is still elementary.

So we can define $j \colon H_{\lambda^+} \to H_{\lambda^+}$ by setting $j(x)$ to be the unique element of $\mathrm{coll}(k(C_{x,f}))$ of maximal rank for some bijection $f \colon |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$. Moreover j is elementary since

$$H_{\lambda^+} \models \varphi(x_1, \ldots, x_n) \iff \hat{H} \models \hat{\varphi}(\mathrm{trcl}(\{x_1\}), \ldots, \mathrm{trcl}(\{x_n\}))$$
$$\iff \tilde{H} \models \tilde{\varphi}(C_{x_1, f_1}, \ldots, C_{x_n, f_n})$$
$$\iff \tilde{H} \models \tilde{\varphi}(k(C_{x_1, f_1}), \ldots, k(C_{x_n, f_n}))$$
$$\iff \hat{H} \models \hat{\varphi}(\mathrm{coll}(k(C_{x_1, f_1})), \ldots, \mathrm{coll}(k(C_{x_n, f_n})))$$
$$\iff H_{\lambda^+} \models \varphi(j(x_1), \ldots, j(x_n)). \qquad \dashv$$

The above theorem shows that the existence of a non-trivial elementary embedding from V to itself under ZFC^- is weaker than I_1, however it does not show that the embedding one obtains has any useful structure. What we shall show is that this embedding must fail one of the most useful fundamental characteristics, that of *cofinality* where

DEFINITION 2.3. An embedding $j \colon M \to N$ is said to be cofinal if for every $y \in N$ there is an $x \in M$ such that $y \in j(x)$.

REMARK 2.4. If M satisfies ZF and $N \subseteq M$ then the cumulative hierarchy for M witnesses that any elementary embedding is cofinal.

§3. Definable embeddings. We begin this section with a standard fact about nontrivial elementary embeddings which is that they must move an ordinal. The only notable thing about the statement is that the proof only requires elementarity for bounded formulae.

PROPOSITION 3.1. *Suppose that* $N \subseteq M$ *are transitive class models of* ZF^- *and* $j \colon M \to N$ *is a non-trivial,* Σ_0-*elementary embedding. Then there exists an ordinal* α *such that* $j(\alpha) > \alpha$.

PROOF. First note that Σ_0-elementarity implies Δ_1-elementarity. Now, since being an ordinal is Σ_0 definable, if α is an ordinal then so is $j(\alpha)$. Next, since $\emptyset$ is definable as the unique set z such that $\forall y \in z \ (y \neq y)$, which is a Σ_0 formula, $j(\emptyset) = \emptyset$. So, by induction, we have that for every ordinal α, $j(\alpha) \geq \alpha$. Now let x be a set of least rank such that $j(x) \neq x$ and let $\delta = \mathrm{rank}(x)$. Then for all $y \in x$, $y = j(y) \in j(x)$ so $x \subseteq j(x)$. Thus there must be some $z \in j(x) \setminus x$. Now suppose that $\mathrm{rank}(j(x)) = \delta$, then we must have that $j(z) = z \in j(x)$ so, by elementarity, $z \in x$ which yields a contradiction. Hence, since the following is Δ_1 definable, we must have that $j(\delta) = \mathrm{rank}(j(x)) > \delta$. $\qquad \dashv$

Therefore, given a non-trivial, Σ_0-elementary $j \colon M \to N$, we will define the critical point of j to be the least ordinal moved and denote it by $\mathrm{crit}(j)$.

It is not a priori obvious that being an elementary embedding should be definable by a single sentence. However, as proven by Gaifman in [5], if M is a model of

a sufficient fragment[2] of ZFC then it suffices to check that a cofinal embedding is elementary for Σ_0 sentences. The version below for the case where $M \models ZF^-$ appears in [7].

THEOREM 3.2 (Gaifman). *Suppose that M is a model of ZF^- and $j \colon M \to N$ is a cofinal, Σ_0-elementary embedding. Then j is fully elementary.*

REMARK 3.3. This theorem does not require any assumptions on N. Moreover, the models M and N need not be transitive.

Using the fact that being Σ_0-elementary is definable by a single formula we obtain a version of Suzuki's theorem on the non-definability of embeddings [22], in the context of ZF^-.

THEOREM 3.4 (Suzuki). *Assume that $V \models ZF^-$. Then there is no non-trivial, cofinal, elementary embedding $j \colon V \to V$ which is definable from parameters.*

PROOF. Formally, this is a theorem scheme asserting that for each formula φ there is no parameter p for which $\varphi(\cdot, \cdot, p)$ defines a non-trivial, cofinal, elementary embedding $j \colon V \to V$. Using Theorem 3.2, it suffices to show that for no parameter p are we able to define a non-trivial, cofinal, Σ_0-elementary embedding $j \colon V \to V$ by

$$j(x) = y \iff \varphi(x, y, p) \text{ holds.}$$

So, seeking a contradiction, let $\sigma(p)$ be the sentence asserting that $\varphi(\cdot, \cdot, p)$ defines a Σ_0-elementary embedding and let $\psi(p)$ asserts that $\varphi(\cdot, \cdot, p)$ defines a total function which is non-trivial, cofinal and Σ_0-elementary. That is,

$$\psi(p) \equiv \forall u \exists! v \varphi(u, v, p) \wedge \exists t \neg \varphi(t, t, p) \wedge \forall y \exists x, z (\varphi(x, z, p) \wedge y \in z) \wedge \sigma(p).$$

Let $\vartheta(p, \kappa)$ assert that κ is the critical point of j. So,

$$\vartheta(p, \kappa) \equiv \kappa \in \text{ORD} \wedge \forall \alpha \in \kappa \, \varphi(\alpha, \alpha, p) \wedge \neg \varphi(\kappa, \kappa, p).$$

Then, by Proposition 3.1,

$$V \models \psi(p) \to \exists! \kappa \, \vartheta(p, \kappa).$$

So denote by crit_p the (unique) κ for which $\vartheta(p, \kappa)$ holds. Now fix p such that crit_p is as small as possible, that is such that

$$V \models \psi(p) \wedge \forall w (\psi(w) \to \text{crit}_p \leq \text{crit}_w).$$

Then, by elementarity,

$$V \models \exists s \big(\varphi(p, s, p) \wedge \psi(s) \wedge \forall w (\psi(w) \to \text{crit}_s \leq \text{crit}_w) \big).$$

But, $V \models \text{crit}_p < \text{crit}_s$ because the critical point of the embedding defined by $\varphi(\cdot, \cdot, s)$ must be $j(\text{crit}_p)$, yielding a contradiction.   ⊣

---

[2]Gaifman's original proof is done under the assumption that M is a model of Zermelo, that is ZF with Separation but not Replacement, plus the existence of a cumulative rank hierarchy of sets, namely $V = \bigcup_{\alpha \in \text{ORD}} V_\alpha$. He then comments that the assumption of Power Set can be replaced by the Collection Scheme plus the existence of Cartesian products.

We remark here that the main elements of the proof that there is no definable embedding of the universe into itself are that any such embedding has a critical point and being fully elementary can be expressed by a single sentence. Therefore, by using Gaifman's original theorem, since the Collection Scheme wasn't used in the proofs of Proposition 3.1 and Theorem 3.4, the above proof also shows that there is no non-trivial cofinal elementary embedding of a model of Zermelo plus a cumulative rank hierarchy of sets into itself which is definable from parameters. There are two obvious questions which appear here about whether or not the assumptions of cofinality and Collection were necessary in the proof that there is no definable embedding. That is;

QUESTION 3.5. *Are either of the following two statements consistent*:

1. *There exists a non-trivial elementary embedding $j : V \to V$ which is definable from parameters where $V \models ZF^-$?*
2. *There exists a non-trivial, cofinal elementary embedding $j : V \to V$ which is definable from parameters where $V \models ZF^-$?*

It has been proven in [7] that one can have cofinal, Σ_1-elementary embeddings of ZF^- which are not Σ_2-elementary, which is to say that Gaifman's Theorem can fail without the Collection Scheme. Therefore proving Suzuki's Theorem in either of these contexts would involve a different approach.

§4. Choosing from classes. In this short section we mention how one can apply Choice to set-length sequences of classes using the Collection Scheme. The standard way to do this in full ZFC is by using Scott's trick to replace each class by the set of elements of least rank of that class. However, if V_α is not a set for each α then this may not be possible so we have to be slightly more careful in our approach.

Let μ be an ordinal and suppose that we have a sequence of non-empty classes $\langle \mathcal{C}_\alpha : \alpha \in \mu \rangle$ which are uniformly defined. This allows us to fix a formula $\varphi(v_0, v_1)$ saying that $v_1 \in \mathcal{C}_{v_0}$. Then, for each $\alpha \in \mu$ there is some set x such that $\varphi(\alpha, x)$. So, by Collection, there is some set b such that for each $\alpha \in \mu$ there is some $x \in b$ such that $\varphi(\alpha, x)$. By well-ordering b, there is some cardinal τ and bijection $h : \tau \leftrightarrow b$. So for each $\alpha \in \mu$ we can define a choice function by taking $x_\alpha \in \mathcal{C}_\alpha$ to be $h(\gamma)$ for the least ordinal $\gamma \in \tau$ such that $\varphi(\alpha, h(\gamma))$.

For example, suppose that $S \subseteq \mu$ were a stationary set which was partitioned into $\tau < \mathrm{cf}(\mu)$ many sets $\langle S_\alpha : \alpha \in \tau \rangle$ and one wanted to show that for some $\alpha \in \tau$, S_α was stationary. Arguing for a contradiction, suppose that none of the S_α were stationary and for each $\alpha \in \tau$ define $\mathcal{C}_\alpha$ to be the non-empty class of clubs $D \subseteq \mu$ for which $D \cap S_\alpha \neq \emptyset$. By the above argument, we can choose a sequence of clubs $\langle D_\alpha : \alpha \in \tau \rangle$ such that for each α, $D_\alpha \in \mathcal{C}_\alpha$. Then $\bigcap_{\alpha \in \tau} D_\alpha \cap S = \emptyset$ yielding the required contradiction.

Using this idea we are able to prove many useful classical results without much change from their standard proofs. For completeness, we give here two such ZFC^- results which we will then use in our proof of the Kunen inconsistency.

THEOREM 4.1 (Fodor). *Let μ be a regular cardinal, $S \subseteq \mu$ stationary and f a regressive function on S.[3] Then there exists some stationary set $T \subseteq S$ and $\gamma \in \mu$ such that for all $\alpha \in T$, $f(\alpha) = \gamma$.*

PROOF. Assume for a contradiction that for each $\gamma \in \mu$ the set $\{\alpha \in S : f(\alpha) = \gamma\}$ was non-stationary. Using the above comments, for each $\gamma \in \mu$ choose a club D_γ such that for each α in $D_\gamma \cap S$, $f(\alpha) \neq \gamma$. Let

$$D = \Delta_{\gamma \in \mu} D_\gamma := \{\alpha : \forall \beta \in \alpha \ (\alpha \in D_\beta)\},$$

and note that this is club in μ. Therefore $S \cap D$ is stationary, so in particular non-empty, and for any $\alpha \in S \cap D$ and $\gamma \in \alpha$, $f(\alpha) \neq \gamma$. So $f(\alpha) \geq \alpha$, contradicting the assumption that f was regressive. ⊣

DEFINITION 4.2. For cardinals $\delta < \mu$ let $S_\delta^\mu = \{\alpha < \mu : \operatorname{cf}(\alpha) = \delta\}$.

THEOREM 4.3 (Solovay). *Suppose that μ is an uncountable, regular cardinal and $S \subseteq S_\omega^\mu$ is stationary. Then there is a partition of S into μ many disjoint stationary sets.*

PROOF. First note that for each $\alpha \in S$ there is some increasing sequence of ordinals $\langle t_n : n \in \omega \rangle$ cofinal in α. Therefore by the comments at the beginning of this section, for each $\alpha \in S$ choose an increasing sequence $\langle a_n^\alpha : n \in \omega \rangle$ cofinal in α. Then, as in the usual proof, using our first example and the regularity of μ we can fix some $n \in \omega$ such that for each $\sigma \in \mu$, $\{\alpha \in S : a_n^\alpha \geq \sigma\}$ is stationary in μ. Now define a regressive function $f : S \to \mu$ by $f(\alpha) = a_n^\alpha$. Using Fodor's Theorem, for each $\sigma \in \mu$ fix some S_σ stationary and $\gamma_\sigma \geq \sigma$ such that for all $\alpha \in S_\sigma$, $f(\alpha) = \gamma_\sigma$. Then if $\gamma_\sigma \neq \gamma_{\sigma'}$, $S_\sigma \cap S_{\sigma'} = \emptyset$ and, by the regularity of μ, $|\{S_\sigma : \sigma \in \mu\}| = \mu$, which gives the required partition. ⊣

§5. Non-existence of embeddings. We are now in the position to prove that there is no non-trivial, cofinal elementary embedding j of ZFC$^-$ with $V_{\operatorname{crit}(j)} \in V$. This shall be done in two parts; first we shall show that Woodin's proof of the Kunen inconsistency, which is the second proof of Theorem 23.12 in [13], goes through in ZFC$^-$ under the additional assumption that $(\sup\{j^n(\operatorname{crit}(j)) : n \in \omega\})^+ \in V$. Then we shall show, by modifying the coding from Theorem 2.2, that no cofinal embedding can exist in any model that sufficiently resembles H_{λ^+}.

In order to prove this formally we shall work in a subtheory of the second order version of ZFC$^-$ which we shall denote as ZFC$_j^-$. Essentially this is the theory ZFC$^-$ along with a class predicate j for the elementary embedding and the assertion that the Collection and Separation Schemes hold when expanded to include formulae with this predicate. To be more precise;

DEFINITION 5.1. Suppose that T is a first order set theory, M is a model of T and j is a class predicate. We say that $M \models T_j$ if M satisfies the axiom schemes of T (for example Induction, Separation and Replacement / Collection / Reflection) in the language expanded to include j.

---

[3]A function $f : S \to \mathrm{ORD}$ is *regressive* if for any non-zero $\alpha \in S$, $f(\alpha) < \alpha$.

THEOREM 5.2. *There is no non-trivial, elementary embedding* $j : V \to V$ *such that* $V \models \mathrm{ZFC}_j^-$ *and* $(\sup\{j^n(\mathrm{crit}(j)) : n \in \omega\})^+ \in V$.

PROOF. Suppose for a contradiction that $j : V \to V$ was a non-trivial elementary embedding with critical point κ and let $\lambda = \sup\{j^n(\kappa) : n \in \omega\}$. Then $j(\lambda) = \lambda$ and, since λ^+ is definable as the least cardinal above λ, $j(\lambda^+) = \lambda^+$. Now, using Theorem 4.3, let $\langle S_\alpha : \alpha \in \kappa \rangle$ be a partition of $S_\omega^{\lambda^+}$ into κ many disjoint stationary sets and let $S = \{\langle \alpha, S_\alpha \rangle : \alpha \in \kappa\}$. Then $j(S) = \{\langle \alpha, T_\alpha \rangle : \alpha \in j(\kappa)\}$ and, by elementarity, $\langle T_\alpha : \alpha \in j(\kappa) \rangle$ is a partition of $S_\omega^{\lambda^+}$ into disjoint sets such that for each α

$$T_\alpha \text{ is a stationary subset of } \lambda^+.$$

Also, we have that for each $\alpha \in \kappa$, $j(S_\alpha) = T_\alpha$. We claim that there is some $\beta \in \kappa$ such that $T_\kappa \cap S_\beta$ is stationary. For suppose not, then by our comments on choosing from set many classes, for each α we can fix a club C_α such that $T_\kappa \cap S_\alpha \cap C_\alpha = \emptyset$. Letting $C = \bigcap_{\alpha \in \kappa} C_\alpha$ we must have that

$$\emptyset = T_\kappa \cap C \cap \bigcup_{\alpha \in \kappa} S_\alpha = T_\kappa \cap C,$$

contradicting the assumption that T_κ was stationary. So fix β such that $T_\kappa \cap S_\beta$ is stationary. Now, let

$$U = \{\gamma \in \lambda^+ : \gamma = j(\gamma)\},$$

and note that U is closed under sequences of length ω. Therefore there exists some $\sigma \in U \cap T_\kappa \cap S_\beta$. But then $\sigma = j(\sigma) \in j(S_\beta) = T_\beta$, contradicting the assumption that the T_α were disjoint. Hence no such embedding can exist. ⊣

REMARK 5.3. The above theorem did not require any assumption about j being cofinal or that $V_{\mathrm{crit}(j)}$ was a set.

THEOREM 5.4. *There is no non-trivial, cofinal, Σ_0-elementary embedding* $j : V \to V$ *such that* $V \models \mathrm{ZFC}_j^-$ *and* $V_{\mathrm{crit}(j)} \in V$.

PROOF. Suppose for a contradiction that $j : V \to V$ was a non-trivial, cofinal, Σ_0-elementary embedding with critical point κ and let $\lambda = \sup\{j^n(\kappa) : n \in \omega\}$. Note that, by an instance of Replacement in the language expanded to include the predicate j, $\lambda \in V$. Now there are two cases:

- CASE 1: λ^+ exists.
- CASE 2: For all $x \in V$, there is an injection $f : x \to \lambda$.

CASE 1: Using Theorem 3.2, any such embedding must be fully elementary. Therefore, this is just a special case of Theorem 5.2.

CASE 2: First note that by elementarity, since $V_\kappa \in V$ so is $V_{j^n(\kappa)}$ for each $n \in \omega$ and therefore $V_\lambda = \bigcup_{n \in \omega} V_{j^n(\kappa)} \in V$. Note also that $\lambda \times \lambda \in V$ and, by the Well-Ordering Principle, for each $x \in V$ there is a bijection

$$f : |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\}).$$

Moreover, since there is an injection of x into λ, we must have that $|\mathrm{trcl}(\{x\})| \leq \lambda$ for each $x \in V$. Now let

$$C_{x,f} := \{\langle \alpha, \beta \rangle \in \lambda \times \lambda : f(\alpha) \in f(\beta)\}.$$

Then $C_{x,f} \in V$ and therefore so is its Mostowski collapse, with $\mathrm{coll}(C_{x,f}) = \mathrm{trcl}(\{x\})$. This means that for any x and bijection $f : |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$,

$$j\big(\mathrm{trcl}(\{x\})\big) = j\big(\mathrm{coll}(C_{x,f})\big) = \mathrm{coll}\big(j(C_{x,f})\big)$$

$$= \mathrm{coll}\big(\bigcup_{\alpha<\lambda} j(C_{x,f} \cap V_\alpha)\big)$$

$$= \mathrm{coll}\big(\bigcup_{\alpha<\lambda} j \restriction V_\lambda(C_{x,f} \cap V_\alpha)\big).$$

That is, j is completely determined by its construction up to V_λ. Now, let $i := j \restriction V_\lambda$ and note that, since $V_\lambda \times V_\lambda \in V$, so is

$$i = \{\langle x, y\rangle \in V_\lambda \times V_\lambda : j(x) = y\}.$$

Therefore, by defining $\varphi(\cdot, \cdot, i, \lambda)$ as

$$\varphi(x, y, i, \lambda) \equiv \exists f, z, C_{x,f} \,\big(\text{``}\mathrm{dom}(f) \text{ is a cardinal''} \wedge \mathrm{ran}(f) = \mathrm{trcl}(\{x\})$$

$$\wedge \text{``}f \text{ is a bijection''} \wedge C_{x,f} := \{\langle \alpha, \beta\rangle \in \lambda \times \lambda : f(\alpha) \in f(\beta)\}$$

$$\wedge \; z = \mathrm{coll}\big(\bigcup_{\alpha<\lambda} i(C_{x,f} \cap V_\alpha)\big)$$

$$\wedge \text{``}y \text{ is the element of } z \text{ of maximal rank''}\big),$$

we have that $\varphi(x, y, i, \lambda)$ holds if and only if $j(x) = y$ so j is definable from the parameters i and λ, both of which lie in V, contradicting Theorem 3.4. ⊣

§6. Removing the assumption that $V_{\mathrm{crit}(j)} \in V$. Assuming that V satisfies the additional assumption of *Dependent Choice of length μ* for every infinite cardinal μ, we are able to remove the assumption that $V_{\mathrm{crit}(j)} \in V$. This will be done by first proving that, in this theory, every proper class must surject onto any given non-zero ordinal. In particular, for $V_{\mathrm{crit}(j)}$ to be a proper class it is necessary for $V_{\mathrm{crit}(j)}$ to surject onto $j(\kappa)$ which we shall show cannot happen. Note that, in the standard ZFC case, the cardinality of $V_{\mathrm{crit}(j)}$ is $\mathrm{crit}(j)$.

In the ZF context the principle of Dependent Choice of length μ, for μ an infinite cardinal is the following statement formulated by Lévy [18].

> Let S be a non-empty set and R a binary relation such that for every $\alpha \in \mu$ and every α-sequence $s = \langle x_\beta : \beta \in \alpha\rangle$ of elements of S there exists some $y \in S$ such that sRy. Then there is a function $f : \mu \to S$ such that for every $\alpha \in \mu$, $(f \restriction \alpha) R f(\alpha)$.

In the more general ZFC$^-$ context we want to consider a natural class version of this where S and R are replaced by definable classes. Such classes can be considered as the collection of sets x which satisfy $\psi(x, u)$ for some formula ψ. This leads to the definition of the DC$_\mu$-*Scheme* as the following:

> Let φ and ψ be formulae and u be a set such that for some y, $\psi(y, u)$ and for every $\alpha \in \mu$ and every α-sequence $s = \langle x_\beta : \beta \in \alpha\rangle$ satisfying $\psi(x_\beta, u)$ for each β, there is a z satisfying $\psi(z, u)$ and

$\varphi(\alpha, s, z, u)$. Then there is a function f with domain μ such that for each $\alpha \in \mu$ $\psi(f(\alpha), u)$ and $\varphi(\alpha, f \restriction \alpha, f(\alpha), u)$.

For $\mu = \aleph_0$ we shall refer to this concept as the DC-Scheme. An equivalent way to view the DC_μ-Scheme is the assertion that if T is a tree that has no maximal element and is μ-closed, that is to say every α-sequence of nodes in T has an upper bound, then T has a branch of order type μ. We note that, as with the set case, if $\delta < \mu$ are infinite cardinals and DC_μ holds then so does DC_δ.

PROPOSITION 6.1 [18]. *If $\delta < \mu$ are infinite cardinals then the DC_μ-Scheme implies the DC_δ-Scheme.*

PROOF. Let φ and ψ be formulae and u be a set such that for some y, $\psi(y, u)$ and for every $\alpha \in \delta$ and every α-sequence $s = \langle x_\beta : \beta \in \alpha \rangle$ satisfying $\psi(x_\beta, u)$ for each β, there is a z satisfying $\psi(z, u)$ and $\varphi(\alpha, s, z, u)$. We define a new formula ϑ extending φ to apply to any α-sequence, s, for $\alpha \in \mu$ by

$$\vartheta(\alpha, s, z, u, \delta) \equiv \big(\alpha < \delta \wedge \forall \beta \in \alpha \; \psi(s(\beta), u) \wedge \varphi(\alpha, s, z, u)\big)$$
$$\vee \big(\alpha \geq \delta \wedge \psi(z, u)\big).$$

Then for any function f with domain μ witnessing this instance of the DC_μ-Scheme, $f \restriction \delta$ witnesses that DC_δ holds for ψ and φ. ⊣

An important strengthening of the Collection Scheme is the Reflection Principle which we define next.

DEFINITION 6.2. The *Reflection Principle* is the assertion that for any formula φ and set a there is a transitive set A such that $a \subseteq A$ and φ is absolute between A and the universe.

The next pair of theorems show how this principle relates to Dependent Choice.

THEOREM 6.3 [7]. *Over ZFC^-, the DC-Scheme is equivalent to the Reflection Principle.*

THEOREM 6.4 [4]. *The Reflection Principle is not provable in ZFC^-.*

REMARK 6.5. To see that the Reflection Principle implies the DC-Scheme, let T be some definable tree of height ω with no maximal element. This statement can be reflected to a transitive set M using the Reflection Principle which can then be well-ordered to provide the witnesses for an infinite branch.

However, this argument does not go through for the DC_μ-Scheme when μ is an uncountable cardinal. The issue here is that the transitive set M need not be closed under infinite sequences definable in the full universe. Therefore, when we reflect the tree it need no longer be the case that the tree is closed under sequences of length less than μ.[4]

This leads to the theory that we shall work in throughout this section,

DEFINITION 6.6. Let $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{ORD}}$ denote the theory ZFC^- plus the DC_μ-Scheme for every cardinal μ.

---

[4]In fact, it is possible to separate the various Dependent Choice Schemes over ZFC^-. This is a joint result of the author and Victoria Gitman which can be found in [19] and will appear in [8].

THEOREM 6.7. *Suppose that* $V \models ZF^- + DC_\mu$ *for μ an infinite cardinal. Then for any proper class $\mathcal{C}$, which is definable over V, there is a subset b of $\mathcal{C}$ of cardinality μ.*

PROOF. Let $\mathcal{C} = \{x \colon \psi(x)\}$ be a proper class. We shall in fact prove the equivalent statement that for any $v \leq \mu$ there is a subset b of $\mathcal{C}$ and a bijection between b and v. Suppose for a contradiction that this were not the case and let γ be the least ordinal for which no such subset of size γ exists. It is obvious that γ must be an infinite cardinal. Let $\varphi(\alpha, s, y)$ be the statement that s is a sequence of length α and $\operatorname{ran}(s) \cup \{y\}$ is a subset of $\mathcal{C}$ and $y \notin \operatorname{ran}(s)$. Then, by assumption, for every $\alpha \in \gamma$ there is a sequence of elements of $\mathcal{C}$ of length α. Also, since $\mathcal{C}$ is a proper class, if s is an α length sequence from $\mathcal{C}$ then there is some $y \in \mathcal{C}$ which is not in s so the hypothesis of DC_γ is satisfied. Therefore, by DC_γ, there is a function f with domain γ and whose range gives a subset of $\mathcal{C}$ of cardinality γ, giving us our desired contradiction. ⊣

COROLLARY 6.8. *Suppose that* $V \models ZFC^- + DC_{<ORD}$. *Then for any proper class $\mathcal{C}$ which is definable over V and any non-zero ordinal γ there is a definable surjection of $\mathcal{C}$ onto γ.*

We can now prove that if j is a non-trivial elementary embedding from V to some class $M \subseteq V$ then $V_{\operatorname{crit}(j)} \in V$.

LEMMA 6.9. *Suppose that* $V \models ZFC^- + DC_{<ORD}$, $M \subseteq V$ *and* $j \colon V \to M$ *is a non-trivial elementary embedding with critical point κ. Then for any $\alpha \in \kappa + 1$, $V_\alpha \in V$.*

PROOF. This is proven by induction on $\alpha \in \kappa + 1$. Clearly limit cases follow by an instance of Collection so it suffices to prove that for $\alpha \in \kappa$, if $V_\alpha \in V$ then so is $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. First note that j fixes every set of rank less than κ so $j \upharpoonright V_{\alpha+1}$ is the identity. Now suppose for sake of a contradiction that $V_{\alpha+1}$ was a proper class. Then, by Corollary 6.8, we could fix a set $b \subseteq V_{\alpha+1}$ and a surjection

$$h \colon b \twoheadrightarrow \kappa.$$

So, by elementarity, there is a surjection

$$j(h) \colon j(b) \twoheadrightarrow j(\kappa)$$

in M. However, since $b \subseteq V_{\alpha+1}$, $j(b)$ is also a subset of $V_{j(\alpha+1)} = V_{\alpha+1}$ and for any $x \in j(b)$, $j(x) = x$. Therefore,

$$x \in j(b) \iff j(x) \in j(b) \iff x \in b,$$

and hence $b = j(b)$. Then, for any $x \in b$,

$$j(h)(x) = j(h)(j(x)) = j(h(x)) = h(x),$$

so $j(h) = h$. But this then contradicts the assumption that $j(h)$ was a surjection onto $j(\kappa)$. Hence $V_{\alpha+1}$ must be a set in V as required. ⊣

Combining this result with Theorem 5.4 gives the Kunen inconsistency for the theory $(ZFC^- + DC_{<ORD})_j$.

COROLLARY 6.10. *There is no non-trivial, cofinal, Σ_0-elementary embedding $j \colon V \to V$ such that $V \models (ZFC^- + DC_{<ORD})_j$.*

However this leaves open the question as to whether or not this result is provable without relying on $DC_{<ORD}$, namely;

QUESTION 6.11.   *Is the existence of a non-trivial, cofinal, Σ_0-elementary embedding $j : V \to V$ such that $V \models ZFC_j^-$ inconsistent?*

The stumbling block that one needs to overcome appears to be the following:

QUESTION 6.12.   *Suppose that $V \models ZFC^-$, $M \subseteq V$ and $j : V \to M$ is a non-trivial elementary embedding. Is $\mathcal{P}(\omega) \in V$? Is $V_{crit(j)} \in V$?[5]*

§7. Collection in symmetric models.

The important tool from the previous section was the fact that in models of $ZFC^- + DC_{<ORD}$ proper classes are "*big*". That is, given any non-zero ordinal, any proper class surjects onto that ordinal. This property is also a feature of models of ZF as shown by the following result:

PROPOSITION 7.1.   *Under ZF, there is a surjection from any proper class onto any non-zero ordinal.*

PROOF.   Given a proper class $\mathcal{C}$, define

$$S := \{\gamma \in ORD : \exists x \in \mathcal{C} \; rank(x) = \gamma\}.$$

Then S must be unbounded in the ordinals so, given an ordinal α, we can take the first α many elements of S, $\{\gamma_\beta : \beta \in \alpha\}$. Then

$$f(x) = \begin{cases} \beta, & \text{if } rank(x) = \gamma_\beta, \\ 0, & \text{otherwise}, \end{cases}$$

defines a surjection of $\mathcal{C}$ onto α.   ⊣

On the other hand, we have the following theorem from [7]:

THEOREM 7.2 [7].   *Suppose that $V \models ZFC$, κ is a regular cardinal with $2^\omega < \aleph_\kappa$ and that $G \subseteq Add(\omega, \aleph_\kappa)$ is V-generic. If $W = \bigcup_{\gamma < \kappa} V[G_\gamma]$ where $G_\gamma = G \cap Add(\omega, \aleph_\gamma)$, (that is G_γ is the first $\aleph_\gamma$ many of the Cohen reals added by G) then $W \models ZFC^-$ has the same cardinals as V and the DC_α-Scheme holds in W for all $\alpha < \kappa$, but the DC_κ-Scheme and the Reflection Principle fail.*

Since V will have the same cardinals as $V[G]$, that is the full extension by all $\aleph_\kappa$ many reals, in $V[G]$ $2^\omega = \aleph_\kappa$ and therefore there is no surjection of $\mathcal{P}(\omega)$ onto $\aleph_{\kappa+1}$. Hence there is no such surjection in W, so W is a model of ZFC^- and the DC_α-scheme for all $\alpha \in \kappa$ in which $\mathcal{P}(\omega)$ is a proper class which is not big.

So it seems natural to ask which sub-theories of ZFC also prove this feature. While we shall not answer this question here, the following results show the difficulty in coming up with a counterexample in the theory ZF^-.

One candidate for a counterexample in this theory is an amorphous class where:

DEFINITION 7.3.   A class A is said to be *amorphous* if it is infinite but not the disjoint union of two infinite subclasses. Namely, if $A = B \cup C$ and $B \cap C = \emptyset$ then either B or C must be a finite set.

---

[5]Assuming the consistency of ZFC plus a measurable cardinal, a negative answer to this question is obtained in [8, 19] .

Now one could imagine having a set $M \models ZF^-$ with $A \subseteq M$ such that M believes that A is an amorphous class. Then A could not surject onto ω since if f were such a surjection, $\{x \in A : f(x) \text{ is even}\}$ and $\{x \in A : f(x) \text{ is odd}\}$ would be a partition of A into two infinite classes. However, we shall show that such a situation can never arise because it is inconsistent with ZF^- to have a definable amorphous class. In fact, the analysis will show that a symmetric submodel of a pretame class forcing will not in general satisfy the Collection Scheme. To begin with we give the basics of symmetric extensions.

7.1. Symmetric extensions. When talking about symmetric extensions we shall follow the notation of Karagila, which can be found in papers such as [14, 15]. The purpose of symmetric extensions is to find a suitable submodel of a generic extension of ZFC in which Choice fails. This allows us to build objects such as Dedekind-finite sets and amorphous sets whose existence would contradict the Axiom of Choice while still using Choice in the ground model to control what happens.

NOTATION 7.4. *Given a set of $\mathbb{P}$-names $\{\dot{x}_i : i \in I\}$ let $\{\dot{x}_i : i \in I\}^{\bullet}$ denote the $\mathbb{P}$-name $\{\langle \dot{x}_i, \mathbb{1} \rangle : i \in I\}$.*

Now, given a forcing notion $\mathbb{P}$ and automorphism π of $\mathbb{P}$, π can be extended to $\mathbb{P}$-names by the following recursion:

$$\pi \dot{x} = \{\langle \pi \dot{y}, \pi p \rangle : \langle \dot{y}, p \rangle \in \dot{x}\}.$$

For $\mathcal{G}$ a group of automorphisms of $\mathbb{P}$ we say that $\mathcal{F}$ is a normal filter of subgroups over $\mathcal{G}$ if:

- $\mathcal{F}$ is a non-empty family of subgroups of $\mathcal{G}$.
- $\mathcal{F}$ is closed under finite intersections and supergroups.
- (*Normality*) For any $H \in \mathcal{F}$ and $\pi \in \mathcal{G}$, $\pi H \pi^{-1} \in \mathcal{F}$.

A symmetric system is then a triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ such that $\mathbb{P}$ is a notion of forcing, $\mathcal{G}$ is a group of automorphisms of $\mathbb{P}$ and $\mathcal{F}$ is a normal filter of subgroups over $\mathcal{G}$.

We also fix the following notation:

- The *stabaliser* of $\dot{x}$ under $\mathcal{G}$ is $\text{sym}_{\mathcal{G}}(\dot{x}) = \{\pi \in \mathcal{G} : \pi \dot{x} = \dot{x}\}$.
- $\dot{x}$ is said to be *$\mathcal{F}$-symmetric* when $\text{sym}_{\mathcal{G}}(\dot{x}) \in \mathcal{F}$.
- $\dot{x}$ is said to be *hereditarily $\mathcal{F}$-symmetric* when it is $\mathcal{F}$-symmetric and for any $\langle p, \dot{y} \rangle$ in $\dot{x}$, $\dot{y}$ is also hereditarily $\mathcal{F}$-symmetric.
- $\text{HS}_{\mathcal{F}}$ denotes the class of hereditarily $\mathcal{F}$-symmetric names.

We then have the following two theorems which summarise the important consequences of this construction and can be found in Chapter 15 of [12].

LEMMA 7.5 (The Symmetry Lemma). *For any $p \in \mathbb{P}$, automorphism π of $\mathbb{P}$, formula $\varphi(v)$ of the forcing language and $\mathbb{P}$-name $\dot{x}$,*

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi \dot{x}).$$

THEOREM 7.6. *If M is a transitive model of ZF, $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system and $G \subseteq \mathbb{P}$ is a generic filter then $N := \text{HS}_{\mathcal{F}}^G = \{\dot{x}^G : \dot{x} \in \text{HS}_{\mathcal{F}}\}$ is a transitive model of ZF with $M \subseteq N \subseteq M[G]$.*

7.2. Class forcing. In order to describe our desired models, we shall need to do a relatively simple class forcing. Because class forcing is not the main goal of this paper, we shall only briefly sketch the construction and refer the reader to works such as [3], [9] or [10] for more details on how to formally define class forcing.

As in set forcing, when trying to formalise the theory of class forcing one often works in a countable, transitive model of some second order theory such as GB^-. Such a model will be of the form $\mathbb{M} = \langle M, \mathcal{C} \rangle$ where M denotes the *sets* of the model and $\mathcal{C}$ the classes. However, primarily for ease of notation, we shall repeatedly only talk about the first order part of the theory, noting that if M is a set model of ZF^- and $\mathcal{C}$ is the collection of classes definable over M then $\langle M, \mathcal{C} \rangle$ is a model of GB^-. We shall say that a class $\dot{\Gamma}$ is a $\mathbb{P}$-*name* if every element of $\dot{\Gamma}$ is of the form $\langle \dot{x}, p \rangle$ where $\dot{x}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. We then define $M^{\mathbb{P}}$ to be the collection of $\mathbb{P}$-names which are elements of M and define $\mathcal{C}^{\mathbb{P}}$ as those names which are in $\mathcal{C}$.

Essentially, the question is which properties of set forcing are still true when the partial order, $\mathbb{P}$, is now assumed to be a proper class. Here one immediately runs into a problem when trying to prove the forcing theorem which comprises of two parts; *truth* and *definability*. The *definability lemma* is the assertion that the forcing relation is definable in the ground model and the *truth lemma* is that anything true in the generic extension is forced to be true by an element of the generic. However, as shown in [9], given any countable, transitive model of GB^- there is a class forcing notion which does not satisfy the forcing theorem for atomic formulae. In fact, it is shown in [6] that, over GB with a global well-order, the statement that the forcing theorem holds for any class forcing is equivalent to elementary transfinite recursion for any recursion of length ORD.

Moreover, even if a class forcing satisfies the forcing theorem it is not always the case that GB^- will be preserved in any generic extension. The simplest such example is $Col(\omega, ORD)$ which generically adds a function collapsing the ordinals onto ω. However, there is a well known collection of class forcings which both satisfy the forcing theorem and preserve all of the axioms of GB^-. This property was first defined by Stanley and, while such forcings are normally characterised combinatorially, for our purposes we shall use a simpler but less enlightening definition which one can prove is equivalent.

DEFINITION 7.7 (Stanley). Let $\mathbb{M}$ be a model of GB^-. A class forcing $\mathbb{P}$ is said to be *pretame* for $\mathbb{M}$ if, for any generic filter $G \subseteq \mathbb{P}$, $\mathbb{M}[G]$ satisfies GB^-.

The class forcing we shall consider is $Add(\omega, ORD)$, the forcing to add a proper class of Cohen reals. It can be proven that this satisfies the forcing theorem because it is *ordinal approachable by projections*,[6] that is to say it can be written as a continuous, increasing union of set-sized forcings, $Add(\omega, ORD) = \bigcup_{\alpha \in ORD} Add(\omega, \alpha)$.

THEOREM 7.8 [9]. *If $\mathbb{P}$ is ordinal approachable by projections then $\mathbb{P}$ satisfies the forcing theorem.*

---

[6]In [9] the authors call this property *approachability by projections*. However we will follow the terminology from [11] where they give a general definition of being approachable by projections. They then comment that this is perhaps how approachability should have originally been defined and that it would be more natural to rename the original definition as being ordinal approachable by projections.

While this property in itself does not ensure that the forcing is pretame, it allows us to use an equivalent characterisation of pretameness from [10]:

THEOREM 7.9 [10]. *Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a model of* GB *and $\mathbb{P}$ is a class forcing notion for $\mathbb{M}$ which satisfies the forcing theorem. Then $\mathbb{P}$ is pretame if and only if there is no set $a \in M$, name $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and condition $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{F} \colon \check{a} \to$* ORD *is cofinal".*

THEOREM 7.10. *Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a model of* GB + AC *and $\mathbb{P}$ is a class forcing notion for $\mathbb{M}$ which satisfies the forcing theorem. If μ is an uncountable cardinal in $\mathbb{M}$ and $\mathbb{P}$ satisfies the μ-cc then $\mathbb{P}$ is pretame.*

PROOF. This will be proven by a variation on a standard set forcing result which uses the μ-cc to approximate functions in the extension:

CLAIM 7.11. *Suppose that $a \in M$ and $F \colon a \to$* ORD *is a class function in $\mathbb{M}[G]$. Then there exists some $f \in M$ such that for all $x \in a$, $F(x) \in f(x)$ and $(f(x) \in$ $[$ORD$]^{<\mu})^{\mathbb{M}}$.*

To prove the claim, let $\dot{F}$ be a name for F and take p such that $p \Vdash \dot{F} \colon \check{a} \to$ ORD. Next, for each $x \in a$, let

$$f(x) = \{\alpha \in \text{ORD} : \exists q \leq p \ (q \Vdash \dot{F}(\check{x}) = \check{\alpha})\}.$$

Now suppose that for some $x \in a$, $f(x)$ did not have cardinality less than μ. By using Collection and set sized Choice, we can choose a subset Y of $f(x)$ of size μ. Then for each $\alpha \in Y$ we can choose $q_\alpha \leq p$ such that $q_\alpha \Vdash \dot{F}(\check{x}) = \check{\alpha}$. But then this must be an antichain of size μ contradicting the assumption of μ-cc.

Using the claim, we have that if $a \in M$, $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ are such that $p \Vdash \dot{F} \colon \check{a} \to$ ORD then, taking f from the conclusion of the claim, $\delta = \sup\{\sup(f(x)) : x \in a\}$ is an ordinal. Therefore

$$p \Vdash \text{``the image of } \dot{F} \text{ is contained in } \delta\text{''}.$$

So, in particular, p cannot force the function to be cofinal which implies that $\mathbb{P}$ is pretame by Theorem 7.9. ⊣

7.3. Breaking collection. We can now produce our class symmetric systems. Solely for simplicity, let $\mathbb{M}$ be a countable, transitive model of GB + AC + CH. The first model we give is a symmetric system to add an amorphous class consisting of sets of Cohen reals. A detailed account of the set version of this forcing can be found as Exercise 8.13.1 of [2] which serves as the basis for our construction.

Let $\mathbb{P} = \text{Add}(\omega, \text{ORD} \times \omega)$ be the poset which adds ORD many ω-blocks of Cohen reals. Now, for π^0 a permutation of ORD and $\{\pi_\alpha : \alpha \in \text{ORD}\}$ a collection of permutations of ω let π be the permutation

$$\pi \colon \text{ORD} \times \omega \to \text{ORD} \times \omega \qquad \pi(\alpha, n) = (\pi^0(\alpha), \pi_\alpha(n)),$$

and let $\mathcal{G}$ be the class of permutations defined in this way. This is known as the *wreath product* of the permutations of ORD and the permutations of ω. Then let $\mathcal{F}$ be the filter generated by $\text{fix}(E) \coloneqq \{\pi \in \mathcal{G} : \pi \restriction E = \text{id}\}$ for finite sets $E \subseteq \text{ORD} \times \omega$. We can extend π to $\mathbb{P}$ by

$$\pi p(\pi(\alpha, n)) = p(\alpha, n).$$

So the idea is that elements of $\mathcal{G}$ first permute the ω-blocks of reals and then permute within the blocks.

Now define

- $\dot{i}_{(\alpha,n)} := \{\langle \check{m}, p \rangle \colon p \in \mathbb{P} \ \wedge \ p(\alpha, n, m) = 1\}$. This is the canonical name for the Cohen real generated by $\mathbb{P}$ restricted to the co-ordinate (α, n).
- $\dot{T}_\alpha := \{\dot{i}_{(\alpha,n)} \colon n \in \omega\}^\bullet$ to be a name for the αth ω-block of reals.
- $\dot{A} := \{\dot{T}_\alpha \colon \alpha \in \text{ORD}\}^\bullet$ to be a name for the collection of all ORD many ω-blocks.

We claim that in the symmetric extension $\text{HS}_{\mathcal{F}}^G$, $A = \dot{A}^G$ is an amorphous proper class. To see why it is amorphous, suppose that $\dot{B}$ were a symmetric name such that for some $p \in \mathbb{P}$, $p \Vdash \dot{B} \subseteq \dot{A}$. Take $E \subseteq \text{ORD} \times \omega$ to be a finite set such that $\text{fix}(E) \subseteq \text{sym}(\dot{B})$ and let

$$y := \{\alpha \in \text{ORD} \colon \exists n \in \omega \ (\alpha, n) \in E \cup \text{supp}(p)\} \in [\text{ORD}]^{<\omega}.$$

The following claim is a standard argument, for example see [2], and therefore we omit its proof.

CLAIM 7.12. *For any $q \leq p$, if there is an $\alpha \notin y$ such that $q \Vdash \dot{T}_\alpha \in \dot{B}$, then $q \Vdash \dot{T}_\beta \in \dot{B}$ for any $\beta \notin y$.*

Therefore, for any $q \leq p$,

$$q \Vdash \{\dot{T}_\alpha \colon \alpha \in \text{ORD} \setminus y\} \subseteq \dot{B} \ \vee \ \{\dot{T}_\alpha \colon \alpha \in \text{ORD} \setminus y\} \subseteq \dot{A} \setminus \dot{B},$$

and so $\dot{A}$ will indeed be forced to be an amorphous class.

The second model is the system with $\mathbb{P} = \text{Add}(\omega, \text{ORD})$, permutations given by $\pi p(\pi \alpha, n) = p(\alpha, n)$ where π is a permutation of ORD and $\mathcal{F}$ is the filter generated by $\text{fix}(E)$ for finite sets $E \subseteq \text{ORD}$. This system will add a Dedekind-finite class of Cohen reals, A, and it can be proven that if B is an infinite subclass of A then B is a proper class.

The fact that both of these symmetric submodels fail to satisfy the Collection Scheme follows from the following slightly more general theorem.

THEOREM 7.13. *Suppose that $\langle M, A \rangle$ satisfies*;

1. $M \models (\text{ZF}^-)_A$,
2. $A \subseteq M$ and $\langle M, A \rangle \models$ "A is a proper class", and
3. $\langle M, A \rangle \models$ "if $B \subseteq A$ is infinite then B is a proper class".

Then the Collection Scheme fails in $\langle M, A \rangle$. Moreover, $\langle M, A \rangle$ does not have a cumulative hierarchy and therefore Power Set also fails.

PROOF. To prove that the Collection Scheme fails consider classes b satisfying

$$\forall n \in \omega \ \exists y \in b \ (|y| = n \ \wedge \ y \subseteq A).$$

Since $\bigcup b \cap A$ is an infinite subclass of A, by the third assumption b must be a proper class. Therefore, while for every $n \in \omega$ there is a y such that $(|y| = n \ \wedge \ y \subseteq A)$ there is no set witnessing this for all n.

For the "moreover" part of the theorem, we shall in fact prove that any well-orderable sequence of sets can only contain finitely many elements of A. To see

this, let $\mathcal{C} = \langle C_\alpha : \alpha \in I \rangle$ be an indexed sequence of sets where I is either ORD or an infinite ordinal. We shall show that $\bigcup \mathcal{C} \cap A$ is finite and therefore that $\mathcal{C}$ cannot be a hierarchy for the universe. Suppose for a contradiction that $\bigcup \mathcal{C} \cap A$ was in fact infinite. First note that for any $\alpha \in I$, $C_\alpha \cap A$ must be finite. Now we define a sequence of ordinals $\delta_n \in I$ inductively as the least ordinal $\alpha \in I$ such that $(C_\alpha \setminus \bigcup_{m \in n} C_{\delta_m}) \cap A \neq \emptyset$. Such an ordinal must exist by the assumption that $\bigcup \mathcal{C} \cap A$ is infinite and that $\bigcup_{m \in n}(C_{\delta_m} \cap A)$ is a union of finite sets. But then $\bigcup_{n \in \omega} C_{\delta_n} \cap A$ is an infinite set, contradicting the third condition of the theorem. ⊣

Karagila [16] claims that, over GB, any symmetric submodel of a tame[7] class forcing will again be a model of GB. However, since it is not necessarily true that the symmetric submodel of a pretame class forcing of a model of GB^- is again a model of GB^- we have the following three natural general questions:

QUESTION 7.14.

- *Suppose that* $\mathbb{M} \models GB^-$. *Let* $\mathbb{P}$ *be a pretame class forcing and* $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *a symmetric system with symmetric submodel* N. *What theory does* N *satisfy?*
- *Is there a combinatorial condition one can place on a symmetric system so that the symmetric submodel will satisfy* GB^-?
- *Which class versions of sets whose existence is incompatible with Choice can exist over models of* GB *or* GB^-?

In general, this final problem seems to be a difficult one to answer because, as shown by Monro in [20], it is consistent to have a Dedekind-finite proper class:

THEOREM 7.15 (Monro). *Let* ZF(K) *be the theory with the language of* ZF *plus a one-place predicate* K *and let* M *be a countable transitive model of* ZF. *Then there is a model* N *such that* N *is a transitive model of* ZF(K) *and*

$$N \models K \text{ is a proper class which is Dedekind-finite}$$

$$\text{and can be mapped onto the universe.}$$

In fact it is possible to have a class symmetric system which adds a Dedekind-finite class out of ORD many Cohen reals, is pretame and such that the symmetric submodel is a model of GB^-. This is obtained as a symmetric submodel of the forcing $Add(\omega, ORD \times \omega)$ which adds a block of ω many Cohen reals for each ordinal. The permutation class is then those permutations which preserve the blocks and then permute within the block and the filter is, as usual, generated by the permutations which fix finite subsets of $ORD \times \omega$.

The reason why the symmetric submodel in this case models both Collection and Separation is that for any pair $\langle \dot{y}, p \rangle \in M^{\mathbb{P}} \times \mathbb{P}$ there is some ordinal α such that for any permutation π, $\langle \pi \dot{y}, \pi p \rangle$ can be seen as an element of the set forcing $Add(\omega, \alpha \times \omega)$. This means that for any H in the filter of subgroups, $\{\langle \pi \dot{y}, \pi p \rangle : \pi \in H\}$ is a set which is a sufficient condition to prove the Collection Scheme.

To conclude this work we outline an approach to solving the first question from Question 7.14, which we will study further in future work. As before, suppose that $\mathbb{M}$ is a countable transitive model of GB^- and that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system

---

[7]A class forcing is said to be *tame* if it is pretame and also preserves Power Set.

where $\mathbb{P}$ is a pretame class forcing. We shall say that a class name $\dot{\Gamma}$ is $\mathcal{F}$-*respected* if $\{\pi \in \mathcal{G} : \mathbb{1} \Vdash \pi\dot{\Gamma} = \dot{\Gamma}\} \in \mathcal{F}$ and *hereditarily $\mathcal{F}$-respected* if this property holds hereditarily. Let HR be the class of names $\dot{x} \in \mathrm{M}^{\mathbb{P}}$ which are hereditarily respected.

The purpose of defining symmetric names in this way is to circumvent the issue that, for an arbitrary element H of the filter, $\{\pi\dot{x} : \pi \in H\}$ may well be a proper class. However, in the set forcing case, the two resulting extensions are equal. That is to say, if $\mathbb{P}$ is a set forcing and $G \subseteq \mathbb{P}$ is generic then $\mathrm{HS}^G = \mathrm{HR}^G$ because any respected name is easily seen to be equivalent to a symmetric name by closing under its symmetry group.

Now, suppose that our symmetric system is *tenacious*.[8] Let $\dot{f}$ be a name for a function, $\dot{a}$ a name for its domain and $\dot{b}$ the obvious class forcing name for $f\text{``}a$, for example take the name defined in Theorem 3.1 of [10]. Then, for an appropriately chosen $q \in G$ and any π fixing q, $\dot{a}$ and $\dot{f}$ we will have that $\mathbb{1} \Vdash \pi\dot{b} = \dot{b}$. Thus Replacement will hold in HR^G. Moreover, one can show that HR^G is a model of GB$^-$. However, either of our earlier class symmetric systems will still witness the failure of the Collection Scheme in this model.

REFERENCES

[1] A. Brooke-Taylor, *Large cardinals and L-like combinatorics*, Ph.D. thesis, University of Vienna, 2007.

[2] F. R. Drake and D. Singh, *Intermediate Set Theory*. John Wiley & Sons Inc, Chichester, 1996.

[3] S.-D. Friedman, *Fine Structure and Class Forcing*, de Gruyter Series in Logic and its Applications, vol. 3, de Gruyter, Berlin and New York, 2000.

[4] S.-D. Friedman, V. Gitman, and V. Kanovei, *A model of second-order arithmetic satisfying AC but not DC*. *Journal of Mathematical Logic*, vol. 19 (2019), no. 1, pp. 1850013-1–1850013-39.

[5] H. Gaifman, *Elementary embeddings of models of set-theory and certain subtheories*, *Axiomatic Set Theory (Proceedings of the Symposium on Pure Mathematics, vol. XIII, part II, University of California, Los Angeles, California, 1967)*, American Mathematical Society, Providence, RI, 1974, pp. 33–101.

[6] V. Gitman, J. D. Hamkins, P. Holy, P. Schlicht, and K. Williams, *The exact strength of the class forcing theorem*, this Journal, vol. 85 (2020), no. 3, pp. 869–905.

[7] V. Gitman, J. D. Hamkins, and T. A. Johnstone, *What is the theory ZFC without power set?* *Mathematical Logic Quarterly*, vol. 62 (2016), nos. 4–5, pp. 391–406.

[8] V. Gitman and R. Matthews, *Dependent choice principles without power set*, 2022, in preparation.

[9] P. Holy, R. Krapf, P. Lücke, A. Njegomir, and P. Schlicht, *Class forcing, the forcing theorem and Boolean completions*, this Journal, vol. 81 (2016), no. 4, pp. 1500–1530.

[10] P. Holy, R. Krapf, and P. Schlicht, *Characterizations of pretameness and the Ord-cc*. *Annals of Pure and Applied Logic*, vol. 169 (2018), no. 8, pp. 775–802.

---

[8]A symmetric system is tenacious if there is a dense class of conditions $p \in \mathbb{P}$ for which $\{\pi \in \mathcal{G} : \pi p = p\} \in \mathcal{F}$. Any of the class symmetric systems we have defined are tenacious.

[11] ———, *Sufficient conditions for the forcing theorem, and turning proper classes into sets*. **Fundamenta Mathematicae**, vol. 246 (2019), pp. 27–44.

[12] T. Jech, **Set Theory**, third millennium ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[13] A. Kanamori, **The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings**, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.

[14] A. Karagila, *Preserving dependent choice*. **Bulletin of the Polish Academy of Sciences Mathematics**, vol. 67 (2019), no. 1, pp. 19–29.

[15] ———, *Approaching a Bristol model*, 2020, to appear.

[16] ———, Personal communications.

[17] K. Kunen, *Elementary embeddings and infinitary combinatorics*, this Journal, vol. 36 (1971), no. 3, pp. 407–413.

[18] A. Lévy, *The interdependence of certain consequences of the axiom of choice*. **Fundamenta Mathematicae**, vol. 54 (1964), no. 2, pp. 135–157.

[19] R. Matthews, **Large cardinals in weakened axiomatic theories**, Ph.D. thesis, University of Leeds, 2021.

[20] G. P. Monro, *Independence results concerning Dedekind-finite sets*. **Journal of the Australian Mathematical Society**, vol. 19 (1975), no. 1, pp. 35 – 46.

[21] R. Solovay, W. Reinhardt, and A. Kanamori, *Strong axioms of infinity and elementary embeddings*. **Annals of Mathematical Logic**, vol. 13 (1978), no. 1, pp. 73–116.

[22] A. Suzuki, *No elementary embedding from V into V is definable from parameters*, this Journal, vol. 64 (1999), no. 4, pp. 1591–1594.

SCHOOL OF MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS, LS2 9JT, UK
E-mail: R.M.A.Matthews@leeds.ac.uk