A NOTE ON GENERALIZED UNIQUE EXTENSION OF MEASURES*

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In Theorem 1, we shall discuss some properties of semifinite measure, that is, the measure μ on a ring R of sets with the property that, for every E in R, $\mu(E)$ is equal to the least upper bound of $\mu(F)$ where F runs over sets such that F is in R ($F \subset E$) and $\mu(F) < \infty$. Let $\sigma(R)$ be the σ-ring generated by R. To prove Theorem 2 we shall use the uniqueness theorem in Luther's paper [2], which is stated as a lemma in this paper. Theorem 2 is to the effect that for measures μ_1 and μ_2 on $\sigma(R)$, $\mu_1 \leqq \mu_2$ on R implies $\mu_1 \leqq \mu_2$ provided that $\overline{\mu_i/R}$ $(i = 1,2)$ is semifinite on $\sigma(R)$. Here $\overline{\mu_i/R}$ is the restriction, on $\sigma(R)$, of the outer measure $(\mu_i/R)^*$ induced by the restricted measure μ_i/R of μ_i on R. Definitions of terms are the same as [1] and [2].

Fix a set X. Let R be a ring of subsets of X and μ a measure on R. Let $\sigma(R)$ be the σ-ring generated by R, μ^* the outer measure induced by μ on the hereditary σ-ring $H(R)$ generated by R and let $\bar{\mu}$ be the restriction of μ^* to $\sigma(R)$, that is, $\bar{\mu} = \mu^*/\sigma(R)$. Then $\bar{\mu}$ is a measure on $\sigma(R)$. In [2] Luther showed that semifiniteness of $\bar{\mu}$ implies that of μ on R and that the semifiniteness of μ can not imply that of $\bar{\mu}$. We can prove the following:

THEOREM 1. *If the measure μ is semifinite on R and if for every $A \in \sigma(R)$ there is an F in R ($F \subset A$) such that $\bar{\mu}(A) = \mu(F)$ then $\bar{\mu}$ is semifinite.*

PROOF. For every A in $\sigma(R)$, there is an F in R ($F \subset A$) such that

$$\begin{aligned}
\bar{\mu}(A) &= \mu(F) \\
&= \sup\{\mu(G) : G \subset F,\ \mu(G) < \infty,\ G \in R\} \\
&\leqq \sup\{\bar{\mu}(G) : G \subset A,\ \bar{\mu}(G) < \infty,\ G \in \sigma(R)\} \\
&\leqq \bar{\mu}(A).
\end{aligned}$$

Hence $\bar{\mu}$ is semifinite.

REMARK. The converse of Theorem 1 is not true. For example, let $X = [0, 1]$,

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$$R_n = \left\{A\colon A \text{ is Lebesgue measurable and } A \supset \left[0, \frac{1}{2} + \frac{1}{n+1}\right] \text{ or } A \subset \left(\frac{1}{2} + \frac{1}{n+1}, 1\right)\right\},$$

$(n = 1, 2, \cdots)$ and let $R = \bigcup_1^\infty R_n$. Then R is a ring (actually is an algebra). Let μ be the Lebesgue measure restricted to R. Then $[0, \frac{1}{2}]$ is in $\sigma(R)$, so $F \subset [0, \frac{1}{2}]$, F in R (and $\mu(F) < \infty$) implies $F = \varnothing$.* Further, semifiniteness of μ can not imply that A in $\sigma(R)$ yields the existence of an E in R such that $\bar{\mu}(A) = \mu(E)$. Moreover, we can not get semifiniteness of μ even if also A in $\sigma(R)$ implies the existence of an F in R ($F \subset A$) satisfying $\bar{\mu}(A) = \mu(F)$. For example, let X be any infinite set and R the ring of all finite subsets of X. Define μ on R by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \varnothing, \\ \infty & \text{if } E \neq \varnothing. \end{cases}$$

The following lemma is due to Luther [2].

LEMMA. *Let μ be a measure on a ring R. If $\bar{\mu}$ is semifinite on $\sigma(R)$ then there exists a unique extension of μ to $\sigma(R)$.*

By using this lemma we shall prove the following:

THEOREM 2. *Let $\mu_i (i = 1, 2, \cdots)$ be measures on $\sigma(R)$. If $\overline{\mu_i/R}$ $(i = 1, 2)$ is semifinite on $\sigma(R)$ and if $\mu_1 \leqq \mu_2$ on R, then $\mu_1 \leqq \mu_2$.*

PROOF. Let $M = \{E \in \sigma(R)\colon \mu_1(E) \leqq \mu_2(E)\}$. Clearly, $M \supset R$. First we note that, if μ_1 and μ_2 are finite measures on $\sigma(R)$, then $\mu_1 \leqq \mu_2$ on $\sigma(R)$. In fact, it is easy to see that M is a monotone class. Hence $M \supset \sigma(R)$. This proves that

(i) for finite measures μ_1 and μ_2, $\mu_1(E) \leqq \mu_2(E)$ for all $E \in \sigma(R)$.

Let $v_i = \mu_i/R$ $(i = 1, 2)$. Then $\bar{v}_i$ is semifinite and $\mu_i = \bar{v}_i$ on R. By the lemma, we can obtain

(ii) $\bar{v}_i = \mu_i$ on $\sigma(R)$ for $i = 1, 2$.

Choose $E \in \sigma(R)$; in proving that $\mu_1(E) \leqq \mu_2(E)$, one may assume that $\mu_2(E) < \infty$. By semifiniteness of $\bar{v}_1$, we can find $F \in \sigma(R)$ $(F \subset E)$ with $\bar{v}_1$-σ-finite measure such that $\bar{v}_1(E) = \bar{v}_1(F)$. Hence there is a sequence $\{F_n\}$ of sets in R such that $F \subset \bigcup_1^\infty F_n$ and $v_1(F_n) < \infty$. Since by (ii)

$$\bar{v}_2(F) = \mu_2(F) \leqq \mu_2(E) < \infty,$$

there is a sequence $\{G_n\}$ of sets in R such that $F \subset \bigcup_1^\infty G_n$ and $v_2(G_n) < \infty$. Hence we can suppose that

* I know this example from Dr. N. Y. Luther.

$$F \subset \bigcup_1^\infty H_n,\ H_n \in R,\ \nu_i(H_n) < \infty\ (i = 1,2;\ n = 1,2,\cdots) \text{ and } H_j \cap H_k = \emptyset\ (j \neq k).$$

Therefore we see $F = \bigcup_1^\infty (H_n \cap F)$ and $\mu_i(H_n) < \infty$ and we get

$$\begin{aligned}\mu_1(F) &= \sum_1^\infty \mu_1(H_n \cap F) = \sum_1^\infty (\mu_1)_{H_n}(F) \leqq \sum_1^\infty (\mu_2)_{H_n}(F) \qquad \text{(by (i))}\\ &= \sum_1^\infty \mu_2(H_n \cap F) = \mu_2(F) \leqq \mu_2(E),\end{aligned}$$

which leads to the required inequality,

$$\mu_1(E) = \bar{\nu}_1(E) = \bar{\nu}_1(F) = \mu_1(F) \leqq \mu_2(E). \qquad \text{(by (ii))}$$

REMARK. If we drop the hypothesis that $\overline{\mu_i/R}$ is semifinite, then the result is false, even though $\bar{\mu}_1$ and $\bar{\mu}_2$ are semifinite or μ_1 and μ_2 are σ-finite, as the following example shows.

EXAMPLE. Let R be a ring of subsets of a countable set X with the property that every non-empty set in R is infinite and such that $\sigma(R)$ is the class of all subsets of X. If, for every subset E of X, $\mu_1(E)$ is the number of points in E and $\mu_2(E) = \frac{1}{2}\mu_1(E)$, then μ_1 and μ_2 are σ-finite on $\sigma(R)$ and $\overline{\mu_1/R}$ and $\overline{\mu_2/R}$ are not semifinite but $\bar{\mu}_i = \mu_i$ $(i = 1,2)$ is σ-finite (hence semifinite) on $\sigma(R)$. In this case $\mu_1 \leqq \mu_2$ on R but $\mu_1 \geqq \mu_2$ and $\mu_1 \neq \mu_2$ on $\sigma(R)$.

COR. 1. *Suppose R is a ring, and μ_1 and μ_2 are measures on $\sigma(R)$ such that (i) $\mu_1(E) \leqq \mu_2(E)$ for all E in R, and (ii) μ_i/R is σ-finite. Then $\mu_1 \leqq \mu_2$ on $\sigma(R)$.*

PROOF. Obviously, $\overline{\mu_i/R}$ is σ-finite and hence semifinite.

COR. 2. *Let μ_i $(i = 1,2)$ be measure on $\sigma(R)$. If μ_i/R $(i = 1,2)$ is semifinite and for every A in $\sigma(R)$ there is an F in R $(F \subset A)$ such that*

$$\overline{\mu_i/R}\,(A) = \mu_i/R\,(F)$$

and if $\mu_1 \leqq \mu_2$ on R, then $\mu_1 \leqq \mu_2$.

PROOF. By Theorem 1, $\overline{\mu_i/R}$ $(i = 1,2)$ is semifinite, and by Theorem 2, we get $\mu_1 \leqq \mu_2$.

References

[1] S. K. Berberian, *Measure and Integration*, Macmillan (New York, 1965).
[2] N. Y. Luther, 'Unique Extension and Product Measures', *Canad. J. Math.* 19 (1967), 757–763.

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