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THE L^2-SINGULAR DICHOTOMY FOR EXCEPTIONAL LIE GROUPS AND ALGEBRAS

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Abstract

We show that every orbital measure, μ_x, on a compact exceptional Lie group or algebra has the property that for every positive integer either $\mu_x^k \in L^2$ and the support of μ_x^k has non-empty interior, or μ_x^k is singular to Haar measure and the support of μ_x^k has Haar measure zero. We also determine the index k where the change occurs; it depends on properties of the set of annihilating roots of x. This result was previously established for the classical Lie groups and algebras. To prove this dichotomy result we combinatorially characterize the subroot systems that are kernels of certain homomorphisms.

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1. Introduction

This paper was motivated by a classical result of Ragozin, which states that whenever G is a compact, connected, simple Lie group, then the convolution of dimension of G, continuous, G-invariant measures is absolutely continuous with respect to Haar measure. In a series of papers, culminating in [4–6], one of the authors, with various coauthors, found the minimum number of convolution powers which gives the absolute continuity property for such measures. The number depends on the Lie type of G, but is roughly the rank of G.

The proof of this result involved the study of orbital measures, the continuous, G-invariant measures supported on conjugacy classes in the group G or on adjoint orbits in the associated Lie algebra $\mathfrak{g}$. A striking L^2-singular dichotomy was discovered for these measures in the case of the *classical* Lie groups and algebras: for every x belonging to G (or $\mathfrak{g}$) there is an integer $k(x)$ such that if μ_x is the orbital measure supported on the conjugacy class $C_x \subseteq G$ (respectively, the adjoint orbit $O_x \subseteq \mathfrak{g}$)

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generated by x, then $\mu_x^k \in L^2$ for all $k \geq k(x)$; while μ_x^k is purely singular to Haar measure if $k < k(x)$. Moreover, the k-fold product of C_x, denoted C_x^k (respectively, the k-fold sum, $(k)O_x$), has non-empty interior when $k \geq k(x)$ and the Haar measure of C_x^k (or of $(k)O_x$) is zero when $k < k(x)$. The value of $k(x)$ was also determined.

In this paper we complete this investigation, showing that the same L^2-singular dichotomy holds for all the orbital measures on all the *exceptional* compact, connected, simple Lie groups and algebras. We also find the value of $k(x)$.

A new ingredient in the proof, which may be of independent interest, is a combinatorial characterization of the subroot systems that are kernels of certain homomorphisms of the root systems of the Lie groups or algebras (and this includes all the maximal subroot systems of the classical root systems). Our characterization is in terms of properties of the nodes of the extended Dynkin diagram which are removed to produce a base for the subroot system. This characterization is relevant to the L^2-singular dichotomy problem because the answers for $k(x)$ depend upon combinatorial properties of the subroot systems consisting of the roots α with $\alpha(x) = 0$ (in the algebra case) or belonging to $\mathbb{Z}$ (in the group case).

Other methods have been used to study sums of orbits and convolutions of orbital measures in [1, 3, 12, 14], for example.

2. Characteristic of a subroot system

Let Φ be an irreducible root system in the Euclidean space E, as defined in [8, Section 9.2], with base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and rank n. Let α_0 be the highest root in Φ with respect to Δ and write

$$\alpha_0 = \sum_{i=1}^{n} h_i \alpha_i.$$

The extended base, $\tilde{\Delta}$, is defined as $\Delta \cup \{-\alpha_0\}$, and the Dynkin diagram determined by this extended base is called the extended Dynkin diagram of Φ. It is well known that if Φ is an irreducible root system of rank n, then the Dynkin diagram of Φ is one of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2. For the convenience of the reader these Dynkin diagrams and some of the basic facts about irreducible root systems can be found in the Appendix. Other basic properties of roots and root systems are discussed in [8, 10].

Every proper subset Δ' of $\tilde{\Delta}$ is the base for a subroot system[1] Ψ, a subset of Φ that is a root system in its own right. Just take $\Psi = \mathrm{span}_{\mathbb{Z}}(\Delta') \cap \Phi$ and check that every element of Ψ can be written as either all positive or all negative linear combinations of roots in Δ'. We will say that Ψ is *obtained by deleting roots D* from the extended base, where $D = \tilde{\Delta} \setminus \Delta'$. The type of Ψ is determined by its Dynkin diagram, which is a subdiagram of the extended Dynkin diagram.

[1] We use the terminology from [9]. Others say 'root subsystem'.

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We denote the usual inner product on E by $(\cdot,\cdot)$. Every subroot system of the form $\Psi = \Phi(t)$, where

$$\Phi(t) = \{\alpha \in \Phi : (\alpha, t) \in \mathbb{Z}\}, \tag{2.1}$$

is $\mathbb{Z}$-closed, meaning $\operatorname{span}_{\mathbb{Z}} \Psi \cap \Phi = \Psi$, and is known to be Weyl conjugate to one obtained by deleting roots from the extended base as described above [9, Section 12.4].

The Borel–de Siebenthal theorem (see [9, Section 12.1]) implies that, up to Weyl equivalence, all the maximal $\mathbb{Z}$-closed subroot systems are those with bases of the form

(1) $\{\alpha_1, \alpha_2, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n\}$ when $h_i = 1$, or
(2) $\{-\alpha_0, \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n\}$ when h_i is prime,

where $\hat{}$ denotes elimination. This motivates the following definition.

Definition 2.1. Suppose that Ψ is a subroot system of Φ that is Weyl conjugate to one obtained by deleting roots D from the extended base $\{-\alpha_0, \alpha_1, \ldots, \alpha_n\}$ of Φ.

(i) If $D = \{-\alpha_0, \alpha_{i_1}, \ldots, \alpha_{i_r}\}$, then Ψ is said to have characteristic $(0, h_{i_1}, \ldots, h_{i_r})$ and we write $\operatorname{char}_\Phi \Psi = (0, h_{i_1}, \ldots, h_{i_r})$.

(ii) If $D = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ with $i_j \neq 0$, then Ψ is said to have characteristic $(h_{i_1}, \ldots, h_{i_r})$ and we write $\operatorname{char}_\Phi \Psi = (h_{i_1}, \ldots, h_{i_r})$.

The characteristic of a subroot system need not be unique, but this will not present a serious complication.

Note that in this language, the Borel–de Siebenthal theorem states that the maximal $\mathbb{Z}$-closed subroot systems are those of characteristic $(0, 1)$ or (p) for p prime.

If Ψ is characteristic $(0, h_{i_1}, \ldots, h_{i_r})$, then $\{\alpha_1, \ldots, \widehat{\alpha_{i_1}}, \ldots, \widehat{\alpha_{i_r}}, \ldots, \alpha_n\}$ is a base for Ψ, and any root $x \in \Psi$ can be written uniquely as $x = \sum_{j=1}^n t_j\alpha_j$, with $t_j \in \mathbb{Z}$ and $t_j = 0$ if $j \in \{i_1, \ldots, i_r\}$. We call t_j the α_j-coefficient of x and denote it by $[x]_j$. Thus $x \in \Psi$ if and only if $[x]_j = 0$ for all $j \in \{i_1, \ldots, i_r\}$.

If Ψ is characteristic $(h_{i_1}, \ldots, h_{i_r})$, then $\{-\alpha_0, \alpha_1, \ldots, \widehat{\alpha_{i_1}}, \ldots, \widehat{\alpha_{i_r}}, \ldots, \alpha_n\}$ is a base for Ψ, and any root $x \in \Psi$ can be written uniquely as $x = -m\alpha_0 + \sum_{j=1}^n t_j\alpha_j$, with $m, t_j \in \mathbb{Z}$ and $t_j = 0$ if $j \in \{i_1, \ldots, i_r\}$. But $-m\alpha_0 = -m\sum h_i\alpha_i$, hence the α_{i_j}-coefficient of x is $[x]_{i_j} = -mh_{i_j}$. Since h_{i_j} is the α_{i_j}-coefficient of the highest root, m can only equal 0 or ± 1. This shows that if $x \in \Psi$, either $[x]_{i_j} = \overset{+}{\underset{(-)}{}} h_{i_j}$ for all $j = 1, \ldots, r$, or $[x]_{i_j} = 0$ for all $j = 1, \ldots, r$. One can easily see that the converse is also true.

Given Ψ of characteristic $(0, h_{i_1}, \ldots, h_{i_r})$ or $(h_{i_1}, \ldots, h_{i_r})$, we define ρ_Ψ on E by

$$\rho_\Psi(x) = \rho(x) = [x]_{i_1} + [x]_{i_2} + \cdots + [x]_{i_r}.$$

Since any root has the property that all $[x]_j$ are of the same sign, these arguments show the following characterization of Ψ.

Proposition 2.2.

(i) *A root x belongs to the subroot system Ψ with characteristic $(0, h_{i_1}, \ldots, h_{i_r})$ if and only if $\rho_\Psi(x) = 0$.*
(ii) *A root x belongs to Ψ with characteristic $(h_{i_1}, \ldots, h_{i_r})$ if and only if either $\rho_\Psi(x) = \pm(h_{i_1} + \cdots + h_{i_r}) = \pm\rho_\Psi(\alpha_0)$ or $\rho_\Psi(x) = 0$.*

This will be a useful criterion for determining whether a given root belongs to Ψ.

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Given $\phi \subseteq \Phi$, we let Z_ϕ denote the $\mathbb{Z}$-span of ϕ and $Q_\phi = Z_\Phi / Z_\phi$, the quotient of abelian groups. For $\mathbb{Z}$-closed subroot systems Ψ, $Z_\Psi \cap \Phi = \Psi$. Of course, $Z_\Phi = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$. If Ψ has characteristic $(0, h_{i_1}, \ldots, h_{i_r})$, then

$$Z_\Psi = \mathbb{Z}\alpha_1 + \widehat{\mathbb{Z}\alpha_{i_1}} + \cdots + \widehat{\mathbb{Z}\alpha_{i_r}} + \cdots + \mathbb{Z}\alpha_n$$

(where, as usual, $\hat{}$ denotes omission), hence $Q_\Psi \simeq \mathbb{Z}^r$. If Ψ has characteristic $(h_{i_1}, \ldots, h_{i_r})$, then

$$Z_\Psi = \mathbb{Z}\alpha_1 + \widehat{\mathbb{Z}\alpha_{i_1}} + \cdots + \widehat{\mathbb{Z}\alpha_{i_r}} + \cdots + \mathbb{Z}\alpha_n + \mathbb{Z}(h_{i_1}\alpha_{i_1} + \cdots + h_{i_r}\alpha_{i_r}),$$

so

$$Q_\Psi \simeq \mathbb{Z}^r / \mathbb{Z}(h_{i_1}, \ldots, h_{i_r}) \simeq \mathbb{Z}^{r-1} \times \mathbb{Z}_m,$$

where $m = \gcd(h_{i_1}, \ldots, h_{i_r})$.

Example 2.3. There are two subroot systems of type A_7 contained in a root system of type E_8, one of which has characteristic $(0, 3)$ and the other characteristic $(2, 4)$. These cannot be Weyl conjugate as the one has a torsion-free quotient, while the other has a quotient with $\mathbb{Z}_2$ as its torsion subgroup.

Subroot systems of characteristic $(0, h_{i_1}, \ldots, h_{i_r})$ are well behaved as they are obtained by deleting roots from the original (unextended) base Δ of Φ. The above observation shows the quotient space, Q_Ψ, is torsion-free (we say that Ψ is torsion-free). The converse is true as well. To prove this, we first record a useful fact about extending bases that was shown to us by Wright.

We will say that Ψ is $\mathbb{R}$-closed if $\operatorname{span}_\mathbb{R} \Psi \cap \Phi = \Psi$.

Lemma 2.4. *If Ψ is $\mathbb{R}$-closed, then any base of Ψ can be extended to a base for Φ.*

Proof. Without loss of generality, we may assume Ψ is of codimension one, for otherwise we proceed by induction.

As Ψ is $\mathbb{R}$-closed, we may choose $y \in E$ such that $(\alpha, y) = 0$ for all $\alpha \in \Psi$, but $(\beta, y) \neq 0$ for any $\beta \in \Phi \setminus \Psi$. Let $\varepsilon = \min_{\beta \in \Phi \setminus \Psi} |(\beta, y)| > 0$. Choose $x \in E$ with $0 < |(\alpha, x)| < \varepsilon/2$ for all $\alpha \in \Phi$.

If $z = x + y$, then $(\alpha, z) = (\alpha, x) \neq 0$ for $\alpha \in \Psi$, while if $\alpha \in \Phi \setminus \Psi$, then $|(\alpha, z)| \geq |(\alpha, y)| - |(\alpha, x)| \geq \varepsilon/2$. Hence if we let $\Phi_z^+ = \{\alpha \in \Phi : (\alpha, z) > 0\}$, then the indecomposable elements of Φ_z^+ (those which are not the sum of two elements of Φ_z^+) form a base of Φ [8, page 48].

The choice of z ensures that if $\alpha \in \Psi$ and $\beta \in \Phi \setminus \Psi$, then $|(\beta, z)| \geq |(\beta, y)| - |(\beta, x)| > \varepsilon/2$, while $|(\alpha, z)| = |(\alpha, x)| < \varepsilon/2$. In particular, if $\alpha \in \Psi$ and $\beta \in \Phi_z^+ \setminus \Psi$, then $(\alpha, z) < (\beta, z)$. This shows that if $\alpha \in \Psi$ is the sum of two roots in $\Phi_z^+ \setminus \Psi$, then both of these roots belong to $\Phi_z^+ \cap \Psi = \Psi_z^+$. Thus the indecomposable roots of Ψ_z^+ are also indecomposable roots of Φ_z^+, and therefore the base of Ψ consisting of the indecomposable roots of Ψ_z^+ extends to the base of Φ consisting of the indecomposable roots of Φ_z^+. $\square$

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EXAMPLE 2.5. The set $\{\alpha \in \Phi : (\alpha, t) = 0\}$ is $\mathbb{R}$-closed and hence a base can be found by removing roots from a base for Φ.

PROPOSITION 2.6. *Suppose that Ψ is $\mathbb{Z}$-closed and Q_Ψ is torsion-free. Then Ψ has characteristic* $(0, h_{i_1}, \ldots, h_{i_r})$.

PROOF. From the previous lemma, it will be enough to check that Ψ is $\mathbb{R}$-closed. We first verify that the torsion-free assumption implies Ψ is $\mathbb{Q}$-closed. So assume that there is a root $x \in \Phi$ which can be written as a $\mathbb{Q}$-linear combination of $\beta_i \in \Psi$, say $x = \sum_{j=1}^{N}(p_j/q_j)\beta_j$, where $p_j, q_j \in \mathbb{Z}$. Put $q = q_1 \cdots q_N$. Then $qx \in Z_\Psi$ and the element $x + Z_\Psi$ in the quotient space Q_Ψ satisfies $q(x + Z_\Psi) = 0$. Since Q_Ψ is torsion-free, this implies $x + Z_\Psi = 0$ in Q_Ψ, hence $x \in Z_\Psi \cap \Phi = \Psi$.

Now we check that any root in the $\mathbb{R}$-span of Ψ is also in the $\mathbb{Q}$-span. To see this, choose a base $\Delta' = \{\beta_1, \ldots, \beta_n\}$ of Ψ and let $\{\lambda_i\}$ be the base of E dual to $\{2\beta_i/(\beta_i, \beta_i)\}$. Given $x \in \operatorname{span}_{\mathbb{R}} \Psi \cap \Phi$, let $x = \sum_i a_i\lambda_i$ with $a_i \in \mathbb{R}$. The dual basis property ensures that $a_i = (x, 2\beta_i/(\beta_i, \beta_i))$, and this is an integer since both x and β_i are roots.

The Cartan matrix is the transition matrix from the basis $\{\beta_1, \ldots, \beta_n\}$ to the basis $\{\lambda_1, \ldots, \lambda_n\}$. By applying the inverse of the Cartan matrix, which has rational coefficients, to the tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, it follows that $x \in \operatorname{span}_{\mathbb{Q}}\{\beta_1, \ldots, \beta_n\} \subseteq sp_{\mathbb{Q}}\Psi$. Hence $sp_{\mathbb{R}}\Psi \cap \Phi = sp_{\mathbb{Q}}\Psi \cap \Phi = \Psi$. □

EXAMPLE 2.7. One can see from the extended Dynkin diagram of type E_6 that, up to Weyl conjugacy, there are at most three subroot systems of type A_5 in E_6 that are obtained by deleting roots from the extended Dynkin diagram. Two have characteristic $(1, 2)$ and the third, characteristic $(0, 2)$. But all three are torsion-free and as there is only one class of characteristic $(0, 2)$, they must coincide.

3. Intersections of subroot systems

DEFINITION 3.1. Let $\Psi \subseteq \Phi$ and H be an abelian group. We say that Ψ is an H-kernel in Φ if there is a homomorphism $\pi : Z_\Phi \to H$, with $\ker \pi \cap \Phi = \Psi$.

An H-kernel is always a $\mathbb{Z}$-closed, subroot system, and Weyl conjugates of H-kernels are again H-kernels.

LEMMA 3.2. *Suppose that $\Omega \subseteq \Phi$ is an irreducible subroot system. If Ψ is an H-kernel in Φ, then $\Psi \cap \Omega$ is an H-kernel in Ω.*

PROOF. Let $\pi : Z_\Phi \to H$ be a homomorphism with $\ker \pi \cap \Phi = \Psi$. Then $\ker \pi|_{Z_\Omega} \cap \Omega = \ker \pi \cap \Omega = (\ker \pi \cap \Phi) \cap \Omega = \Psi \cap \Omega$. Thus $\pi|_{Z_\Omega} : Z_\Omega \to H$ provides the desired homomorphism. □

EXAMPLE 3.3. If $\operatorname{char}_\Phi \Psi = (2)$, then Ψ is a $\mathbb{Z}_2$-kernel. To see this, suppose that Ψ is obtained by deleting the root $\alpha_i \in \Delta$ with $h_i = 2$. Define $\pi(x) = [x]_i \bmod 2 = \rho_\Psi(x) \bmod 2$. By Proposition 2.2, the root x belongs to Ψ if and only if $\rho(x) = 0$ or ± 2. As $|[y]_i| \le h_i = 2$ for all $y \in \Phi$, $\rho(y) = 0, \pm 2$ if and only if $\pi(y) = 0$, hence $\Psi = \ker \pi \cap \Phi$.

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Example 3.4. If Ψ is a $\mathbb{Z}_2$-kernel, then Ψ has the property that the sum of two roots belongs to Ψ if and only if either both roots belong to Ψ or neither root belongs to Ψ. The converse is also true. To see this, let $\Delta = \{\alpha_i\}_{i=1}^n$ be a base for Φ and define π_0 on Δ by $\pi_0(\alpha) = 0$ if $\alpha \in \Psi$ and 1 otherwise. Extend π_0 by linearity to Z_Φ and put $\pi(x) = \pi_0(x) \bmod 2$. As Δ is a base, this is a well-defined homomorphism. If $x \in \Phi$, we can write x as $x = \alpha_{i_1} + \cdots + \alpha_{i_k}$, where all the partial sums, $\alpha_{i_1} + \cdots + \alpha_{i_j}$, $j \le k$, are roots. An inductive argument can be applied to show that $x \in \Psi$ if and only if $\pi(x) = 0$.

One example of a pair Ψ, Φ with the property described in the second example is the set $\Psi = \{e_i \pm e_j : i, j \ne 1, \ldots, 8\}$ in the root system Φ of type E_8. The root system Ψ is of type D_8 and this has characteristic (2) in E_8. Another example is $\Psi = \{e_i \pm e_j : i, j = 1, \ldots, 4\}$ in Φ of type D_5. This Ψ is type D_4 and has characteristic $(0, 1)$ in D_5. We show in the next proposition that these two characteristics are the only possibilities for a $\mathbb{Z}_2$-kernel.

Proposition 3.5. *The following are equivalent for a subroot system Ψ of Φ:*

(1) Ψ *is a* $\mathbb{Z}_2$*-kernel in* Φ;
(2) $\Psi = \Phi$ *or* $\operatorname{char}_\Phi \Psi = (2)$ *or* $(0, 1)$.

Remark 3.6. We observe that all the maximal $\mathbb{Z}$-closed subroot systems of the classical root systems are of this form. (See [9, page 136] for a complete list.)

Proof. $(2 \Rightarrow 1)$ In Example 3.3 we saw that characteristic (2) subroot systems are $\mathbb{Z}_2$-kernels. The argument is similar if Ψ has characteristic $(0, 1)$. Assume that Ψ is obtained by deleting $\{-\alpha_0, \alpha_i\}$ from a base for Φ, with $h_i = 1$. Define $\pi : Z_\Phi \to \mathbb{Z}_2$ by $\pi(x) = [x]_i \bmod 2$. Since $|[x]_i| \le 1$, it follows that $x \in \Psi$ if and only if $\rho_\Psi(x) = 0$ if and only if $\pi(x) = 0$.

Of course, if $\Psi = \Phi$, just take the trivial homomorphism.

$(1 \Rightarrow 2)$ Let $\pi : Z_\Phi \to \mathbb{Z}_2$ be a homomorphism with $\Psi = \ker \pi \cap \Phi$. Suppose that $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a base for Φ and let $\{\lambda_1, \ldots, \lambda_n\}$ be the dual basis. Define k_i by

$$k_i = \begin{cases} 0 & \text{if } \pi(\alpha_i) = 0, \\ 1/2 & \text{if } \pi(\alpha_i) = 1, \end{cases}$$

and put $t = \sum_{i=1}^n k_i \lambda_i \in E$. If $x \in \Phi$, then $x = \sum [x]_i \alpha_i$ and

$$\pi(x) = \sum [x]_i \pi(\alpha_i) \bmod 2 = \sum [x]_i 2k_i \bmod 2 = 2(x, t) \bmod 2.$$

Thus $(x, t) \in \mathbb{Z}$ if and only if $\pi(x) = 0$. This shows Ψ is the root system $\Phi(t)$, as defined in (2.1), and therefore is Weyl conjugate to one obtained by deleting roots D from the extended basis $\widetilde{\Delta}$. There are two cases to consider.

Case 1. The highest root is one of the deleted roots, i.e., $-\alpha_0 \in D$.

If $D = \{-\alpha_0\}$, then $\Psi = \Phi$. Otherwise, $D = \{-\alpha_0, \alpha_{i_1}, \ldots, \alpha_{i_r}\}$ with $r \ge 1$ and $\operatorname{char}_\Phi \Psi = (0, h_{i_1}, \ldots, h_{i_r})$. By definition, $\rho_\Psi(\alpha_0) = h_{i_1} + \cdots + h_{i_r}$.

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Suppose, first, that $\rho(\alpha_0) \geq 2$. Write the highest root as $\alpha_0 = \alpha_{k_1} + \cdots + \alpha_{k_N}$ so that each partial sum is a root. As $\rho(\alpha_j) = 0$ or 1, there must be a partial sum $x = \alpha_{k_1} + \cdots + \alpha_{k_m} \in \Phi$ such that $\rho(x) = \sum_{j=1}^{r} [x]_{i_j} = 2$.

Since $\Psi = \ker \pi \cap \Phi$, we have $\pi(\alpha_{i_k}) = 0$ for $k = 1, \ldots, r$ and $\pi(\alpha_j) = 1$ for $\alpha_j \in \widetilde{\Delta} \setminus D$. Consequently,

$$\pi(x) = \sum_{j=1}^{r} [x]_{i_j} \bmod 2 = \rho(x) \bmod 2 = 0.$$

As $x \in \Phi$, this implies $x \in \Psi$. But the fact that $\rho(x) = 2$ contradicts Proposition 2.2(i).

As $\rho(\alpha_0)$ is integer-valued, this implies $\rho(\alpha_0) \leq 1$, and since each $h_i \geq 1$ it follows that $r = 1$ and $h_{i_1} = 1$. In other words, $\mathrm{char}_\Phi \Psi = (0, 1)$.

Case 2. $\alpha_0 \notin D$, say $D = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ and $\mathrm{char}\,\Psi = (h_{i_1}, \ldots, h_{i_r})$ for some $r \geq 1$.

If $\rho(\alpha_0) \geq 3$, the same reasoning as above shows there is some $x \in \Phi$ with $\rho(x) = 2$. But then $\pi(x) = 0$, so $x \in \Psi$. But $\rho(x)$ is neither 0 nor $\pm\rho(\alpha_0)$ and that contradicts Proposition 2.2(ii).

Hence $\rho(\alpha_0) \leq 2$. As $h_{i_j} \geq 1$, $r \leq 2$. If $r = 2$ then $h_{i_1} = h_{i_2} = 1$, $\rho(\alpha_0) = 2$ and $\mathrm{char}_\Phi \Psi = (1, 1)$. In this case, Q_Ψ is torsion-free and by Proposition 2.6 it also has characteristic $(0, h_{j_1}, \ldots, h_{j_k})$. This reduces to the first case.

If, instead, $r = 1$, then $h_{i_1} = 1$ or 2. If $h_{i_1} = 1$, again we conclude that Q_Ψ is torsion-free and we reduce to the first case. Otherwise $\mathrm{char}_\Phi \Psi = (2)$. □

Corollary 3.7. *Suppose that* $\Omega \subseteq \Phi$ *is an irreducible subroot system. If* $\mathrm{char}_\Phi \Psi = (2)$ *or* $(0, 1)$*, then either* $\Omega \subseteq \Psi$ *or* $\mathrm{char}_\Omega(\Psi \cap \Omega) = (2)$ *or* $(0, 1)$.

Proof. Combine Lemma 3.2 and the previous proposition. □

This can be improved when $\mathrm{char}_\Phi \Psi = (0, 1)$.

Corollary 3.8. *If* $\mathrm{char}_\Phi \Psi = (0, 1)$*, then either* $\Omega \subseteq \Psi$ *or* $\mathrm{char}_\Omega(\Psi \cap \Omega) = (0, 1)$.

Proof. We just need to show that $\mathrm{char}_\Omega(\Psi \cap \Omega) \neq (2)$. We proceed by contradiction and assume that $\Psi \cap \Omega$ is obtained from Ω by deleting the root β_k from a base $\{\beta_1, \ldots, \beta_m\}$ of Ω, where the highest root β_0 of Ω has 2 as its β_k coefficient.

As $\mathrm{char}_\Phi \Psi = (0, 1)$, we can assume that Ψ is obtained from Φ by deleting $-\alpha_0$ and α_i, from the extended base of Φ, with $h_i = 1$. Since $\beta_k \notin \Psi$ and $\beta_j \in \Psi$ for $j \neq k$, $[\beta_j]_i = 0$, while $[\beta_k]_i \neq 0$. Thus $|[\beta_0]_i| = 2|[\beta_k]_i| \geq 2$. But this contradicts the fact, implied by $h_i = 1$, that $|[x]|_i \leq 1$ for all roots $x \in \Phi$. □

Corollary 3.9. *Suppose that* $\Omega \subseteq \Phi$ *is irreducible,* $\mathrm{char}_\Phi \Psi_1 = (2)$ *and* $\mathrm{char}_\Phi \Psi_2 = (0, 1)$*. Lower bounds on the cardinalities of* $\Psi_1 \cap \Omega$ *and* $\Psi_2 \cap \Omega$ *are as given below, where the notation* $\lfloor s \rfloor$ *denotes the least integer greater than or equal to* s.

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Type Ω	A_n	B_n or C_n	D_n	E_6	E_7	E_8
$\min\lvert\Psi_1 \cap \Omega\rvert$	$\left\lfloor \frac{n^2-1}{2} \right\rfloor$	$n^2 - n$	$n^2 - 2n$	32	56	112
$\min\lvert\Psi_2 \cap \Omega\rvert$	$\left\lfloor \frac{n^2-1}{2} \right\rfloor$	$n^2 - n$	$n^2 - n$	40	72	–

Proof. The subroot system $\Psi_1 \cap \Omega$ is either equal to Ω or has characteristic (2) or (0, 1) in Ω. By looking at the extended Dynkin diagrams it is easy to determine what the possibilities are for $\Psi_1 \cap \Omega$. For instance, when Ω is type D_n, the characteristic (2) subroot systems are type $D_k \times D_{n-k}$ with $k, n-k \geq 2$ (where D_2 means type $A_1 \times A_1$) and the characteristic (0, 1) subroot systems are type D_{n-1} or type A_{n-1}. When Ω is type A_n, they are type A_{n-1} and $A_j \times A_{n-j-1}$ for $j, n-j-1 \geq 1$. For $\Omega = E_6$ the only possibilities are type $A_5 \times A_1$ and D_5. The other types are similar. It is a simple calculus exercise to determine the minimal cardinality in each case.

The arguments for Ψ_2 are similar. □

We will also characterize $\mathbb{Z}_3$-kernels. First, we state two elementary combinatorial facts.

Lemma 3.10.

(i) *If* $(\alpha_{j_1}, \ldots, \alpha_{j_r})$ *is a connected string of nodes in a Dynkin diagram, then* $\alpha_{j_1} + \cdots + \alpha_{j_r} \in \Phi$.

(ii) *Suppose that some nodes of a connected diagram are coloured* 1*, some* 2 *and the rest (possibly none) are coloured* 0*. Then there is a connected string of nodes such that one of the end nodes of the string is coloured* 1*, the other end node is coloured* 2 *and the interior of the string are nodes coloured* 0*.*

Proof. (i) We prove this by induction on the length r of the string. It is clearly true if $r = 1$. Assume that the result is true for length $r-1$; hence $\alpha_{j_1} + \cdots + \alpha_{j_{r-1}} \in \Phi$. Since a Dynkin diagram contains no cycle and an edge between two nodes means a negative inner product between the two simple roots, we see that

$$(\alpha_{j_1} + \cdots + \alpha_{j_{r-1}}, \alpha_{j_r}) < 0,$$

and this implies that their sum belongs to Φ.

(ii) Just pick one node of colour 1 and the other of colour 2, and connect them. If there is any node of colour 1 or 2 in the interior of the string, pick the string with that endpoint as one node and the original endpoint of the other colour as the other node. This string has shorter length than the original. Repeat this process until we get the desired string after finitely many applications. □

Proposition 3.11. *Let* Ψ *be a subroot system of* Φ*. The following are equivalent:*

(1) Ψ *is a* $\mathbb{Z}_3$*-kernel in* Φ;

(2) $\Psi = \Phi$ *or* $\mathrm{char}_\Phi \Psi = (3)$, $(0, 2)$, $(0, 1, 1)$ *or* $(0, 1)$.

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Proof. ($2 \Rightarrow 1$) Take $\pi(x) = \rho_\Psi(x) \bmod 3$ (or π to be the trivial homomorphism if $\Psi = \Phi$). In each case one can check that $\pi(x) = 0$ if and only if $x \in \Psi$ by arguments similar to those used in the proof of Proposition 3.5.

($1 \Rightarrow 2$) We begin in a similar fashion to the proof of Proposition 3.5. Let $\pi : Z_\Phi \to \mathbb{Z}_3$ be a homomorphism with $\Psi = \ker \pi \cap \Phi$, suppose that $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is a base for Φ and let $\{\lambda_1, \dots, \lambda_n\}$ be the dual base. Define k_i by

$$k_i = \begin{cases} 0 & \text{if } \pi(\alpha_i) = 0, \\ 1/3 & \text{if } \pi(\alpha_i) = 1, \\ 2/3 & \text{if } \pi(\alpha_i) = 2, \end{cases}$$

and put $t = \sum_{i=1}^n k_i \lambda_i \in E$. As before, $\Psi = \Phi(t)$ and hence can be obtained by deleting roots D from the extended base for Φ.

Case 1. $-\alpha_0 \in D$. If $D = \{-\alpha_0\}$, then $\Psi = \Phi$. Otherwise, $D = \{-\alpha_0, \alpha_{i_1}, \dots, \alpha_{i_r}\}$ with $r \geq 1$. First, suppose that π is not constant on $\{\alpha_{i_1}, \dots, \alpha_{i_r}\}$. Think of colouring each $\alpha \in \Delta$ by the value of $\pi(\alpha)$. Since $\pi(\alpha_{i_j}) \neq 0$ we are in the situation of Lemma 3.10(ii) and we choose a connected string whose endpoints are not in Ψ, but all of whose interior nodes are. By the first part of Lemma 3.10, the sum of the nodes is a root z. Notice that $\pi(z) = 1 + 2 + 0 \equiv 0 \bmod 3$, so $z \in \Psi$. But $\rho_\Psi(z) = 2$ since precisely two deleted simple roots appear in the expression for z. This contradicts Proposition 2.2.

Thus π is constant on $\{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, say $\pi(\alpha_{i_j}) = c$ for all j. Of course, $c = 1$ or 2. If $\rho(\alpha_0) \geq 3$, then as in the proof of Proposition 3.5, there must exist some root $x \in \Phi$, such that $\rho(x) = 3$. Proposition 2.2 implies $x \notin \Psi$. But $\pi(x) = c\rho(x) \bmod 3 \equiv 0 \bmod 3$, which implies $x \in \Psi$.

Hence $\rho(\alpha_0) \leq 2$ and this gives only the possibilities $\mathrm{char}_\Phi \Psi = (0, 2)$, $(0, 1)$ or $(0, 1, 1)$.

Case 2. $-\alpha_0 \notin D$, say $D = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$. As in Case 1, if π is not constant on D we can obtain a root $z \in \Psi$ with $\rho(z) = 2$. Proposition 2.2 requires $\rho(z) = \pm\rho(\alpha_0)$ or 0, so we must have $\rho(\alpha_0) = h_{i_1} + \cdots + h_{i_r} = 2$. Notice that $r \geq 2$ as π takes on two different values on D. Thus $r = 2$ and $h_{i_1} = h_{i_2} = 1$. But then Q_Ψ is torsion-free and this reduces the problem to Case 1.

Hence we can assume that π is constant on D. If $\rho(\alpha_0) \geq 4$, there exists a root y such that $\rho(y) = 3$. But then $\pi(y) = 0$, so $y \in \Psi$ and this is a contradiction as $\rho(y) \neq \pm\rho(\alpha_0)$ or 0. Hence $\rho(\alpha_0) \leq 3$. If any $h_{i_j} = 1$, then Q_Ψ is torsion-free and this reduces to the first case. Otherwise, $r = 1$ and $h_{i_1} = 2$ or 3. However, if $h_{i_1} = 2$, then $\pi(\alpha_0) \equiv 2\pi(\alpha_{i_1}) \equiv 2$ or $4 \bmod 3$, implying $\alpha_0 \notin \Psi$. As this is false, we must have $\mathrm{char}_\Phi \Psi = (3)$. □

Corollary 3.12. *Suppose that Ψ, Ω are subroot systems of Φ and Ω is irreducible. Assume that $\Psi \cap \Omega \neq \Omega$.*

(i) *If* $\mathrm{char}_\Phi \Psi = (3)$, *then* $\mathrm{char}_\Omega(\Psi \cap \Omega) = (3), (0, 2), (0, 1, 1)$ *or* $(0, 1)$.
(ii) *If* $\mathrm{char}_\Phi \Psi = (0, 2)$, *then* $\mathrm{char}_\Omega(\Psi \cap \Omega) = (0, 2), (0, 1, 1)$ *or* $(0, 1)$.
(iii) *If* $\mathrm{char}_\Phi \Psi = (0, 1, 1)$, *then* $\mathrm{char}_\Omega(\Psi \cap \Omega) = (0, 1, 1)$ *or* $(0, 1)$.

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Proof. (i) is immediate from the previous result.

For (ii), assume that Ψ is obtained from Φ by deleting the roots $\{-\alpha_0, \alpha_i\}$ with $h_i = 2$, and that $\mathrm{char}_\Omega(\Psi \cap \Omega) = (3)$. Then $\Psi \cap \Omega$ is obtained from Ω by deleting one root, β_s, from the extended base $\{-\beta_0, \beta_1, \ldots, \beta_m\}$ for Ω (with highest root β_0) where the β_s-coefficient of β_0 is equal to 3. Since $\beta_j \in \Psi$ if and only if $j \neq s$, $[\beta_j]_i = 0$ for $j \neq s$ and $[\beta_s]_i \neq 0$. Thus $|[\beta_0]_i| = 3|[\beta_s]_i| \geq 3$ and that contradicts the fact that $h_i = 2$.

The arguments for (iii) are similar. □

From this we can also deduce the following corollary that will be useful later.

Corollary 3.13. *Suppose that $\Omega \subseteq \Phi$ is irreducible and $\mathrm{char}_\Phi \Psi = (0, 2)$ or $(0, 1, 1)$. Lower bounds on the cardinality of $\Psi \cap \Omega$ are given below.*

Type Ω	A_n	D_n	E_6	E_7	E_8
$\min \lvert\Psi \cap \Omega\rvert$	$\left\lfloor \dfrac{n^2 - n - 2}{3} \right\rfloor$	$\left\lfloor \dfrac{2(n^2 - 2n)}{3} \right\rfloor$	22	42	84

Proof. The subroot system $\Psi \cap \Omega$ either is equal to Ω or is a $\mathbb{Z}_3$-kernel in Ω. If Ω is type A_n, $\mathrm{char}_\Omega \Psi = (0, 1)$ or $(0, 1, 1)$. Those having characteristic $(0, 1)$ in A_n are listed in Corollary 3.9, and those of characteristic $(0, 1, 1)$ are type A_{n-2}, $A_j \times A_{n-j-2}$, or $A_j \times A_k \times A_{n-j-k-2}$ where all indices are positive. It is a routine exercise to check the minimal cardinality. The other types are similar. □

One can check that all the choices listed in Corollary 3.12 can be obtained; some examples are given below.

Example 3.14. Take $\Phi = \{e_i \pm e_j : i, j = 1, \ldots, 6\}$ of type D_6 and $\Psi = \{e_1 - e_2, e_i \pm e_j : i, j = 3, \ldots, 6\}$ a subroot system of Φ of type $A_1 \times D_4$. Then $\mathrm{char}_\Phi \Psi = (0, 2)$.

(a) If $\Omega = \{e_i \pm e_j : i, j = 2, \ldots, 6\}$ is of type D_5, then $\Omega \cap \Psi = \{e_i \pm e_j : i, j = 3, \ldots, 6\}$ is of type D_4 and $\mathrm{char}_\Omega \Omega \cap \Psi = (0, 1)$.

(b) If $\Omega = \{e_i \pm e_j : i, j = 1, 2, 4, 5, 6\}$, then $\Omega \cap \Psi$ is type $A_1 \times D_3$ and $\mathrm{char}_\Omega \Omega \cap \Psi = (0, 2)$.

(c) Finally, if Ω is type A_5 with base $\{e_1 + e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6\}$, then $\Omega \cap \Psi$ is type A_3 and $\mathrm{char}_\Omega \Omega \cap \Psi = (0, 1, 1)$.

Example 3.15. Take Φ of type E_8. If the node $e_2 - e_1$ is removed from the standard base of E_7 (in the notation of [8, page 65]), a type $A_5 \times A_2$ is obtained. Adding the additional root $\frac{1}{2}(e_8 + e_7 - \sum_{i=1}^{6} e_i)$ gives an A_8 in E_8. Since E_7 is not a subsystem of A_8 and $A_5 \times A_2$ is maximal in E_7, this $A_5 \times A_2$ must be $A_8 \cap E_7$. Thus $\mathrm{char}_{E_7}(A_8 \cap E_7) = \mathrm{char}_{E_7}(A_5 \times A_2) = (3)$ and $\mathrm{char}_{E_8} A_8 = (3)$.

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4. L^2-singular dichotomy

4.1. Terminology and statement of the dichotomy. In this section we will use the results on intersections of subroot systems to prove that the L^2-singular dichotomy holds for the exceptional Lie groups and algebras. Throughout this section, G will denote a compact, connected, simple Lie group, with centre $Z(G)$ and Lie algebra $\mathfrak{g}$. We let $\mathbb{T}$ denote the maximal torus of G and let $\mathfrak{t}$ be its Lie algebra. The set of roots of the complexification of $\mathfrak{g}$ with respect to the complexified torus will be denoted by Φ. These sets Φ are irreducible root systems and any two semisimple Lie algebras with the same root system are isomorphic.

The notation $\mathrm{Ad}_G(\cdot)$ will denote both the adjoint action of G on $\mathfrak{g}$ and the conjugation action of G on itself; the meaning will be clear from the context. We let $O_X \subseteq \mathfrak{g}$ and $C_x \subseteq G$ denote the orbits of $X \in \mathfrak{g}$ or $x \in G$ respectively, under the (appropriate) action of Ad_G. Of course, C_x is the conjugacy class of $x \in G$. Being proper submanifolds, every orbit has zero Haar measure. Moreover, every orbit contains a torus element.

Definition 4.1. Given $X \in \mathfrak{g}$, the *orbital measure*, μ_X, is the Borel measure on $\mathfrak{g}$ defined by the rule

$$p\int_{\mathfrak{g}} f\, d\mu_X = \int_G f(\mathrm{Ad}_G(g)X)\, dm_G(g)$$

for any continuous, compactly supported function f on $\mathfrak{g}$. (Here m_G denotes the Haar measure on G.)

The orbital measure μ_X is G-invariant, that is, $\mu_X(E) = \mu_X(\mathrm{Ad}_G(g)E)$ for all $g \in G$ and Borel sets $E \subseteq \mathfrak{g}$. When $X \neq 0$, μ_X is the unique G-invariant, probability measure, compactly supported on the adjoint orbit O_X. One can similarly define the orbital measure μ_x, for $x \in G$ as the unique G-invariant, probability measure, supported on the conjugacy class C_x. Since adjoint orbits and conjugacy classes have measure zero, orbital measures are purely singular with respect to Haar measure.

Definition 4.2. We will say that the root $\alpha \in \Phi$ *annihilates* element $X \in \mathfrak{t}$ or $x \in \mathbb{T}$ if $(\alpha, X) = 0$ or $(\alpha, x) \in \mathbb{Z}$. By the *type* of X or x we will mean the type of its set of annihilating roots.

Except when $X = 0 \in \mathfrak{t}$ or $x \in Z(G)$, the set of annihilating roots is a proper subroot system of Φ. When $x \in \mathbb{T}$, the set of annihilating roots is the subroot system $\Phi(x)$ in the notation of the previous section, and thus is Weyl conjugate to one obtained by deleting roots from the extended base for Φ. When $X \in \mathfrak{t}$, the set of annihilating roots is $\mathbb{R}$-closed and hence by Lemma 2.4 is Weyl conjugate to one obtained by deleting roots from the base for Φ. We will still use the notation $\Phi(X)$ for its set of annihilating roots. Of course, all torus elements in the same orbit have the same type.

Suppose that G is one of the compact, connected, simple, exceptional Lie groups, those of Lie type E_6, E_7, E_8, F_4 or G_2. Our dichotomy result can be stated in terms of

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the following constant. For $z \in \mathbb{T} \setminus Z(G)$ or $0 \neq z \in \mathfrak{t}$, put

$$k(z) = \begin{cases} 4 & \text{if } z \text{ is type } B_4 \text{ in } F_4 \\ 3 & \text{if } z \text{ is type } \begin{cases} E_7 \text{ or } E_7 \times A_1 \text{ in } E_8 \\ E_6, D_6 \text{ or } D_6 \times A_1 \text{ in } E_7 \\ D_5, A_5 \text{ or } A_5 \times A_1 \text{ in } E_6 \\ A_2 \text{ in } G_2 \end{cases} \\ 2 & \text{otherwise.} \end{cases}$$

We remark that there are no non-zero elements $z \in \mathfrak{t}$ with $\Phi(z)$ of full rank. Hence $z \in \mathfrak{t}$ cannot be of type B_4 in F_4, $E_7 \times A_1$ in E_8, $D_6 \times A_1$ in E_7, $A_5 \times A_1$ in E_6 or A_2 in G_2. All the other types do occur.

We use the notation μ^k to denote the k-fold convolution product of μ, C_x^k for the k-fold product of C_x, and $(k)O_X$ for the k-fold sum of O_X. With this notation we can now state the L^2-singular dichotomy.

Theorem 4.3 (L^2-singular dichotomy). *Suppose that $z \in \mathbb{T} \setminus Z(G)$ or $0 \neq z \in \mathfrak{t}$, and $k(z)$ is as stated above.*

(i) *For all $k \geq k(z)$, the convolution product $\mu_z^k \in L^2$ and C_z^k (or $(k)O_z$) has non-empty interior.*
(ii) *For all $k < k(z)$, μ_z^k is purely singular to Haar measure m and $m(C_z^k) = 0$ (or $m((k)O_z) = 0$).*

The measure μ_x^k is supported on C_x^k when $x \in G$, thus μ_x^k is singular if the Haar measure of C_x^k is zero. Conversely, if $\mu_x^k \in L^2(G)$, then C_x^k has positive measure, and it is well known that for these submanifolds positive measure implies non-empty interior [11]. Consequently, for $x \in G$ it will be enough to prove $\mu_x^k \in L^2(G)$ for all $k \geq k(x)$ and $m_G(C_x^k) = 0$ for all $k < k(x)$. Analogous statements hold for $(k)O_X$ and μ_X^k.

If $X = 0$, then the orbit and the orbital measure are trivial, as are all k-fold sums and convolution powers. If $x \in Z(G)$, then C_x is a singleton and μ_x is the point mass measure at x. Hence all convolution powers of μ_x are discrete measures, and therefore singular, and all C_x^k are finite sets, hence of measure zero. We also call these trivial orbital measures.

4.2. Proof of L^2-singular dichotomy theorem, part (i): if $k \geq k(z)$, then $\mu_z^k \in L^2$. Building on the work of [7], it was shown in [6] that $\mu_z^4 \in L^2$ for all non-trivial orbital measures on F_4 and $\mu_z^3 \in L^2$ for all non-trivial orbital measures on the other exceptional Lie groups/algebras. Thus it will be enough to prove that $\mu_z^2 \in L^2$ for those z where we claim that $k(z) = 2$. First, we remark that it will be enough to prove that $\mu_x^2 \in L^2(G)$ whenever $x \in \mathbb{T} \setminus Z(G)$ and $k(x) = 2$ because transference arguments can then be used to deduce the corresponding result on the Lie algebra, as explained in [5, Proof of Theorem 8.2].

4.2.1. Combinatorial criterion. To prove that $\mu_x^2 \in L^2(G)$ when $k(x) = 2$, we rely on a combinatorial criterion established in [5], which is valid for all simple, compact, connected Lie groups G. As in the previous sections, assume that $\{\alpha_1, \ldots, \alpha_n\}$ is a base for the root system Φ of rank n and $\{\lambda_1, \ldots, \lambda_n\}$ is the dual base. Put

$$S_j = \{\alpha \in \Phi : (\alpha, \lambda_j) \neq 0\}.$$

Observe that S_j is the complement of the subroot system, S_j^c, which is obtained by removing root α_j from the base of Φ and thus has characteristic $(0, h_j)$. A summary of basic facts about the sets S_j, when G is one of the five exceptional Lie groups, can be found in the Appendix.

Given $\Psi \subseteq \Phi$ and a set of l integers $i_1, \ldots, i_l$ satisfying $n \geq i_1 > i_2 > \cdots > i_l \geq 1$, we inductively define

$$T_j = S_{i_j} \Big\backslash \bigcup_{k=1}^{j-1} S_{i_k} = \Big\{\alpha \in \Phi \Big\backslash \bigcup_{k=1}^{j-1} T_k : (\alpha, \lambda_{i_j}) \neq 0\Big\}$$

and

$$U_j = \Big\{\alpha \in \Psi \Big\backslash \bigcup_{k=1}^{j-1} U_k : (\alpha, \lambda_{i_j}) \neq 0\Big\}$$

for $j = 1, \ldots, l$. Let $\kappa(i_1, \ldots, i_l, \Psi)$ be the minimum integer k such that

$$\sum_{j=1}^{l} ((k-1)|T_j| - k|U_j|) > l.$$

or, equivalently,

$$k|\Psi \cap S_{i_1} \cup \cdots \cup S_{i_l}| < (k-1)|S_{i_1} \cup \cdots \cup S_{i_l}| - l. \tag{4.1}$$

Let $r_{i_1,\ldots,i_l}$ denote the rank of $S_{i_1}^c \cap \cdots \cap S_{i_l}^c$. Since l is the corank of $S_{i_1}^c \cap \cdots \cap S_{i_l}^c$, inequality (4.1) holds for $k = 2$ if and only if

$$2|\Psi| - |\Phi| + n < 2|\Psi \cap S_{i_1}^c \cap \cdots \cap S_{i_l}^c| - |S_{i_1}^c \cap \cdots \cap S_{i_l}^c| + r_{i_1,\ldots,i_l}. \tag{4.2}$$

The combinatorial criterion for μ_x^k to belong to $L^2(G)$ is the content of the next theorem. The proof, which can be found in [5], uses the Weyl character and degree formulas, and the Peter–Weyl theorem.

Theorem 4.4 ([5, Theorem 6.1]). *Suppose that G is a compact, connected, simple Lie group of* rank n, *$x \in \mathbb{T}$ and $\Phi(x)$ is the set of annihilating roots of x. Let $\kappa_0(x) = \max\{\kappa(i_1, \ldots, i_l, \Psi)\}$ where the maximum is taken over all $l \in \{1, \ldots, n\}$, all sets of indices $n \geq i_1 > i_2 > \cdots > i_l \geq 1$ and all root subsystems, Ψ, that are conjugate under the Weyl group to $\Phi(x)$. Then $\mu_x^{\kappa_0(x)} \in L^2(G)$.*

Thus it will be enough to show that $\kappa_0(x) = 2$ whenever $k(x) = 2$, and this is equivalent to proving that (4.2) holds for all $n \geq i_1 > i_2 > \cdots > i_l \geq 1$ and all subroot

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systems, Ψ, that are Weyl conjugate to $\Phi(x)$. Consequently, it will be helpful to have good lower bounds on $|\Psi \cap S_{i_1}^c \cap \cdots \cap S_{i_l}^c|$. Whenever one of Ψ or $S_{i_1}^c \cap \cdots \cap S_{i_l}^c$ is irreducible and the other is a $\mathbb{Z}_2$ or $\mathbb{Z}_3$-kernel in Φ, we will be able to use the results of Section 3 to obtain such bounds.

It is clear that if (4.1) holds for $\Psi \subseteq \Phi$, then it also holds for any $\Psi' \subseteq \Psi$, thus it will suffice to prove that inequality (4.2) is satisfied for the maximal subroot systems $\Phi(x)$, other than those for which $k(x) > 2$, as well as for those which are maximal within the $\Phi(x)$ with $k(x) > 2$.

We begin with a sufficient condition that can be applied to $\mathbb{Z}_2$-kernels, $\Phi(x)$, and is easy to check.

Proposition 4.5. *Suppose that Φ is a root system of* rank $n > 1$ *and $\Phi(x)$ is a $\mathbb{Z}_2$-kernel in Φ. If $2|\Phi(x)| \leq |\Phi| - 2n$, then $\kappa_0(x) = 2$ and $\mu_x^2 \in L^2(G)$.*

Proof. Because $2|\Phi(x)| - |\Phi| + n \leq -n$, to prove that $\kappa_0(x) = 2$, it will be enough to show that

$$2|\Psi \cap C| - |C| + \operatorname{rank} C > -n$$

whenever $C = S_{i_1}^c \cap \cdots \cap S_{i_l}^c$ and Ψ is Weyl conjugate to $\Phi(x)$. Of course, being Weyl conjugate, Ψ will also be a $\mathbb{Z}_2$-kernel.

For any such subroot system C, let

$$\sigma(C) \equiv 2 \min |\Psi \cap C| - |C| + \operatorname{rank} C$$

where the minimum is taken over all $\mathbb{Z}_2$-kernels Ψ. When C is irreducible, then C is either one of the four classical types, A_j, B_j, C_j or D_j with $j < n$, or $C = E_6$ or E_7. It follows from Corollary 3.9 that

$$\begin{aligned}\sigma(A_j) &= 2 \min |\Psi \cap A_j| - |A_j| + \operatorname{rank} A_j \\ &\geq 2\left\lfloor \frac{j^2 - 1}{2} \right\rfloor - j(j+1) + j = -1 > -n.\end{aligned}$$

If C is not type A_j, then similar calculations show that

$$\sigma(C) \geq -\operatorname{rank} C > -n.$$

The subroot systems $C = S_{i_1}^c \cap \cdots \cap S_{i_l}^c$ are obtained by removing roots from the base of Φ and hence can only be of type $A_{j_1} \times \cdots \times A_{j_t}$ or $\Omega \times A_{j_1} \times \cdots \times A_{j_t}$, where Ω is one of type B_j, C_j, D_j, E_6 or E_7 and $t \leq n - 1 - \operatorname{rank} \Omega$. We note that if C is not irreducible, say $C = \Omega_1 \times \Omega_2$, then $\sigma(C) \geq \sigma(\Omega_1) + \sigma(\Omega_2)$. Thus

$$\sigma(A_{j_1} \times \cdots \times A_{j_t}) \geq -t > -n$$

and

$$\begin{aligned}\sigma(\Omega \times A_{j_1} \times \cdots \times A_{j_t}) &\geq \sigma(\Omega) + \sigma(A_{j_1} \times \cdots \times A_{j_t}) \\ &\geq -\operatorname{rank} \Omega - (n - 1 - \operatorname{rank} \Omega) > -n.\end{aligned}$$

This completes the argument. □

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EXAMPLE 4.6. This can be used to give a new proof that if x is type $B_{n/2} \times D_{n/2}$ in B_n or type A_{n-1} in B_n or C_n, then $\mu_x^2 \in L^2$. (See [4, Theorem 9.1].)

We now consider each exceptional group separately.

4.2.2. The exceptional group E_8. Since all the proper subroot systems of E_7 are contained in maximal subroot systems of $E_7 \times A_1$, it will be sufficient to prove that (4.2) holds for the maximal subroot systems of E_8 other than other than $E_7 \times A_1$, and the maximal subroot systems of $E_7 \times A_1$. These are D_8,

$$\begin{gathered} A_8,\, E_6 \times A_2,\, A_4 \times A_4,\, E_6 \times A_1,\, D_6 \times A_1 \times A_1, \\ A_7 \times A_1 \quad \text{and} \quad A_5 \times A_2 \times A_1. \end{gathered} \tag{4.3}$$

We remark that $D_6 \times A_1 \times A_1 \neq \Phi(x)$ for any $x \in E_8$ since it is not obtained by removing roots from the extended Dynkin diagram of E_8. However, we can still formally verify that (4.2) is satisfied for $\Psi = D_6 \times A_1 \times A_1$, and then (4.2) will also hold for all $\Phi(x)$ that are subsets of $D_6 \times A_1 \times A_1$.

Since $\mathrm{char}_{E_8} D_8 = (2)$ and D_8 consists of 112 of the 240 roots in the root system of type E_8, Proposition 4.5 implies $\kappa_0(x) = 2$ when x is type D_8.

The basic facts about the sets S_j summarized in the Appendix will be useful in checking (4.2) in the remaining cases. Of course, if

$$2|\Psi| - |S_{i_1} \cup \cdots \cup S_{i_l}| + l < 0, \tag{4.4}$$

then (4.2) is clearly satisfied for Ψ; we call this the trivial inequality.

One can easily confirm that the trivial inequality holds for all choices of $S_{i_1} \cup \cdots \cup S_{i_l}$ if Ψ is either type $A_4 \times A_4$ or $A_5 \times A_2 \times A_1$ since $|A_4 \times A_4| = 40$, $|A_5 \times A_2 \times A_1| = 38$ and the minimum cardinality of $S_{i_1} \cup \cdots \cup S_{i_l}$ is 114. In fact, the minimum cardinality of $S_{i_1} \cup \cdots \cup S_{i_l}$, other than S_1 or S_8, is 166 and the maximum cardinality of any root system of the types listed in (4.3) is $|E_6 \times A_2| = 78$, so the trivial inequality also holds for all the other subroot systems listed above, except when $l = 1$ and $S_{i_1} = S_1$ or S_8. Since S_1^c and S_8^c both have characteristic $(0, 2)$, Corollary 3.13 provides lower bounds on $|\Omega^+ \cap S_j^c|$ whenever Ω is an irreducible subroot system. We summarize the relevant information from that corollary in the chart below.

Type Ω	A_8	E_6	D_6	A_7
$\min\lvert\Omega \cap S_j^c\rvert$	18	22	16	14

If Ψ is not irreducible, say $\Psi = \Omega_1 \times \Omega_2$, we obtain lower bounds on cardinalities by noting that $|\Omega_1 \times \Omega_2 \cap S_j^c| = |\Omega_1 \cap S_j^c| + |\Omega_2 \cap S_j^c|$. With this we can quickly check that (4.2) holds when Ψ is any of the remaining types from (4.3). For example, if Ψ is type $E_6 \times A_2$, then since S_8^c is type E_7,

$$2|\Psi| - |E_8| + 8 = 2(72 + 6) - 240 + 8 = -76$$

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while

$$\begin{aligned} 2|\Psi \cap S_8^c| - |S_8^c| + 7 &\geq 2|E_6 \cap S_8^c| - 126 + 7 \\ &\geq 2 \cdot 22 - 126 + 7 = -75. \end{aligned}$$

4.2.3. The exceptional group E_7. It suffices to check that (4.2) holds for the maximal subroot systems of E_7 other than E_6 or $D_6 \times A_1$, and the maximal subroot systems of E_6 and $D_6 \times A_1$, other than D_6. These are

$$A_7, A_5 \times A_2, D_5 \times A_1, D_4 \times A_2 \times A_1, A_3 \times A_3 \times A_1, A_2 \times A_2 \times A_2. \tag{4.5}$$

Since $\mathrm{char}_{E_7} A_7 = (2)$ and $2|A_7| = |E_7| - 2 \cdot 7$, we can appeal to Proposition 4.5 for type A_7.

For the other subroot systems listed in (4.5), we first observe that the trivial condition (4.4) is satisfied for all $S_{i_1} \cup \cdots \cup S_{i_l}$ when Ψ is type $A_3 \times A_3 \times A_1$ or $A_2 \times A_2 \times A_2$. (We refer the reader to the Appendix for helpful information.) For the remaining cases, we will need to verify (4.2) for $l = 1$, with $S_{i_1} = S_1$ or S_7. We also need to verify that (4.2) is satisfied when $\Psi = D_5 \times A_1$ for the cases S_2, S_6, $S_1 \cup S_7$ and $S_6 \cup S_7$.

For these, we use the fact that S_1^c, S_2^c and S_6^c have characteristic $(0, 2)$ in E_7 and S_7^c has characteristic $(0, 1)$ to obtain lower bounds on $|\Psi \cap S_j^c|$. Furthermore, $D_5 \times A_1$ has characteristic $(0, 2)$ in E_7, while $S_1^c \cap S_7^c$ and $S_6^c \cap S_7^c$ are both of type D_5. Thus we can appeal to Corollary 3.13 to obtain the estimates we need. For example,

$$2|D_5 \times A_1 \cap (S_1^c \cap S_7^c)| - |S_1^c \cap S_7^c| + \mathrm{rank}(S_1^c \cap S_7^c) \geq -15,$$

while $2|D_5 \times A_1| - |E_7| + 7 = -35$.

4.2.4. The exceptional group E_6. It suffices to show that (4.2) is satisfied for the subroot systems D_4, $A_4 \times A_1$, $A_3 \times A_1 \times A_1$ and $A_2 \times A_2 \times A_2$. An appeal to the trivial inequality shows we will only need to consider the cases S_1 or S_6 (for all the root systems above), S_2 for the subroot systems D_4 and $A_4 \times A_1$, and $S_1 \cup S_6$ for D_4. As S_1^c and S_6^c are characteristic $(0, 1)$, S_2^c is characteristic $(0, 2)$ and $S_1^c \cap S_6^c$ is characteristic $(0, 1, 1)$, our standard arguments can be applied.

4.2.5. The exceptional group F_4. The conclusion for $\Phi(x)$ of type $A_1 \times C_3$ follows from Proposition 4.5, thus we only need consider the subroot systems $A_2 \times A_2$, $A_3 \times A_1$, B_3 and $B_2 \times A_1$.

The trivial inequality is satisfied for all these subroot systems except for type B_3 with $l = 1$ and the sets S_1, S_4. Since B_3 has characteristic $(0, 2)$ in F_4, S_1^c is type C_3, and C_3 contains no subroot systems with characteristic $(0, 1, 1)$, it follows that $S_1^c \cap B_3$ has characteristic $(0, 2)$ or $(0, 1)$ in C_3. Thus $\min |S_1^c \cap B_3| \geq 4$ and this is good enough for (4.2).

For S_4 a slightly more delicate argument is required. Notice that the set of long roots in any subroot system Ψ of type B_3 are a subsystem of type A_3. Since any Weyl conjugate must preserve length, the long roots of Ψ are always contained in the

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set $\{e_i \pm e_j : i, j = 1, \ldots, 4\}$ and this set is of type D_4. Since A_3 has characteristic (0, 1) in D_4, appealing to Corollary 3.9 it follows that the intersection of any two type A_3s in D_4 will have cardinality at least four. Thus we can also conclude that $\min |S_4^c \cap B_3| \geq 4$.

4.2.6. The exceptional group G_2. The annihilating sets $\Phi(x)$ are either A_2, $A_1 \times A_1$ or A_1. Since both S_1^c and S_2^c are type A_1 and G_2 has 12 roots, the trivial inequality holds with $k = 2$ for both the latter two subroot systems.

4.3. Proof of L^2-singular dichotomy theorem, part (ii): if $k < k(z)$, then $m(C_z^k) = 0$ (or $m((k)O_x) = 0$). It is a well-known geometric fact (see [6, Lemma 1]) that if

$$|\Phi| - |\Phi(z)| < \dim G/k,$$

then the measure of $m(C_z^k) = 0$ (or $m((k)O_z) = 0$). With this property it is simple to check that $m(C_z^k) = 0$ (or $m((k)O_z) = 0$) whenever $k < k(z)$, *except* if z is type A_5 or $A_5 \times A_1$ in E_6.

First, assume that $X \in \mathfrak{t}$ is type A_5 in the Lie algebra of type E_6. It was noted in Example 2.7 that all subroot systems $\Phi(X)$ of type A_5 in E_6 are Weyl conjugate, so there is no loss of generality in assuming that $\Phi(X)$ is the subroot system with base $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ where $\alpha_1 = 1/2(e_8 + e_1 - \sum_{j=2}^{7} e_i)$ and $\alpha_j = e_{j-1} - e_{j-2}$ for $j \geq 3$. We will take $\omega(\Phi(X))$ to be the subroot system of type A_5 with base $\{\alpha_0, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$ where $\alpha_2 = e_1 + e_2$ and $\alpha_0 = 1/2(e_8 - e_7 - e_6 + \sum_{j=1}^{5} e_i)$ is the highest root.

We will use the notation $P_{j,k,l}$ to denote the root $1/2(e_8 - e_7 - e_6 + \sum_{j=1}^{5} s_i e_i)$ with $s_j = s_k = s_l = 1$ and $s_i = -1$ otherwise. It is straightforward to check that $\Phi(X)^c \cap \omega(\Phi(X))^c$ is contained in

$$\Psi = \{e_j \pm e_k, P_{1,j,k} : 2 \leq j < k \leq 5\} \cup \{\alpha_1, \alpha_0\}.$$

Furthermore, Ψ is the subroot system of type D_5 and corank one, with base $\{P_{1,2,4}, e_2 + e_4, -e_2 - e_3, e_2 - e_4, e_3 - e_5\}$. It follows from a result of Wright [14, Theorem 1.4] that $(2)O_X$ does not contain an open set and this implies $m((2)O_X) = 0$.

As $A_5 \times A_1$ has full rank, there is no $X \in \mathfrak{t}$ of type $A_5 \times A_1$.

For the group case, we will make use the fact that if $x = \exp X$, then $C_x^2 \subseteq (2)O_X$ (see [2] or [13]). If $x \in \mathbb{T}$ is type A_5 there is again no loss of generality in assuming that $\Phi(x)$ has base $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. This implies that

$$x = \exp(\theta, \ldots, \theta, -\theta + 2m/3, -\theta + 2m/3, \theta - 2m/3), \tag{4.6}$$

where $\theta \notin \mathbb{Z}/4$ and $m \in \mathbb{Z}$. If we let $y = \exp(0, \ldots, 0, -2m/3, -2m/3, 2m/3)$, then $(\alpha, y) \in \mathbb{Z}$ for all roots α and thus y belongs to the centre of the group E_6. Consequently, $C_{xy}^2 = y^2 C_x^2$, and therefore $m(C_x^2) = 0$ if and only if $m(C_{xy}^2) = 0$. But $xy = \exp X$ where $X = (\theta, \ldots, \theta, -\theta, -\theta, \theta)$. As $\alpha(X) = 0$ if and only if $\alpha \in \Phi(x)$, X is also of type A_5 and so $m((2)O_X) = 0$ by the previous part of the argument. It follows that $m(C_x^2) = m(C_{xy}^2) = 0$.

Finally, suppose that $x \in \mathbb{T}$ is type $A_5 \times A_1$. We claim that up to Weyl conjugacy there is also only one subroot system $\Phi(x)$ of type $A_5 \times A_1$. To see this, note that if there are two, then their irreducible components of type A_5 must be Weyl conjugate and the Weyl element that gives the conjugation must map the unique

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positive root orthogonal to the first subsystem of type A_5 to the root orthogonal to the second type A_5. Thus without loss of generality, we can assume that $\Phi(x)$ has base $\{\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and therefore x is as in (4.6), but with $\theta = 1/4$ or $3/4$. Again we argue that $xy = \exp X$ where X is type A_5, and therefore $m(C_x^2) = 0$.

Acknowledgement

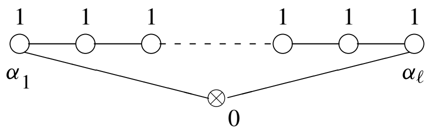
We are grateful to A. Wright for helpful conversations.

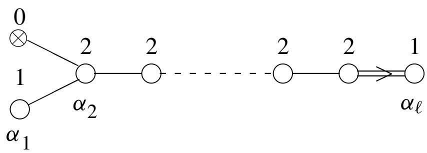
Appendix.

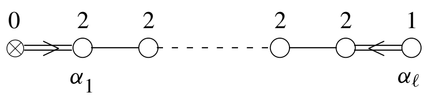
Basic facts about irreducible root systems

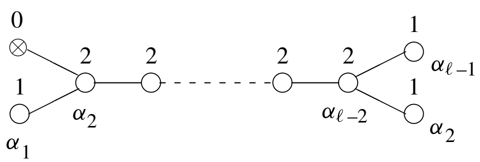
Type	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\lvert\Phi\rvert$	$2\binom{n+1}{2}$	$2n^2$	$2n^2$	$4\binom{n}{2}$	72	126	240	48	12
Rank	n	n	n	n	6	7	8	4	2

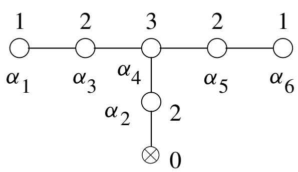
The dimension of the associated simple Lie group or algebra is $|\Phi| + \text{rank}\,\Phi$.

(A_ℓ)  $(\ell \geq 1)$

(B_ℓ)  $(\ell \geq 2)$

(C_ℓ)  $(\ell \geq 2)$

(D_ℓ)  $(\ell \geq 4)$

(E_6) 

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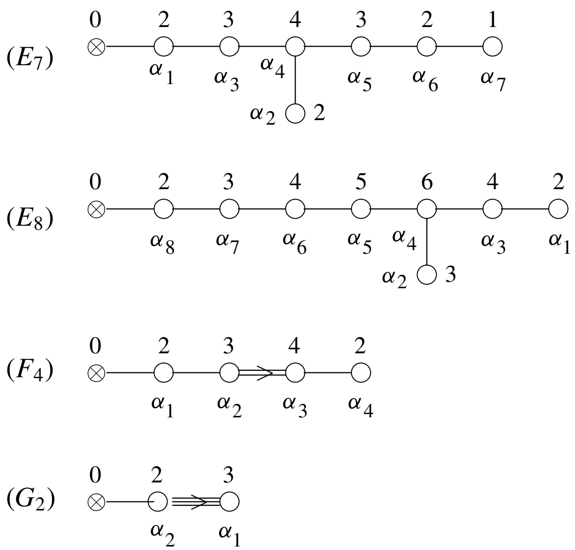

Extended Dynkin diagrams showing coefficients of highest roots and base $\{\alpha_j\}$.

Information about sets S_j for type E_8

Ω	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	$S_1 \cup S_8$
Type Ω^c	D_7	A_7	$A_6 \times A_1$	$A_4 \times A_2 \times A_1$	$A_4 \times A_3$	$A_5 \times A_2$	$E_6 \times A_1$	E_7	D_6
$\mathrm{char}_{E_8}\ \Omega^c$	(0, 2)	(0, 3)	(0, 4)	(0, 6)	(0, 5)	(0, 4)	(0, 3)	(0, 2)	(0, 2, 2)
$\|\Omega\|$	156	184	196	212	208	194	166	114	180

$|\Omega| \geq 198$ for all other $\Omega = S_{i_1} \cup \cdots \cup S_{i_\ell}$.

Information about sets S_j for type E_7

Ω	S_1	S_2	S_3	S_4	S_5	S_6	S_7	$S_1 \cup S_7$ $S_6 \cup S_7$
Type Ω^c	D_6	A_6	$A_5 \times A_1$	$A_3 \times A_2 \times A_1$	$A_4 \times A_2$	$D_5 \times A_1$	E_6	D_5
$\mathrm{char}_{E_7}\ \Omega^c$	(0, 2)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 2)	(0, 1)	(0, 2, 2)
$\|\Omega\|$	66	84	94	106	100	84	54	86

$|\Omega| \geq 96$ for all other $\Omega = S_{i_1} \cup \cdots \cup S_{i_\ell}$.

Information about sets S_j for type E_6

Ω	S_1	S_2	S_3	S_4	S_5	S_6	$S_1 \cup S_6$
Type Ω^c	D_5	A_5	$A_4 \times A_1$	$A_2 \times A_2 \times A_1$	$A_4 \times A_1$	D_5	D_4
$\mathrm{char}_{E_6}\ \Omega^c$	(0, 1)	(0, 2)	(0, 2)	(0, 3)	(0, 2)	(0, 1)	(0, 1, 1)
$\|\Omega\|$	32	42	50	58	50	32	48

$|\Omega| \geq 52$ for all other $\Omega = S_{i_1} \cup \cdots \cup S_{i_\ell}$.

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Information about sets S_j for type F_4

Ω	S_1	S_2	S_3	S_4
Type Ω^c	C_3	$A_2 \times A_1$	$A_2 \times A_1$	B_3
$\mathrm{char}_{F_4}\,\Omega^c$	(0, 2)	(0, 3)	(0, 4)	(0, 2)
$\lvert\Omega\rvert$	30	40	40	30

$|\Omega| \geq 40$ for all other $\Omega = S_{i_1} \cup \cdots \cup S_{i_\ell}$.

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