CAMBRIDGE
UNIVERSITY PRESS

ARTICLE

A generalization of Bondy's pancyclicity theorem

Nemanja Draganić[1,*], David Munhá Correia[2,†] and Benny Sudakov[2,†]

[1]Mathematical Institute, University of Oxford, Oxford, UK and [2]Department of Mathematics, ETH, Zürich, Switzerland
Corresponding author: Nemanja Draganić; Email: nemanja.draganic@maths.ox.ac.uk

Abstract

The *bipartite independence number* of a graph G, denoted as $\tilde{\alpha}(G)$, is the minimal number k such that there exist positive integers a and b with $a + b = k + 1$ with the property that for any two disjoint sets $A, B \subseteq V(G)$ with $|A| = a$ and $|B| = b$, there is an edge between A and B. McDiarmid and Yolov showed that if $\delta(G) \geq \tilde{\alpha}(G)$ then G is Hamiltonian, extending the famous theorem of Dirac which states that if $\delta(G) \geq |G|/2$ then G is Hamiltonian. In 1973, Bondy showed that, unless G is a complete bipartite graph, Dirac's Hamiltonicity condition also implies pancyclicity, i.e., existence of cycles of all the lengths from 3 up to n. In this paper, we show that $\delta(G) \geq \tilde{\alpha}(G)$ implies that G is pancyclic or that $G = K_{\frac{n}{2}, \frac{n}{2}}$, thus extending the result of McDiarmid and Yolov, and generalizing the classic theorem of Bondy.

1. Introduction

The notion of Hamiltonicity is one of the most central and extensively studied topics in combinatorics. Since the problem of determining whether a graph is Hamiltonian is NP-complete, a central theme in combinatorics is to derive sufficient conditions for this property. A classic example is Dirac's theorem [14] which dates back to 1952 and states that every n-vertex graph with minimum degree at least $n/2$ is Hamiltonian. Since then, a plethora of interesting and important results about various aspects of Hamiltonicity has been obtained, see e.g. [1, 11–13, 19, 24, 26, 27, 34], and the surveys [21, 31].

Besides finding sufficient conditions for containing a Hamilton cycle, significant attention has been given to conditions which force a graph to have cycles of other lengths. Indeed, *the cycle spectrum of a graph*, which is the set of lengths of cycles contained in that graph, has been the focus of study of numerous papers and in particular gained a lot of attention in recent years [2, 3, 8, 16, 20, 23, 29, 30, 33, 36]. Among other graph parameters, the relation of the cycle spectrum to the minimum degree, number of edges, independence number, chromatic number, and expansion properties of the graph have been studied.

We say that an n-vertex graph is *pancyclic* if the cycle spectrum contains all integers from 3 up to n. Bondy suggested that in the cycle spectrum of a graph, it is usually hardest to guarantee the existence of the longest cycle, i.e. a Hamilton cycle. This intuition was captured by his famous meta-conjecture [5] from 1973, which asserts that any non-trivial condition which implies

Hamiltonicity, also implies pancyclicity (up to a small class of exceptional graphs). As a first example, he proved in ref. [6], an extension of Dirac's theorem, showing that minimum degree at least $n/2$ implies that the graph is either pancyclic or that it is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Further, Bauer and Schmeichel [4], relying on previous results of Schmeichel and Hakimi [35], showed that the sufficient conditions for Hamiltonicity given by Bondy [7], Chvátal [10] and Fan [18] all imply pancyclicity, up to a certain small family of exceptional graphs.

Another classic Hamiltonicity result is the Chvátal-Erdős theorem, which states that $\kappa(G) \geq \alpha(G)$ implies that G is Hamiltonian, where $\kappa(G)$ is the connectivity of G, and $\alpha(G)$ its independence number. Motivated by Bondy's meta-conjecture, Jackson and Ordaz [22] suggested thirty years ago that $\kappa(G) > \alpha(G)$ already implies pancyclicity. The first progress towards this problem was obtained by Keevash and Sudakov [23], who showed pancyclicity when $\kappa(G) \geq 600\alpha(G)$. Recently, in ref. [15], we were able to resolve the Jackson–Ordaz conjecture asymptotically, proving that $\kappa(G) \geq (1 + o(1))\alpha(G)$ is already enough for pancyclicity. Building on our work in ref. [15], Letzter [28] proved the Jackson–Ordaz conjecture for large enough graphs. It is worth mentioning that, in all the listed work, the proof that the Hamiltonicity condition also implies pancyclicity is usually significantly harder than just proving Hamiltonicity, and requires new ideas and techniques.

An interesting sufficient condition for Hamiltonicity was given by McDiarmid and Yolov [32]. To state their result, we need the following natural graph parameter. For a graph G, its *bipartite independence number* $\tilde{\alpha}(G)$ is the minimal number k, such that there exist positive integers a and b with $a + b = k + 1$, such that between any two disjoint sets $A, B \subseteq V(G)$ with $|A| = a$ and $|B| = b$, there is an edge between A and B. Notice that we always have that $\alpha(G) \leq \tilde{\alpha}(G)$. Indeed, if $\tilde{\alpha}(G) = k$, then G does not contain independent sets I of size at least $k + 1$, since if it did contain such an I, then evidently for every $a + b = k + 1$, there would exist disjoint sets $A, B \subset I$, so that $|A| = a$ and $|B| = b$ and with no edge between A and B. Let us now state the result of McDiarmid and Yolov.

Theorem 1.1 ([32]). *If $\delta(G) \geq \tilde{\alpha}(G)$, then G is Hamiltonian.*

This result implies Dirac's theorem, because if $\delta(G) \geq n/2$, then $\lceil n/2 \rceil \geq \tilde{\alpha}(G)$, as for every $|A| = 1$ and $|B| = \lceil n/2 \rceil$ there is an edge between A and B. Hence also $\delta(G) \geq \lceil n/2 \rceil \geq \tilde{\alpha}(G)$, so G is Hamiltonian by Theorem 1.1.

Naturally, the immediate question which arises is whether the McDiarmid–Yolov condition implies that the graph satisfies the stronger property of pancyclicity. As a very preliminary step in this direction, Chen [9] was able to show that for any given positive constant c, for sufficiently large n it holds that if G is an n-vertex graph with $\tilde{\alpha}(G) = cn$ and $\delta(G) \geq \frac{10}{3}cn$, then G is pancyclic. In this paper we completely resolve this problem, showing that $\delta(G) \geq \tilde{\alpha}(G)$ implies that G pancyclic or $G = K_{\frac{n}{2},\frac{n}{2}}$. This generalizes the classical theorem of Bondy [6], and gives additional evidence for his meta-conjecture, mentioned above.

Theorem 1.2. *If $\delta(G) \geq \tilde{\alpha}(G)$, then G is pancyclic, unless G is a balanced complete bipartite graph $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Our proof is completely self-contained and relies on a novel variant of Pósa's celebrated rotation-extension technique, which is used to extend paths and cycles in expanding graphs (see, e.g., [34]). Define the graph $\tilde{C}_\ell$, to be the cycle of length ℓ together with an additional vertex which is adjacent to two consecutive vertices on the cycle (thus forming a triangle with them). For each $\ell \in [3, n - 1]$, our goal is to either find a $\tilde{C}_\ell$ or a $\tilde{C}_{\ell+1}$, which is clearly enough to show pancyclicity. The proof is recursive in nature, as we will derive the existence of a $\tilde{C}_\ell$ or a $\tilde{C}_{\ell+1}$ from the existence of a $\tilde{C}_{\ell-1}$. In our setting, we would like to apply the rotation-extension technique to the $\tilde{C}_{\ell-1}$ with the additional requirement that the extended cycle preserves the attached triangle. However, this is not possible in general and from the existence of a $\tilde{C}_{\ell-1}$ we will in turn derive the

existence of a gadget denoted as a *switch*, which is a path with triangles attached to it, to which we can apply our rotation-extension technique. One of the key ideas is to consider the switch which is optimal with respect to how close the triangles are to the beginning of the path (see Definition 2.5). The application of the rotation-extension technique to such an optimal switch will then result in either a $\tilde{C}_\ell$, a $\tilde{C}_{\ell+1}$, or a better switch, contradicting the optimality of the original switch. The details are given in the next section.

2. The proof

We first recall the definition of $\tilde{\alpha}(G)$.

Definition 2.1. *For a graph G, let $\tilde{\alpha}(G)$ denote the minimal number k, such that there exist positive integers a and b with $a + b = k + 1$, such that between any two disjoint sets $A, B \subseteq V(G)$ with $|A| = a$ and $|B| = b$, there is an edge between A and B.*

We will also need the following definition of a cycle which has one triangle attached to one of its edges.

Definition 2.2. *Define the graph $\tilde{C}_\ell$, to be the cycle of length ℓ together with an additional vertex which is adjacent to two consecutive vertices on the cycle.*

Proof of Theorem 1.2. Let $n := |V(G)|$, denote $k := \tilde{\alpha}(G)$ and suppose that for $a \leq b$ and $a + b = k + 1$, between every two disjoint vertex sets of sizes a and b, there is an edge. By assumption, we have $\delta(G) \geq k$. Note also that as observed before G has independence number $\alpha(G) \leq k$. Note further that since $\delta(G) \geq \alpha(G)$, we have that if G is bipartite then it must be isomorphic to $K_{n/2,n/2}$. Finally, note also that G is connected. Indeed, consider two non-adjacent vertices u, v; if their neighbourhoods intersect, there clearly exists a uv-path; otherwise, since both neighbourhoods have size at least $\delta(G) \geq k \geq a, b$, there exists an edge between them and thus also a uv-path. □

Claim 2.3. *Either G contains a triangle or G is bipartite.*

Proof. For sake of contradiction, suppose it does not contain a triangle nor is it bipartite and consider any vertex $v \in G$. If its neighbourhood is of size at least $k + 1$, then as observed above it contains an edge, which together with v creates a triangle. Therefore, every vertex has degree k and its neighbourhood does not contain an edge.

Furthermore, observe that every two non-adjacent vertices u, v must have at least $b \geq \frac{k+1}{2}$ common neighbours. Indeed, suppose u has fewer than b neighbours in $N(v)$ and consider a set $S \subseteq N(v) \cup \{u\}$ of size precisely b which contains u and all of its neighbours in $N(v)$, and possibly some other vertices in $N(v)$. Now, by the assumption on the graph, there is an edge between S and $N(v) \setminus S$, since the sizes of these are b and a, respectively. However, this is a contradiction, since there are no edges between u and $N(v) \setminus S$ and any edge contained in $N(v)$ creates a triangle.

To finish, recall that G is non-bipartite and thus contains an odd cycle, which is then not a triangle. Further, it contains an induced odd cycle – indeed, the shortest odd cycle must be induced. Since this cycle is not a triangle, it must then contain three vertices x, y, z such that yz is an edge and y, z are not adjacent to x. Since by the previous paragraph we have that both y, z have at least $\frac{k+1}{2} > |N(x)|/2$ neighbours in $N(x)$, they have a common neighbour (in $N(x)$), which together with the edge yz creates a triangle, a contradiction. □

We will now continue with the proof *assuming that $b \geq 3$*, and in the end we will deal with the few simple remaining cases when $b \leq 2$. So assume G is not isomorphic to $K_{n/2,n/2}$, so it is not bipartite. Note that G contains a $\tilde{C}_3$ or a $\tilde{C}_4$. Indeed, we get this by considering the neighbourhoods of any two vertices lying on a triangle xyz, whose existence is guaranteed by the previous claim; if any two neighbourhoods intersect in a vertex outside of the triangle, this gives $\tilde{C}_3$. Otherwise,
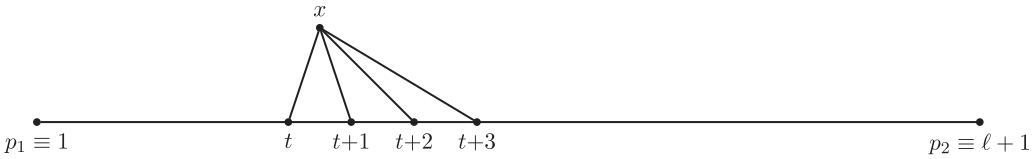
Figure 1. A $(t, 3)$-switch.

between $N(x) - \{y, z\}$ which is of size at least $k - 2 \geq a$, and the set $N(y) - \{x\} + \{y\}$ which is of size at least $k \geq b$, there is an edge which gives the required $\tilde{C}_3$ or $\tilde{C}_4$. Theorem 1.2 will now follow from the following lemma.

Lemma 2.4. *If G contains a copy of $\tilde{C}_\ell$ for some $\ell < n - 1$, then it also contains $\tilde{C}_{\ell+1}$ or $\tilde{C}_{\ell+2}$.*

Indeed, to finish the proof, note that since G contains $\tilde{C}_3$ or $\tilde{C}_4$, we can iteratively apply the lemma to get a family $\{\tilde{C}_\ell \mid \ell \in I\}$ of graphs which are all contained in G, such that for every pair $(i, i+1)$ of consecutive integers in $[3, n]$, one of the two is in I. Since each $\tilde{C}_\ell$ contains both C_ℓ and $C_{\ell+1}$ we are done.

Proof of Lemma 2.4. Suppose for sake of contradiction that G contains a $\tilde{C}_\ell$, but does not contain a $\tilde{C}_{\ell+1}$, nor a $\tilde{C}_{\ell+2}$. We fix such an ℓ until the end of the proof. The central gadget we use in our proof is given by the following definition. □

Definition 2.5. *A (t, s)-switch in G is a subgraph R which consists of a path $P = (1, 2, \ldots, \ell + 1)$ together with the vertex x adjacent to vertices $t, t + 1, \ldots, t + s$ with $t, s \geq 1$ (see Fig. 1). We also write $(t, \cdot)$-switch to denote a switch for which the s is not specified.*

Note first that a (t, s)-switch exists for some t and s. Indeed, since we have a $\tilde{C}_\ell$ and G is connected, there is an edge between $\tilde{C}_\ell$ and a vertex v outside of the $\tilde{C}_\ell$ - this evidently produces a $(t, 1)$-switch whose path starts at v.

Let us now take a (t, s)-switch R such that t is minimized and s is maximized with respect to t and consider the following ordering of its vertices: $\pi = (1, 2, \ldots t, x, t + 1, t + 2 \ldots \ell + 1)$, i.e. the natural order of the path P with x inserted between t and $t + 1$. Denote $p_1 := 1$ and $p_2 := \ell + 1$. Given $v \in V(R)$, we define v^+ to be the vertex which comes after v in the ordering π. Given a set of vertices $T \subset V(R)$, we define T^+ to be the vertices obtained by shifting T to the right by one, i.e., $T^+ = \{v^+ \mid v \in T\}$; similarly define T^-. We start with the following simple claim.

Claim 2.6. *If $t > 1$, then p_2 has no neighbours outside of $V(R)$. If $t = 1$ then p_2 has fewer than a neighbours outside of $V(R)$.*

Proof. First, note that if $t > 1$ and p_2 has a neighbour outside of R, we could add that neighbour to R, and remove p_1 from R, thus obtaining a $(t - 1, s)$-switch, a contradiction. Now, suppose that $t = 1$. Observe that p_1 has fewer than a neighbours outside of R. Indeed, let $A = N(p_1) \setminus V(R)$ and let T be the set of neighbours of $p_2 - 1$ in $R - \{p_2\}$, and let T_{out} be the set of neighbours of $p_2 - 1$ outside of $R - \{p_2\}$. Then, the set $T_{out} \cup T^+$ is of size at least $\delta(G) \geq k \geq b$ and does not contain any vertices in A, since this creates a $\tilde{C}_{\ell+1}$. If $|A| \geq a$, then there is an edge (i, j) between A and $T_{out} \cup T^+$, which creates either a $\tilde{C}_{\ell+1}$ or a $\tilde{C}_{\ell+2}$. Indeed, if $j \in T_{out}$ then obviously we get a $\tilde{C}_{\ell+2}$, if $j \in T^+ \setminus \{2, x\}$ then we get a $\tilde{C}_{\ell+1}$ as in Fig. 2(b), and if $j = 2$ then we get a $\tilde{C}_{\ell+1}$ whose triangle contains $1, i, 2$, while if $j = x$ then we get a $\tilde{C}_{\ell+1}$ whose triangle contains $1, i, x$. Hence, $|A| < a$.

To conclude the case when $t = 1$, suppose that p_2 has at least a neighbours outside R and denote the set of these by B. Since p_1 has fewer than a neighbours outside $V(R)$ by the previous paragraph, we can take a set T of at least $k - (a - 1) \geq b$ neighbours of p_1 in $V(R)$. Hence, there is an edge

(a)

The vertex left to the neighbour of p_1 is adjacent to the vertex to the right of the neighbour of p_2.

(b)

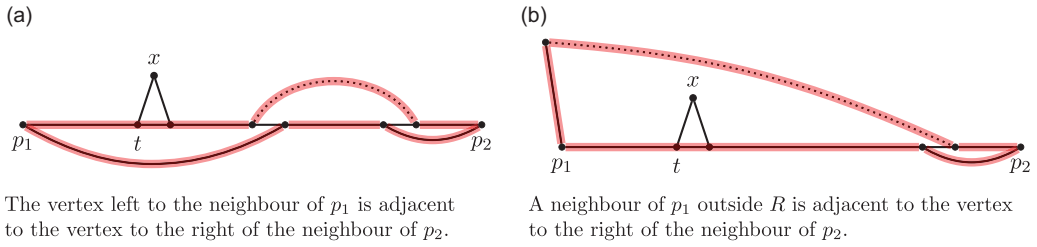A neighbour of p_1 outside R is adjacent to the vertex to the right of the neighbour of p_2.

Figure 2. In the first case we get a copy of $\tilde{C}_{\ell+1}$, and in the second a copy of $\tilde{C}_{\ell+2}$, whose respective cycles $C_{\ell+1}$ and $C_{\ell+2}$ are depicted in red.
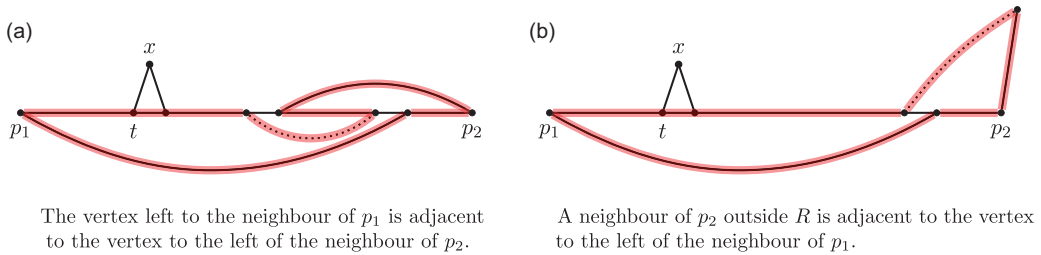
(a)

The vertex left to the neighbour of p_1 is adjacent to the vertex to the left of the neighbour of p_2.

(b)

A neighbour of p_2 outside R is adjacent to the vertex to the left of the neighbour of p_1.

Figure 3. In the first case we get a copy of $\tilde{C}_{\ell+1}$, and in the second a copy of $\tilde{C}_{\ell+2}$, whose respective cycles $C_{\ell+1}$ and $C_{\ell+2}$ are depicted in red.
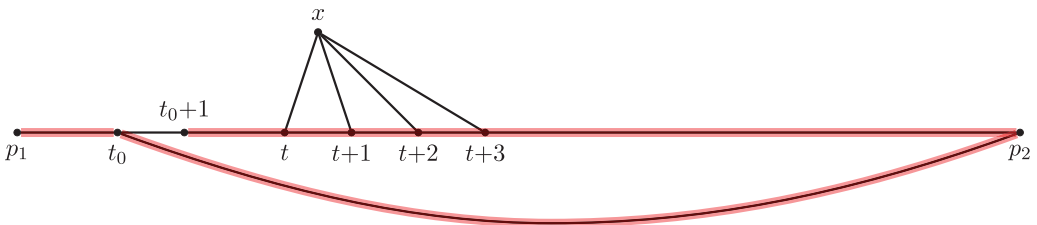

Figure 4. If p_2 has a neighbour before t then we can use the red path to create a (t_0, s)-switch.

(i, j) between T^- and B, which creates a $\tilde{C}_{\ell+2}$ (this is easy to see when $i = p_1$ or $i = x$; otherwise we get the same situation as illustrated in Fig. 3(b), a contradiction. □

Claim 2.7. *If $t > 1$, then p_2 has no neighbours t_0 with $t_0 < t$.*

Proof. Otherwise, their exists a $(t - t_0, s)$-switch, as depicted in Fig. 4, thus contradicting the optimality of R. □

Now, define the set S to consist of the last a neighbours of p_2 in π. Observe that by Claim 2.6 this set exists and as usual, let $\min(S)$ denote the smallest element of S in the ordering π. We then have the following.

Claim 2.8. $\min(S) \geq t + 1$.

Proof. If $t = 1$, note that p_2 is not adjacent to any of 1 or x since any such case would create a $\tilde{C}_{\ell+1}$, a contradiction. Therefore, $\min(S) \geq 2$. If $t > 1$, then Claims 2.6 and 2.7 imply that all of the at least k neighbours of p_2 are in $V(R)$ and all of them are larger or equal to t in π. Hence, at least $k - 2 \geq a$ (recall that we are assuming that $b \geq 3$) neighbours are larger or equal to $t + 1$ in π, which completes the proof. □
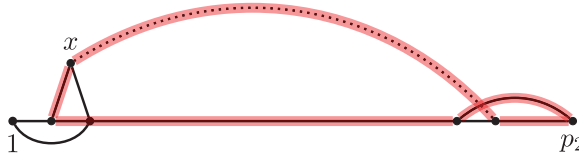
Figure 5. The case of $i = x$. We get a $\tilde{C}_{\ell+1}$ where the triangle consists of vertices 1, 2, 3.

From now on, we will differentiate between two scenarios:

(A) p_1 has fewer than a neighbours in the interval $[\min(S) + 1, p_2]$.
Then, denote by T the set of neighbours of p_1 in $[p_1, \min(S)]$, and by T_{out} the neighbours of p_1 outside of $V(R)$. Note that $|T| + |T_{out}| \geq k - (a - 1) = b$.

(B) p_1 has at least a neighbours in the interval $[\min(S) + 1, p_2]$.
Then, denote this set of neighbours by A, denote by T the set of neighbours of p_2 in $[p_1, \min(S)]$, by T_{out} the set of neighbours of p_2 outside of $V(R)$. Note that by definition of S, p_2 has precisely $a - 1$ neighbours in $[\min(S) + 1, p_2]$ and so we have that $|T| + |T_{out}| \geq k - (a - 1) = b$. Recall further that $T_{out} = \emptyset$ if $t > 1$ by Claim 2.6.

We will now consider a few cases, depending on the parameters s and t. We will argue that besides the edges of R, there exist additional edges in $G[R]$ which would imply the existence of a better switch, or a copy of $\tilde{C}_{\ell+1}$ or $\tilde{C}_{\ell+2}$ in G, thus giving a contradiction. For example, note that p_1 and p_2 are not adjacent, since this would create a copy of $\tilde{C}_{\ell+1}$ in G. In the figures below, we give some more complex examples of edges which we may find in G. In the following subsections, we will consider each one of these situations and we recommend the reader to focus on the figures below only when they are referred to in the proof. We recommend reading case (A) in all sections first, and subsequently case (B) in all sections.

2.1. Triangle at the start: $t = 1$

(A) holds:
Since $|T^-| + |T_{out}| = |T| + |T_{out}| \geq b$ and $|S^+| = |S| = a$, there is an edge between $T^- \cup T_{out}$ and S^+. This creates either a $\tilde{C}_{\ell+1}$ or a $\tilde{C}_{\ell+2}$ (see Fig. 2), so we are done.

(B) holds: Let $T_0 = T - \{2\}$. Note that none of p_1 and x are in T since otherwise a $\tilde{C}_{\ell+1}$ exists, and thus, $\min(T_0) > 2$. Now, $T_0^- \cup T_{out} \cup \{p_2\}$ is of size at least b and so, there is an edge between this set and A^-, which always creates either a $\tilde{C}_{\ell+1}$ or a $\tilde{C}_{\ell+2}$ (see Fig. 3).

2.2. Triangle starts at the second vertex: $t = 2$

(A) holds: Note that $x \notin T$ since otherwise there would exist a $(1, \cdot)$-switch, starting with the triangle $(1, x, 2)$. Consider then the set $T^- \cup T_{out}$ which is of size at least b. Between $T^- \cup T_{out}$ and S^+ (which is of size a) there is an edge (i, j), which creates a $\tilde{C}_{\ell+1}$ or a $\tilde{C}_{\ell+2}$. Indeed, if $i \neq x$ then we can proceed as in Fig. 2(a), and if $i = x$ we proceed as in Fig. 5.

(B) holds: Recall that $|T| \geq b$ since $T_{out} = \emptyset$, and that $p_1 \notin T$. If not both edges (x, p_2) and $(3, p_2)$ are present, then for each vertex (if it is in T) we can assign a unique vertex as follows: $2 \to 1$, $x \to 4$, and for all other vertices $v \to v + 1$. If T^* is the set of vertices assigned to T, then there is an edge (i, j) between T^* and A^+ (note that these are disjoint since $\min(A) > \max(T) = \min(S) \geq t + 1 = 3$, implying that all elements in A^+ are larger than $\max(T^*)$). This edge always creates a copy of $\tilde{C}_{\ell+1}$ when $i \neq 1$ (as in Fig. 6a), and when $i = 1$ then it creates a $(1, \cdot)$-switch (as in Fig. 6b), a contradiction.
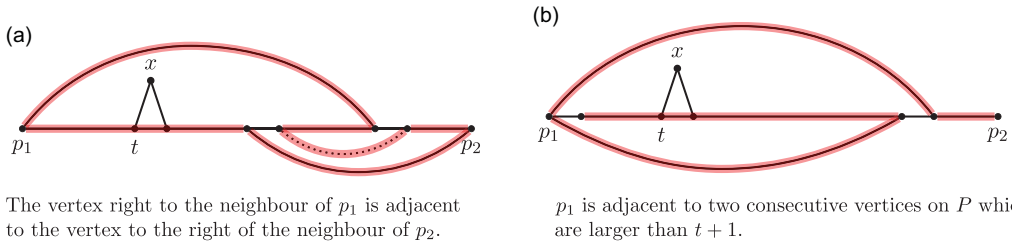
(a)

The vertex right to the neighbour of p_1 is adjacent
to the vertex to the right of the neighbour of p_2.

(b)

p_1 is adjacent to two consecutive vertices on P which
are larger than $t + 1$.

Figure 6. In the first case we get a copy of $\tilde{C}_{\ell+1}$, and in the second a $(t-1, 1)$-switch. The cycle $C_{\ell+1}$ and the switch are depicted in red.
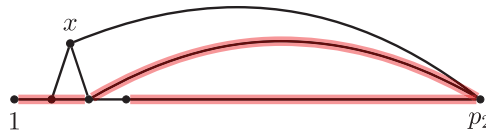

Figure 7. Obtaining a $(t, 2)$-switch, when p_2 is adjacent to both x and 3.

(a)

The $(t-1, \cdot)$-switch where the path is red,
and two blue edges to complete the triangle.

(b)

The $(t-2, \cdot)$-switch where the path is red,
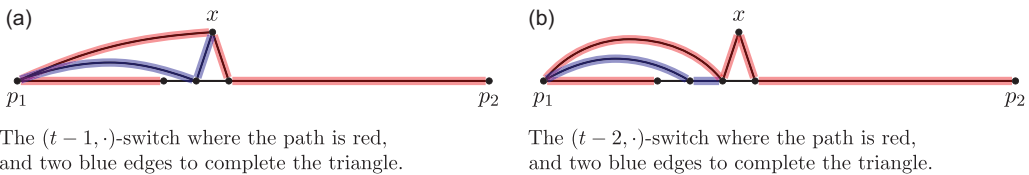and two blue edges to complete the triangle.

Figure 8. p_1 is not adjacent to both t and x, and p_1 is not adjacent to both t and $t-1$ as in both cases we create a better switch.

Otherwise, if both (x, p_2) and $(3, p_2)$ exist, then it must be that $s \geq 2$, since these edges can be used to form a $(t, 2)$-switch (see Fig. 7).

Now we define T^* differently as follows: $x \to p_1$ and $v \to v+1$ (recalling that $v+1$ is not the same as v^+) for all other v in T, and we can proceed as before, by finding an edge (i, j) between T^* and A^+. Note that if now $i = 3$, after doing rotation (as in Fig. 6a), we do destroy the triangle $2, x, 3$, but the triangle $3, x, 4$ is preserved, so it was crucial that $s \geq 2$.

2.3. Triangle in the middle: $t > 2$

(A) holds: First, note that p_1 is not adjacent to both t and x, and p_1 is not adjacent to both t and $t-1$, as in both cases we get a better switch (see Fig. 8).

Now, to each vertex in T we assign a unique vertex as follows, depending on the adjacencies between p_1 and the set $Q = \{t, t+1, x\}$:

   (i) If p_1 is adjacent to at most one vertex in Q, then assign: $x \to t-1, t+1 \to t-1$ and $v \to v^-$ for all other vertices in T.

   (ii) If p_1 is adjacent to only $x, t+1$ in Q, then: $x \to t, t+1 \to t-1$ and $v \to v^-$ for all other vertices in T.

   (iii) If p_1 is adjacent only to $t, t+1$ in Q, then by the observation above it is not adjacent to $t-1$. We then take: $t \to t-1, t+1 \to t-2$ and $v \to v^-$ for all other vertices in T.

As shown before, p_1 cannot be adjacent to both x, t and so, one of the options above must hold. Let T^* be the set of assigned vertices, and note that $|T^*| = |T|$ and thus $|T^* \cup T_{out}| \geq b$. Hence we
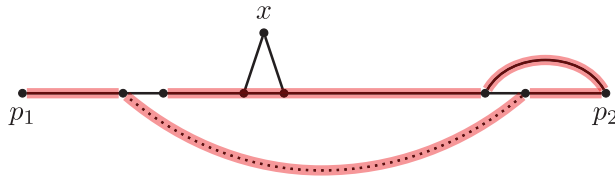
Figure 9. The red line represents the path of a switch with the triangle closer to p_1.
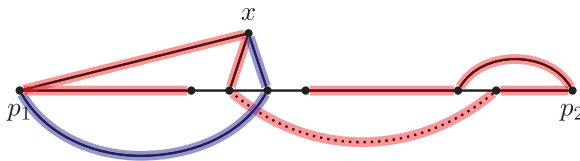

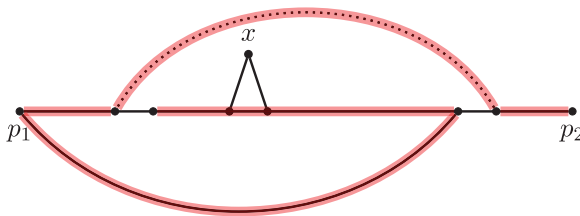**Figure 10.** The red line represents the path of a switch with the (blue) triangle closer to p_1.

Figure 11. The red line represents the path of a switch with the triangle closer to p_1.

have an edge (i, j) between T^* and S^+, and we check that in each case we either get a (t', s')-switch with some $t' < t$ or a $\tilde{C}_{\ell+1}$.

Indeed, for (i) we have a situation as depicted in Fig. 9 if $i = t - 1$, and Fig. 2 otherwise. For (ii) we have a situation as in Fig. 10 if $i = t$, as in Fig. 9 if $i = t - 1$ and otherwise we have again the situation in Fig. 2. For (iii) we have the situation of Fig. 9 if $i \in \{t - 1, t - 2\}$ and the situation of Fig. 2 otherwise.

(B) holds: Recall that by Claim 2.7, the vertex p_2 has no neighbours before t in the ordering π and that $T_{out} = \emptyset$. To each vertex in T we can then assign a unique vertex as follows: $x \to t - 2$, $t \to t - 1$ and $v \to v + 1$ for all other $v \in T$ (which must satisfy $v \geq t + 1$). Let T^* be the set of assigned vertices, and note then that $|T^*| = |T| \geq b$. Hence, we have an edge (i, j) between T^* and A^+, which are disjoint, as we already explained in Section 2.2, Part (B). In each case we either get a $(t', \cdot)$-switch with $t' < t$ or we create a $\tilde{C}_{\ell+1}$. Indeed, if $i = t - 1$ or $i = t - 2$ then we are done by Fig 11, otherwise we are done by Fig. 6a.

This completes the proof of Lemma 2.4.

2.4. Completing the proof: $b \leq 2$

When $a \leq b \leq 2$, the proof is significantly shorter. Indeed, this implies that there is an edge between any two disjoint sets of size at least 2 in G. If $|G| \leq 7$ one can check by hand that the statement holds and we leave this as an exercise to the reader (one can already assume that G is Hamiltonian, as guaranteed by Theorem 1.1, and then analyse what cycles are created by adding edges between pairs of vertices in the Hamilton cycle).

Otherwise, first note that by Theorem 1.1, G contains a Hamilton cycle. We will now show that if G contains a cycle of length ℓ with $6 \leq \ell \leq n - 2$, then it contains a cycle of length

$\ell + 1$. This reduces pancyclicity to only finding cycles of length 3, 4, 5 and 6 in G. Consider then a cycle C_ℓ in G and suppose for sake of contradiction that there is no $C_{\ell+1}$. Let x, y be two vertices outside of the cycle C_ℓ. Trivially, since $l \geq 6$, it must be that at least one of x, y has at least 3 neighbours in C_ℓ (otherwise there would be two vertices in C_ℓ not adjacent to $\{x, y\}$, contradicting the assumption on G). Without loss of generality, assume that x is adjacent to $z_1, z_2, z_3 \in C_\ell$. If any pair of vertices z_i, z_j is consecutive in the cycle C_ℓ, then we can extend this cycle using x to create a $C_{\ell+1}$. Otherwise, fix some orientation of C_ℓ and for each $v \in C_\ell$, denote by v^- the vertex before $v \in C_\ell$ in this orientation. By assumption, there is then an edge between the sets $\{z_1^-, x\}$ and $\{z_2^-, z_3^-\}$. In turn, it is easy to check that any such edge creates a cycle $C_{\ell+1}$ on the vertex set of $C_\ell + x$.

We now prove the existence of cycles of lengths from 3, 4, 5, 6. Note that we have already shown that G contains a triangle xyz in Claim 2.3. Now, if two vertices on the triangle have a common neighbour outside then we also have a C_4. Otherwise, note that since by assumption there is an edge between every two disjoint sets of size two, it must be that every pair of vertices has a vertex with degree at least $\frac{n-3}{2} > 2$ – therefore, two vertices in the triangle have a disjoint neighbourhood of size at least 2 outside the triangle. Between those two neighbourhoods there is an edge, which again gives a C_4 (and a C_5). If G does not have a C_5, then we are in the former case with two triangles sharing an edge (C_4 with a diagonal). Now, all except one vertex outside of these four vertices have at least two neighbours inside of it. The only way not to create a C_5 is to have all of these (at least two) vertices be adjacent only to the two vertices of degree 3. But then there is no edge between those vertices outside, and the remaining vertices of our C_4.

Finally, if we have a C_5 then again all but at most one vertex outside of it, are adjacent to at least two vertices in C_5. One can check that since we have at least 2 of them, this always gives a C_6 as well. This completes our proof.

3. Concluding remarks

Bondy's meta-conjecture states that every non-trivial condition which implies Hamiltonicity, also implies pancyclicity, up to a certain small collection of exceptional graphs. Clearly, there are some cases of natural Hamiltonicity conditions for which this statement fails. For example, it is well known (see [25]) that pseudo-random graphs are Hamiltonian, but even rather dense pseudo-random graphs might have no short cycles to be pancyclic. On the other hand, in addition to the results presented in this paper, we know by now that several well-known Hamiltonicity theorems can be extended to give pancyclicity, for example see [4, 6, 15, 28]. Hence, it would be interesting to explore other interesting Hamiltonicity conditions and understand whether they indeed imply pancyclicity.

References

[1] Ajtai, M., Komlós, J. and Szemerédi, E. (1985) First occurrence of Hamilton cycles in random graphs. In *Cycles in Graphs (Burnaby, B.C 1982)*, Vol. **115**. North-Holland Mathematical Studies, pp. 173–178.

[2] Alon, N. and Krivelevich, M. (2021) Divisible subdivisions. *J. Graph Theor.* **98**(4) 623–629.

[3] Alon, Y., Krivelevich, M. and Lubetzky, E. (2022) Cycle lengths in sparse random graphs. *Random Struct. Algor.* **61**(3) 444–461.

[4] Bauer, D. and Schmeichel, E. (1990) Hamiltonian degree conditions which imply a graph is pancyclic. *J. Comb. Theory Ser. B* **48**(1) 111–116.

[5] Bondy, J. A. (1973) *Pancyclic Graphs: Recent Results, Infinite and Finite Sets*. Colloquia Mathematica Societatis János Bolyai, pp. 181–187.

[6] Bondy, J. A. (1971) Pancyclic graphs I. *J. Comb. Theory Ser. B* **11**(1) 80–84.

[7] Bondy, J. A. (1980) *Longest Paths and Cycles in Graphs of High Degree*. Department of Combinatorics and Optimization, University of Waterloo.

[8] Bucić, M., Gishboliner, L. and Sudakov, B. (2022) Cycles of many lengths in Hamiltonian graphs. *Forum Math. Sigma* **10** E70.
[9] M (2022) Chen Hamilton-connected, vertex-pancyclic and bipartite holes. *Discrete Math.* **345**(12) 113158.
[10] Chvátal, V. (1972) On Hamilton's ideals. *J. Comb. Theory Ser. B* **12**(2) 163–168.
[11] Chvátal, V. and Erdős, P. (1972) A note on Hamiltonian circuits. *Discrete Math.* **2**(2) 111–113.
[12] Csaba, B., Kühn, D., Lo, A., Osthus, D. and Treglown, A. (2016) *Proof of the 1-factorization and Hamilton decomposition conjectures*. Memoirs of the American Mathematical Society, 244, Monograph 1154, 170 pages.
[13] Cuckler, B. and Kahn, J. (2009) Hamiltonian cycles in Dirac graphs. *Combinatorica* **29**(3) 299–326.
[14] Dirac, G. A. (1952) Some theorems on abstract graphs. *Proc. Lond. Math. Soc.* **3**(1) 69–81.
[15] Draganić, N., Munhá Correia, D.and Sudakov, B. (2023) Chvátal-Erdős condition for pancyclicity. J. Assoc. Math. Res. (in press).
[16] Draganić, N., Munhá Correia, D.and Sudakov, B. Pancyclicity of Hamiltonian graphs. *J. Eur. Math. Soc.* (in press).
[17] Erdős, P. (1972) Some problems in graph theory. In *Hypergraph Seminar*. Springer, pp. 187–190.
[18] Fan, G. H. (1984) New sufficient conditions for cycles in graphs. *J. Comb. Theory Ser. B* **37**(3) 221–227.
[19] Ferber, A., Long, E. and Sudakov, B. (2018) Counting Hamilton decompositions of oriented graphs. *Int. Math. Res. Notices* **2018**(22) 6908–6933.
[20] Friedman, L. and Krivelevich, M. (2021) Cycle lengths in expanding graphs. *Combinatorica* **41**(1) 53–74.
[21] Gould, R. J. (2014) Recent advances on the Hamiltonian problem: Survey III. *Graph Comb.* **30**(1) 1–46.
[22] Jackson, B. and Ordaz, O. (1990) Chvátal-Erdős conditions for paths and cycles in graphs and digraphs. A survey. *Discrete Math.* **84**(3) 241–254.
[23] Keevash, P. and Sudakov, B. (2010) Pancyclicity of Hamiltonian and highly connected graphs. *J. Comb. Theory Ser. B* **100**(5) 456–467.
[24] Krivelevich, M. (2011) The critical bias for the Hamiltonicity game is $(1 + o(1))n/\ln n$. *J. Am. Math. Soc.* **24**(1) 125–131.
[25] Krivelevich, M. and Sudakov, B. (2003) Sparse pseudo-random graphs are Hamiltonian. *J. Graph Theory* **42**(1) 17–33.
[26] Krivelevich, M., Lee, C. and Sudakov, B. (2014) Robust Hamiltonicity of Dirac graphs. *Trans. Am. Math. Soc.* **366**(6) 3095–3130.
[27] Kühn, D. and Osthus, D. (2013) Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments. *Adv. Math.* **237** 62–146.
[28] Letzter, S. (2023) Pancyclicity of highly connected graphs, arXiv preprint arXiv: 2306.12579.
[29] Liu, C.-H. and Ma, J. (2018) Cycle lengths and minimum degree of graphs. *J. Comb. Theory Ser. B* **128** 66–95.
[30] Liu, H. and Montgomery, R. (2023) A solution to Erdős and Hajnal's odd cycle problem. *J. Am. Math. Soc.* **36** 1191–1234.
[31] Kühn, D. and Osthus, D. (2014) Hamilton cycles in graphs and hypergraphs: an extremal perspective. In Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. IV, pp. 381–406. *Kyung Moon Sa, Seoul*.
[32] McDiarmid, C. and Yolov, N. (2017) Hamilton cycles, minimum degree, and bipartite holes. *J. Graph Theory* **86**(3) 277–285.
[33] Milans, K. G., Pfender, F., Rautenbach, D., Regen, F. and West, D. B. (2012) Cycle spectra of Hamiltonian graphs. *J. Comb. Theory Ser. B* **102**(4) 869–874.
[34] Pósa, L. (1976) Hamiltonian circuits in random graphs. *Discrete Math.* **14**(4) 359–364.
[35] Schmeichel, E. F. and Hakimi, S. L. (1988) A cycle structure theorem for Hamiltonian graphs. *J. Comb. Theory Ser. B* **45**(1) 99–107.
[36] Verstraëte, J. (2016) Extremal problems for cycles in graphs. In *Recent trends in combinatorics*. Springer, pp. 83–116.