Math. Proc. Camb. Phil. Soc. (2013), **155**, 13–37
doi:10.1017/S0305004113000169
First published online 25 April 2013

Curvature, cones and characteristic numbers

BY MICHAEL ATIYAH†

School of Mathematics, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ.
e-mail: m.atiyah@ed.ac.uk

AND CLAUDE LEBRUN‡

Department of Mathematics, SUNY, Stony Brook, NY
11794-3651*, U.S.A.*
e-mail: claude@math.sunysb.edu

(*Received* 22 *April* 2012; *revised* 29 *January* 2013)

Abstract

We study Einstein metrics on smooth compact 4-manifolds with an edge-cone singularity of specified cone angle along an embedded 2-manifold. To do so, we first derive modified versions of the Gauss–Bonnet and signature theorems for arbitrary Riemannian 4-manifolds with edge-cone singularities, and then show that these yield non-trivial obstructions in the Einstein case. We then use these integral formulæ to obtain interesting information regarding gravitational instantons which arise as limits of such edge-cone manifolds.

1. *Introduction*

Recall [**8**] that a Riemannian manifold (M, g) is said to be *Einstein* if it has constant Ricci curvature; this is equivalent to requiring that the Ricci tensor r of g satisfy

$$r = \lambda g$$

for some real number λ, called the Einstein constant of g. While one typically requires g to be a smooth metric on M, it is sometimes interesting to consider generalizations where g is allowed to have mild singularities. In the Kähler case, beautiful results [**11**, **20**, **32**] have recently been obtained regarding the situation in which g has specific conical singularities along a submanifold of real codimension 2. Einstein manifolds with such *edge-cone singularities* are the main focus of this paper.

Let M be a smooth n-manifold, and let $\Sigma \subset M$ be a smoothly embedded $(n-2)$-manifold. Near any point $p \in \Sigma$, we can thus find local coordinates $(y^1, y^2, x^1, \ldots, x^{n-2})$ in which Σ is given by $y^1 = y^2 = 0$. Given any such adapted coordinate system, we then introduce an associated *transversal polar coordinate* system $(\rho, \theta, x^1, \ldots, x^{n-2})$ by setting $y^1 = \rho\cos\theta$ and $y^2 = \rho\sin\theta$. We define a Riemannian edge-cone metric g of cone angle $2\pi\beta$ on (M, Σ) to be a smooth Riemannian metric on $M - \Sigma$ which, for some $\varepsilon > 0$, can be expressed as

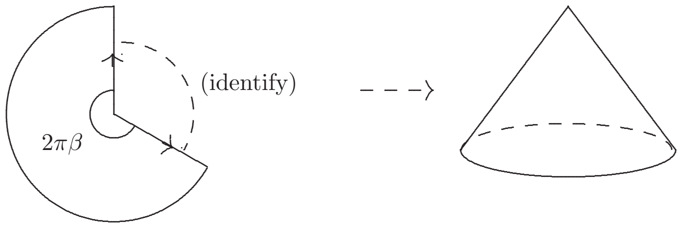
$$g = \bar{g} + \rho^{1+\varepsilon}h \tag{1·1}$$

† Research supported in part by the Simons Center for Geometry and Physics.
‡ Research supported in part by NSF grants DMS-0905159 and DMS-1205953.

near any point of Σ, where the symmetric tensor field h on M has infinite conormal regularity along Σ, and where

$$\bar{g} = d\rho^2 + \beta^2\rho^2(d\theta + u_j dx^j)^2 + w_{jk}dx^j dx^k \tag{1·2}$$

in suitable transversal polar coordinate systems; here $w_{jk}(x)dx^j dx^k$ and $u_j(x)dx^j$ are a smooth metric and a smooth 1-form on Σ. (Our conormal regularity hypothesis means that the components of h in (y, x) coordinates have infinitely many continuous derivatives with respect to $\partial/\partial x^j$, $\partial/\partial\theta$, and $\rho\,\partial/\partial\rho$.) Thus, an edge-cone metric g behaves like a smooth metric in directions parallel to Σ, but is modelled on a 2-dimensional cone



in the transverse directions.

If an edge-cone metric on (M, Σ) is Einstein on $M - \Sigma$, we will call it an Einstein edge-cone metric. For example, when $\beta < 1/3$, any Kähler–Einstein edge metric of cone angle $2\pi\beta$, in the sense of [**11**, **20**, **32**], can be shown [**32**, proposition 4·3] to be an Einstein edge-cone metric. Another interesting class, with $\beta = 1/p$ for some integer $p \geqslant 2$, is obtained by taking quotients of non-singular Einstein manifolds by cyclic groups of isometries for which the fixed-point set is purely of codimension 2. Some explicit Einstein edge-cone metrics of much larger cone angle are also known, as we will see in Section 5 below. Because of the wealth of examples produced by such constructions, it seems entirely reasonable to limit the present investigation to edge-cone metrics, as defined above. However, we do so purely as a matter of exigency. For example, it remains unknown whether there exist Einstein metrics with analogous singularities for which the cone angle varies along Σ. This is an issue which clearly merits thorough exploration.

The Hitchin–Thorpe inequality [**8**, **26**, **45**] provides an important obstruction to the existence of Einstein metrics on 4-manifolds. If M is a smooth compact oriented 4-manifold which admits a smooth Einstein metric g, then the Euler characteristic χ and signature τ of M must satisfy the two inequalities

$$(2\chi \pm 3\tau)(M) \geqslant 0$$

because both expressions are represented by Gauss–Bonnet-type integrals where the integrands become non-negative in the Einstein case. Note that this inequality hinges on several peculiar features of 4-dimensional Riemannian geometry, and that no analogous obstruction to the existence of Einstein metrics is currently known in any dimension $\geqslant 4$.

In light of the current interest in Einstein metrics with edge-cone singularities, we believe it is interesting and natural to look for obstructions to their existence which generalize our understanding of the smooth case. Our main objective here will be to prove the following version of the Hitchin–Thorpe inequality for edge-cone metrics:

THEOREM A. *Let (M, Σ) be a pair consisting of a smooth compact 4-manifold and a fixed smoothly embedded compact oriented surface. If (M, Σ) admits an Einstein edge-cone*

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

metric with cone angle $2\pi\beta$ along Σ, then (M, Σ) must satisfy the two inequalities

$$(2\chi \pm 3\tau)(M) \geqslant (1-\beta)\left(2\chi(\Sigma) \pm (1+\beta)[\Sigma]^2\right).$$

To show this, we will cast our net a good deal wider. In Sections 2–3, we prove edge-cone generalizations of the 4-dimensional Gauss–Bonnet and signature formulæ; these results, Theorems 2·1 and 2·2, do not involve the Einstein condition, but rather apply to arbitrary edge-cone metrics on compact 4-manifolds. We then zero in on the Einstein case in Section 4, proving Theorem A and exploring some of its implications. But Theorems 2·1 and 2·2 have broader ramifications. In Section 5, we apply them to the study of some explicit self-dual edge-cone metrics, and explore the remarkable way that certain gravitational instantons arise as limits of edge-cone manifolds as $\beta \to 0$.

2. *Curvature integrals and topology*

The Euler characteristic χ and signature τ of a smooth compact 4-manifold M may both be calculated by choosing any smooth Riemannian metric g on M, and then integrating appropriate universal quadratic polynomials in the curvature of g. When g has an edge-cone singularity, however, correction terms must be introduced in order to compensate for the singularity of the metric along the given surface $\Sigma \subset M$.

THEOREM 2·1. *Let M be a smooth compact oriented 4-manifold, and let $\Sigma \subset M$ be a smooth compact oriented embedded surface. Then, for any edge-cone metric g on (M, Σ) with cone angle $2\pi\beta$,*

$$\chi(M) - (1-\beta)\chi(\Sigma) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu. \tag{2·1}$$

THEOREM 2·2. *Let M be a smooth compact oriented 4-manifold, and let $\Sigma \subset M$ be a smooth compact oriented embedded surface. Then, for any edge-cone metric g on (M, Σ) with cone angle $2\pi\beta$,*

$$\tau(M) - \frac{1}{3}(1-\beta^2)[\Sigma]^2 = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu. \tag{2·2}$$

Here s, $\mathring{r}$, and W are the scalar curvature, trace-free Ricci tensor, and Weyl curvature of g, while W_+ and W_- are the self-dual and anti-self-dual parts of W, and $d\mu$ is the metric volume 4-form. We follow standard conventions [**8**] by defining

$$|\mathring{r}|^2 := \mathring{r}_{jk}\mathring{r}^{jk}, \qquad |W|^2 := \frac{1}{4}W_{jk\ell m}W^{jk\ell m}$$

where the factor of $1/4$ arises from treating W as an element of $\Lambda^2 \otimes \Lambda^2$; the point-wise norms of $W_\pm$ are defined analogously, so that

$$|W|^2 = |W_+|^2 + |W_-|^2.$$

The expression $[\Sigma]^2$ denotes the self-intersection of the homology class of Σ in $H_2(M) \cong H^2(M)$, and coincides with the Euler class of the normal bundle of Σ, paired with the fundamental cycle of the surface.

We first discovered Theorems 2·1 and 2·2 in the context of global-quotient orbifolds, and we outline a method for deducing the general case from this special one in Section 3 below.

Another workable strategy, which we leave to the interested reader, would be to apply the Gauss–Bonnet and signature theorems with boundary [**6**] to the complement of a tubular neighbourhood of $\Sigma \in M$, and then take limits as the radius of the tube tends to zero. However, we will instead begin here by giving a complete and self-contained proof by yet a third method, of a purely differential-geometric flavor.

For this purpose, observe that, by taking linear combinations, equations (2·1) and (2·2) are equivalent to the pair of equations

$$(2\chi + 3\tau)(M) - \mathrm{def}_+(\Sigma, \beta) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \tag{2·3}$$

$$(2\chi - 3\tau)(M) - \mathrm{def}_-(\Sigma, \beta) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \tag{2·4}$$

provided we define the *defects* $\mathrm{def}_\pm(M, \Sigma, \beta)$ to be

$$\mathrm{def}_\pm(\Sigma, \beta) = 2(1-\beta)\chi(\Sigma) \pm (1-\beta^2)[\Sigma]^2\,. \tag{2·5}$$

However, equations (2·3) and (2·4) are interchanged by simply reversing the orientation of M. To prove Propositions 2·1 and 2·2, it therefore suffices to prove that (2·3) holds for any edge-cone metric on (M, Σ) with cone angle $2\pi\beta$, with defect $\mathrm{def}_+(\Sigma, \beta)$ given by (2·5). We now simplify the problem further by showing that, for each (M, Σ) and β, it suffices to check that (2·3) holds for a single edge-cone metric on (M, Σ) of cone angle $2\pi\beta$.

LEMMA 2·1. *Let (M, Σ) be a smooth compact oriented 4-manifold equipped with a smooth compact oriented embedded surface, and let β be a positive real number. Fix two real constants a and b, and consider the formula for $(a\chi + b\tau)(M)$ obtained by taking the corresponding linear combination of* (2·1) *and* (2·2)*. If this formula holds for one edge-cone metric on (M, Σ) of cone angle $2\pi\beta$, it also holds for every other edge-cone metric on (M, Σ) of the same cone angle.*

Proof. The Gauss–Bonnet and signature integrands are multiples of the 4-forms

$$\Phi_{abcd} = (\star\mathcal{R})_{jk[ab}\mathcal{R}^{jk}{}_{cd]}$$
$$\Psi_{abcd} = \mathcal{R}_{jk[ab}\mathcal{R}^{jk}{}_{cd]}$$

corresponding to the Pfaffian and second symmetric polynomial of curvature. Given a one-parameter family

$$g_t := g + t\dot{g} + O(t^2),$$

of Riemannian metrics, the t-derivative (at $t = 0$) of the curvature operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$ is given by

$$\dot{\mathcal{R}}_{cd}^{ab} = -2\nabla_{[c}\nabla^{[a}\dot{g}_{d]}^{b]} + \dot{g}^{e[a}\mathcal{R}^{b]}{}_{ecd},$$

where ∇ is the Levi–Civita connection of g, and where indices are raised and lowered with respect to g. Thus

$$\dot{\Phi}_{abcd} = -2\nabla_{[a|}\left(\epsilon_{mnjk}\mathcal{R}^{mn}{}_{|cd}\nabla^j\dot{g}^k_{b]}\right)$$
$$\dot{\Psi}_{abcd} = -4\nabla_{[a|}\left(\mathcal{R}_{jk|cd}\nabla^j\dot{g}^k_{b]}\right)$$

where the cancellation of the purely algebraic terms is a nice exercise in the representation theory of $\mathbf{SO}(4)$. In other words,

$$\dot{\Phi} = d\phi \qquad\qquad \dot{\Psi} = d\psi$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

for 3-forms

$$\phi_{bcd} = -(\star\mathcal{R})_{jk[bc}\nabla^j \dot{g}^k_{d]}$$
$$\psi_{bcd} = -\mathcal{R}_{jk[bc}\nabla^j \dot{g}^k_{d]}$$

which obviously satisfy

$$|\phi|, |\psi| \leqslant |\mathcal{R}||\nabla\dot{g}| \,. \tag{2·6}$$

Now we have defined an edge-cone metric g on (M, Σ) of cone angle $2\pi\beta$ to be a tensor field which is smooth on $M - \Sigma$, and which can be written as $\bar{g} + \rho^{1+\varepsilon}h$ for some $\varepsilon > 0$, where h has infinite conormal regularity at Σ, and where the background metric $\bar{g}$ takes the form

$$\bar{g} = d\rho^2 + \beta^2\rho^2(d\theta + \nu)^2 + g_\Sigma$$

in suitable transverse polar coordinates, where ν and g_Σ are the pull-backs to a tubular neighbourhood of Σ of a smooth 1-form and a smooth Riemannian metric on Σ. While ρ and θ can be related to a system of smooth coordinates (y^1, y^2, x^1, x^2) by $y^1 = \rho\cos\theta$, $y^2 = \rho\sin\theta$, we now set $\tilde{\theta} = \beta\theta$, and introduce new local coordinates on transversely wedge-shaped regions by setting $\tilde{y}^1 = \rho\cos\tilde{\theta}$, $\tilde{y}^2 = \rho\sin\tilde{\theta}$. In these coordinates, $\bar{g}$ simply appears to be a smooth metric with $\mathbf{SO}(2)$-symmetry around Σ, while the tensor field h still has the same infinite conormal regularity as before. Thus the first derivatives of the components of g in $(\tilde{y}, x)$ coordinates are smooth plus terms of order ρ^ε, while the second derivatives are no worse than $\rho^{-1+\varepsilon}$. Since g^{-1} is also continuous across Σ, it follows that the Christoffel symbols $\Gamma^j_{k\ell}$ of g in $(\tilde{y}, x)$ coordinates are bounded, and that the norm $|\mathcal{R}|_g$ of the curvature tensor at worst blows up like $\rho^{-1+\varepsilon}$.

Given two edge-cone metrics g and g' on (M, Σ) of the same cone angle $2\pi\beta$, we first apply a diffeomorphism to (M, Σ) in order to arrange that the two choices of radius functions ρ and identifications of the normal bundle of Σ with a tubular neighborhood agree; thus, without interfering with our curvature integrals, we can assume that the two given choices of $\bar{g}$ differ only insofar as they involve different choices of ν and g_Σ. The 1-parameter family $g_t = (1 - t)g + tg'$, $t \in [0, 1]$ is then a family of edge-cone metrics on (M, Σ) of fixed cone angle $2\pi\beta$.

We will now show that $\int_M \Phi_{g_t}$ and $\int_M \Psi_{g_t}$ are independent of t. To see this, let us write

$$\frac{d}{dt}\Phi_{g_t} = d\phi_t \qquad\qquad \frac{d}{dt}\Psi_{g_t} = d\psi_t$$

as above, and notice that (2·6) tells us that

$$|\phi_t|, |\psi_t| \leqslant C\rho^{-1+\varepsilon}$$

for some positive constants C and ε determined by g and g', since the first derivatives of $\dot{g} = g' - g$ in $(\tilde{y}, x)$ coordinates are bounded. Now let M_δ denote the complement of a tube $\rho < \delta$ around Σ. Then

$$\left|\frac{d}{dt}\int_{M_\delta}\Phi_{g_t}\right| = \left|\int_{M_\delta}\frac{d}{dt}\Phi_{g_t}\right| = \left|\int_{\partial M_\delta}\phi_{g_t}\right| < C\delta^{-1+\varepsilon}\mathrm{Vol}^{(3)}(\partial M_\delta, g_t) < \tilde{C}\delta^\varepsilon$$

for some t-independent constant $\tilde{C}$. Integrating in $t \in [0, 1]$, we thus have

$$\left|\int_{M_\delta}\Phi_{g'} - \int_{M_\delta}\Phi_g\right| < \tilde{C}\delta^\varepsilon,$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

and taking the limit $\delta \to 0$ therefore yields

$$\int_M \Phi_{g'} = \int_M \Phi_g.$$

Replacing ϕ_t with ψ_t similarly proves that $\int_M \Psi_{g'} = \int_M \Psi_g$. Thus, if a given linear combination of (2·1) and (2·2) is true for some edge-cone metric g, it is also true for any other edge-cone metric g' of the same cone angle.

We may thus focus our attention on proving (2·3) for some particular edge-cone metric for each (M, Σ) and $\beta > 0$. For any given metric, let us therefore adopt the provisional notation

$$\Upsilon = \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \tag{2·7}$$

for the 4-form appearing as the integrand in (2·3). If g is an edge-cone metric of cone angle $2\pi\beta$ on (M, Σ) and if g_0 is a smooth metric on M, then (2·3) is equivalent to the claim that

$$\int_M \left(\Upsilon_g - \Upsilon_{g_0}\right) = -4\pi^2 \operatorname{def}_+(\Sigma, \beta), \tag{2·8}$$

since integral of Υ_{g_0} is $4\pi^2(2\chi + 3\tau)(M)$ by the standard Gauss–Bonnet and signature theorems. Of course, if we can actually arrange for g and g_0 to exactly agree on the complement of some tubular neighbourhood $\mathcal{V}$ of Σ, this reduces to the statement that

$$\int_{\mathcal{V}-\Sigma} \left(\Upsilon_g - \Upsilon_{g_0}\right) = -4\pi^2 \operatorname{def}_+(\Sigma, \beta)$$

since $\Upsilon_g - \Upsilon_{g_0}$ is then supported in $\mathcal{V}$, and Σ has 4-dimensional measure zero. In the proof that follows, we will not only choose g and g_0 to be related in this manner, but also arrange for both of them to be Kähler on $\mathcal{V}$. Now

$$|W_+|^2 = \frac{s^2}{24}$$

for any Kähler metric on a compatibly oriented 4-manifold, so (2·7) simplifies in the Kähler case to become

$$\Upsilon = \frac{1}{2}\left(\frac{s^2}{4} - |\mathring{r}|^2\right) d\mu = \varrho \wedge \varrho \tag{2·9}$$

where ϱ is the Ricci form. This will reduce the problem to showing that

$$\int_{\mathcal{V}-\Sigma} \left(\varrho_g \wedge \varrho_g - \varrho_{g_0} \wedge \varrho_{g_0}\right) = -4\pi^2(1-\beta)\left[2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right] \tag{2·10}$$

and, by exploiting some key properties of the Ricci form, this will follow by an application of Stokes' theorem.

Our choice of g_0 will involve an auxiliary function $F(t)$, which we will now construct. First let $f : \mathbb{R} \to \mathbb{R}^+$ be a smooth positive function with

$$\int_0^1 f(t)dt = \frac{1}{\beta} \tag{2·11}$$

such that

$$f(t) = \begin{cases} 1/\beta & \text{when } t \leqslant \frac{1}{2} \text{ and} \\ t^{\beta-1} & \text{when } t \geqslant 1. \end{cases}$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

We next define $F(t)$ up to a constant of integration by setting

$$\frac{dF}{dt} = \frac{1}{t}\int_0^t f(x)\,dx, \tag{2·12}$$

where this definition of course entails that $F' \equiv 1/\beta$ for $t \leqslant 1/2$. Because F consequently solves the differential equation

$$\frac{d}{dt}\left(t\,\frac{dF}{dt}\right) = f(t) \tag{2·13}$$

it follows that the smooth, rotationally symmetric metric on the ζ-plane $\mathbb{C}$ with Kähler form

$$i\partial\bar{\partial}F(|\zeta|^2) = if(|\zeta|^2)\,d\zeta\wedge\bar{\zeta} \tag{2·14}$$

is Euclidean near the origin and coincides with a standard cone of perimeter angle $2\pi\beta$ outside the unit disk. Condition (2·11) guarantees that $F'(1) = 1/\beta$, so inspection of the differential equation (2·13) tells us that

$$F(t) = \frac{t^\beta}{\beta^2} + B$$

for all $t \geqslant 1$, for a constant of integration B which we may take to vanish.

PROPOSITION 2·1. *For any compact oriented pair* (M^4, Σ^2) *and any positive real number* β*, there is an edge-cone metric* g *on* (M, Σ) *with cone angle* $2\pi\beta$ *such that* (2·3) *holds, with defect* def_+ *given by* (2·5).

Proof. Choose any metric g_Σ on Σ, and remember that, in conjunction with the given orientation, its conformal class makes Σ into a compact complex curve; in particular, the area form α of g_Σ then becomes its Kähler form. Let $\varpi : E \to \Sigma$ be the normal bundle of $\Sigma \subset M$, and choose an inner product $\langle\ ,\ \rangle$ on E; this reduces the structure group of E to $\mathbf{SO}(2) = \mathbf{U}(1)$, and so makes it into a complex line bundle. Next, we choose a $\langle\ ,\ \rangle$-compatible connection ∇ on E whose curvature is a constant multiple of α on each connected component of Σ. Viewing $\nabla^{0,1}$ as a $\bar{\partial}$-operator on E then makes it into a holomorphic line bundle over Σ, in a unique manner that identifies ∇ with the Chern connection induced by $\langle\ ,\ \rangle$ and the holomorphic structure. Thus the curvature of ∇ is $-i\kappa\alpha$, where the locally constant real-valued function κ takes the value

$$\kappa|_{\Sigma_j} = \frac{2\pi\int_{\Sigma_j} c_1(E)}{\int_{\Sigma_j}\alpha} = \frac{2\pi\,[\Sigma_j]^2}{\int_{\Sigma_j}\alpha}$$

on the jth connected component Σ_j of Σ.

We will let $t : E \to \mathbb{R}$ denote the square-norm function $t(v) = \|v\|^2$, and our computations will involve various closed $(1,1)$-forms expressed as $i\partial\bar{\partial}u(t)$ for various functions $u(t)$. To understand such expressions explicitly, first choose a local coordinate z on Σ so that near the origin

$$\alpha = i[1 + O(|z|^2)]\,dz\wedge d\bar{z},$$

and then choose a local trivialization of E determined by a local holomorphic section ξ with vanishing covariant derivative and unit norm at the origin. Then the function $\hbar := \|\xi\|^2$ satisfies

$$\hbar = 1 + O(|z|^2), \qquad \partial\bar{\partial}\hbar = -\kappa\,dz\wedge d\bar{z} + O(|z|),$$

because $-\partial\bar{\partial}\log \hbar = -i\kappa\alpha$ is the curvature of E. Thus, introducing a fiber coordinate ζ associated with the local trivialization, then, near the ζ-axis which represents the fiber over $z = 0$, we have

$$i\partial\bar{\partial}u(t) = i(tu')'d\zeta \wedge d\bar{\zeta} - i\kappa(tu')dz \wedge d\bar{z} + O(|z|)$$

where $t = |\zeta|^2$ along the ζ-axis. Since the chosen point $z = 0$ of Σ was in fact arbitrary, this calculation of course actually computes $i\partial\bar{\partial}u(t)$ along *any* fiber. For example, consider the (1, 1)-forms

$$\omega = \lambda\varpi^*\alpha + i\partial\bar{\partial}\left(\beta^{-2}t^{\beta}\right) \tag{2·15}$$

and

$$\omega_0 = \lambda\varpi^*\alpha + i\partial\bar{\partial}F(t) \tag{2·16}$$

on E, for some large positive constant λ, where F is given by (2·11). Explicitly, these are given along the ζ-axis of our coordinate system by

$$\begin{aligned}\omega &= it^{\beta-1}d\zeta \wedge d\bar{\zeta} + i\left(\lambda - \frac{\kappa}{\beta}t^{\beta}\right) dz \wedge d\bar{z} + O(|z|)\\ \omega_0 &= if(t)d\zeta \wedge d\bar{\zeta} + i(\lambda - \kappa t F'(t))\, dz \wedge d\bar{z} + O(|z|)\end{aligned}$$

so, for λ sufficiently large, these are the Kähler forms of Kähler metrics $\tilde{g}$ and $\tilde{g}_0$ defined on, say, the region $0 < t < 3$. Notice that $\tilde{g}_0$ is smooth across $t = 0$, and that we have arranged for $\tilde{g}$ and $\tilde{g}_0$ to coincide when $t > 1$. Also observe that $\tilde{g}$ becomes a genuine edge-cone metric on (M, Σ) after making the coordinate change $\zeta = (\beta\rho)^{1/\beta}e^{i\theta}$. Note that, for the purposes of systematically comparing the integrals associated with g and $\tilde{g}$, we have, in effect, pulled $\tilde{g}$ back with respect to a fixed self-diffeomorphism of $M - \Sigma$, although this is of course completely harmless for present purposes.

We now identify E with a tubular neighbourhood $\mathcal{U}$ of Σ via some diffeomorphism, and, for any real number $T > 0$, we let $\mathcal{U}_T \subset \mathcal{U}$ denote the closed tubular neighbourhood corresponding to the region $t \leqslant T$ of E. We then use a cut-off function to extend $\tilde{g}$ and $\tilde{g}_0$ to M as Riemannian metrics, in such a way that they exactly agree on the complement of $\mathcal{U}_1$, but are undamaged by the cut-off on $\mathcal{U}_2$. Calling these extensions g and g_0, respectively, we then see that our basic desiderata have all been fulfilled: g is an edge-cone metric of cone angle $2\pi\beta$ on (M, Σ), g_0 is a smooth Riemannian metric on M, both are Kähler on a tubular neighbourhood $\mathcal{V} = \operatorname{Int}\mathcal{U}_2$ of Σ, and the two metrics agree on the complement of a smaller tubular neighbourhood $\mathcal{U}_1$ of Σ.

Since i times the Ricci form is the curvature of the canonical line bundle,

$$\varrho - \varrho_0 = d\varphi \tag{2·17}$$

where the 1-form

$$\varphi = i\partial\log(V/V_0) \tag{2·18}$$

is defined in terms of the ratio $V/V_0 = d\mu_g/d\mu_{g_0}$ of the volume forms of g and g_0. Thus

$$\varrho^2 - \varrho_0^2 = (\varrho - \varrho_0) \wedge (\varrho + \varrho_0) = d\left(\varphi \wedge [2\varrho_0 + d\varphi]\right)$$

and Stokes' theorem therefore tells us that

$$\int_{M-\mathcal{U}_\epsilon} \left(\Upsilon_g - \Upsilon_{g_0}\right) = \int_{\mathcal{U}_2-\mathcal{U}_\epsilon} d\left(\varphi \wedge [2\varrho_0 + d\varphi]\right) = -\int_{S_\epsilon} \varphi \wedge [2\varrho_0 + d\varphi] \tag{2·19}$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

where the level set S_ϵ defined by $t = \epsilon$ has been given the outward pointing orientation relative to Σ. However, relative to the basis provided by the 4-form $-dz \wedge d\bar{z} \wedge d\zeta \wedge d\bar{\zeta}$ along the ζ-axis, the volume forms of g and g_0 are represented by the component functions

$$V = t^{\beta-1}(\lambda - \frac{\kappa}{\beta}t^\beta) \quad \text{and} \quad V_0 = (\lambda - \kappa t F'(t)) f(t)$$

where the latter simplifies when $t < 1/2$ to become

$$V_0 = \left(\lambda - \frac{\kappa}{\beta}t\right)\beta^{-1}.$$

Hence the 1-form defined by (2·18) is given by

$$\varphi = i\left((\beta - 1) - \frac{\kappa t^\beta}{\lambda - \frac{\kappa}{\beta}t^\beta} + \frac{\kappa t/\beta}{\lambda - \frac{\kappa}{\beta}t}\right)\partial \log t$$

in the region $\mathcal{U}_{1/2} - \Sigma$. Restricting this to S_ϵ, then along the ζ-axis this expression is just

$$\begin{aligned}\varphi &= \left((1 - \beta) + \frac{\kappa\epsilon^\beta}{\lambda - \frac{\kappa\epsilon}{\beta}\epsilon^\beta} - \frac{\kappa\epsilon/\beta}{\lambda - \frac{\kappa}{\beta}\epsilon}\right) d\theta \\ &= \left[(1 - \beta) + O(\epsilon^{\min(1,\beta)})\right] d\theta\end{aligned} \qquad (2{\cdot}20)$$

since $\zeta = \sqrt{\epsilon}e^{i\theta}$ along the intersection of S_ϵ and the ζ-axis. On the other hand,

$$\begin{aligned}d\varphi &= -i\partial\bar{\partial}\log(V/V_0) \\ &= i\kappa\left((\beta - 1) - \frac{\kappa t^\beta}{\lambda - \frac{\kappa}{\beta}t^\beta} + \frac{\kappa t/\beta}{\lambda - \frac{\kappa}{\beta}t}\right) dz \wedge d\bar{z} + U(t)\, d\zeta \wedge d\bar{\zeta}\end{aligned}$$

along the ζ-axis, for some function $U(t)$. Since this calculation is valid along any fiber, it follows that, for $0 < \epsilon < 1$,

$$j_\epsilon^* d\varphi = \left[(\beta - 1) + O(\epsilon^{\min(1,\beta)})\right] j_\epsilon^* \varpi^*(\kappa\alpha)$$

where $j_\epsilon : S_\epsilon \hookrightarrow M$ denotes the inclusion map, and $\varpi : E \to \Sigma$ once again denotes the bundle projection. Similarly, letting $j : \Sigma \hookrightarrow M$ be the inclusion, we have

$$j_\epsilon^* \varrho_0 = j_\epsilon^* \varpi^* j^* \varrho_0 + O(\epsilon)$$

since ϱ_0 is smooth across Σ and is invariant under the action of $S^1 = \mathbf{U}(1)$ on E. Integration over the fibers of $S_\epsilon \to \Sigma$ therefore yields

$$\int_{S_\epsilon} \varphi \wedge [2\varrho_0 + d\varphi] = 2\pi(1 - \beta)\int_\Sigma [2\varrho_0 + (\beta - 1)\kappa\alpha] \quad + O(\epsilon^{2\min(1,\beta)}) \qquad (2{\cdot}21)$$

by virtue of (2·20). But since $\varrho_0/2\pi$ represents $c_1(T^{1,0}E) = c_1(T^{1,0}\Sigma) + c_1(E)$ in deRham cohomology, and since $c_1(E)$ is similarly represented by $\kappa\alpha/2\pi$, we have

$$\begin{aligned}\int_\Sigma [2\varrho_0 + (\beta - 1)\kappa\alpha] &= 2\pi\left[2\left(\mathbf{c}_1(T^{1,0}\Sigma) + \mathbf{c}_1(E)\right) + (\beta - 1)\mathbf{c}_1(E)\right] \\ &= 2\pi\left[2\mathbf{c}_1(T^{1,0}\Sigma) + (\beta + 1)\mathbf{c}_1(E)\right] \\ &= 2\pi\left(2\chi(\Sigma) + (\beta + 1)[\Sigma]^2\right)\end{aligned}$$

where the boldface Chern classes have been evaluated on the homology class of Σ. Plugging this into (2·21), we obtain

$$\int_{S_\epsilon} \varphi \wedge \left[2\varrho_0 + d\varphi\right] = 4\pi^2(1-\beta)\left(2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right) \quad + O(\epsilon^{2\min(1,\beta)}).$$

Substituting this into (2·19) and taking the limit as $\epsilon \to 0$ thus yields

$$\int_M \left(\Upsilon_g - \Upsilon_{g_0}\right) = -4\pi^2(1-\beta)\left[2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right] \tag{2·22}$$

which is exactly the sought-after identity (2·8) for the particular metrics g and g_0. Applying the Gauss–Bonnet and signature fomulæ to the smooth metric g_0 now transforms (2·22) into

$$\frac{1}{4\pi^2}\int_M \Upsilon_g = (2\chi + 3\tau)(M) - (1-\beta)\left[2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right]$$

which is exactly (2·3), with defect $\mathrm{def}_+(M, \Sigma)$ given by (2·5).

Theorems 2·1 and 2·2 now follow. Indeed, given any (M, Σ) and β, Proposition 2·1 shows that (2·3) holds for some edge-cone metric g on (M, Σ) of cone angle $2\pi\beta$, and Lemma 2·1 thus shows that the same is true of any edge-cone metric on any (M, Σ) for any β. Applying this conclusion to the orientation-reversed manifold $\overline{M}$ shows that (2·4) also holds, and taking appropriate linear combinations then proves Theorems 2·1 and 2·2.

So far, we have assumed that Σ is oriented, but this is not essential. Of course, (2·1) makes perfectly good sense even if Σ is non-orientable, but more must be said about (2·2). As long as M is oriented, the Euler class of the normal bundle of $\Sigma \subset M$ will have the twisted coefficients needed to be consistently integrated on Σ, and this normal bundle will therefore have a well-defined Euler number which counts the zeroes, with multiplicities, of a generic section. When $\Sigma \subset M$ is non-orientable, we now decree that $[\Sigma]^2$ is to be interpreted in (2·2) as meaning the Euler number of its normal bundle rather than being defined in terms of homology classes on M. Now this Euler number can be calculated by passing to an oriented double cover of Σ, integrating the Euler class of the pull-back, and then dividing by 2; meanwhile, the correction term in (2·2) is represented by an integral supported in a tubular neighbourhood of Σ, so one can similarly compute it by passing to a double cover of a tubular neighbourhood of Σ and then dividing by 2. This covering trick allows us to prove Theorem 2·2 even when Σ is non-orientable, and Theorem 2·1 even if neither M nor Σ is orientable.

This observation has a useful corollary. Suppose that M admits an almost-complex structure J and that $\Sigma \subset M$ is totally real with respect to J, in the sense that $T\Sigma \cap J(T\Sigma) = 0$ at every point of Σ. We now give M the orientation induced by J, but emphasize that Σ might not even be orientable. If e_1, e_2 is a basis $T\Sigma$ at some point, we now observe that (e_1, e_2, Je_1, Je_2) is then a *reverse-oriented* basis for TM. Under these circumstances, the Euler number of the normal bundle of Σ, in the sense discussed above, therefore equals $-\chi(\Sigma)$. In light of our previous remarks, we thus obtain a close cousin of Theorem 2·2:

PROPOSITION 2·2. *Let (M, J) be an almost-complex 4-manifold, equipped with the orientation induced by J, and let $\Sigma \subset M$ be a (possibly non-orientable) surface which is totally real with respect to J. Then, for any edge-cone metric g of cone angle $2\pi\beta$ on (M, Σ),*

$$\tau(M) + \frac{1}{3}(1-\beta^2)\chi(\Sigma) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu. \tag{2·23}$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

3. *Indices and orbifolds*

While the calculations used in Section 2 suffice to prove Theorems 2·1 and 2·2, they hardly provide a transparent explanation of the detailed structure of equations (2·1) and (2·2). In this section, we will describe another method for obtaining these formulæ that makes the edge-cone corrections seem a great deal less mysterious. For brevity and clarity, we will confine ourselves to providing a second proof of Theorem 2·2. The same method can also be used to prove Theorem 2·1, but several more elementary proofs are also possible in this case.

Our approach is based on the $\mathbf{G}$-index theorem [**5**], so we begin by recalling what this tells us about the signature operator in four dimensions. If a finite group $\mathbf{G}$ acts on a compact oriented connected 4-manifold X, we can choose a $\mathbf{G}$-invariant decomposition

$$H^2(M,\mathbb{R}) = H^+ \oplus H^-$$

of the second cohomology into subspaces on which the intersection form is positive- and negative-definite; for example, we could equip X with a $\mathbf{G}$-invariant Riemannian metric $\hat{g}$, and let $H^\pm$ consist of those de Rham classes which have self-dual or anti-self-dual harmonic representatives, respectively, with respect to this metric. For each $\mathfrak{g} \in \mathbf{G}$, we then let $\mathfrak{g}_*$ denote the induced action of $\mathfrak{g}$ on $H^2(X,\mathbb{R})$, and set

$$\tau(\mathfrak{g}, X) = \mathrm{tr}(\mathfrak{g}_*|_{H^+}) - \mathrm{tr}(\mathfrak{g}_*|_{H^-}) \,.$$

In particular, $\tau(1, X)$ coincides with the usual signature $\tau(X)$. By contrast, when $\mathfrak{g} \neq 1$, $\tau(\mathfrak{g}, X)$ is instead expressible in terms of the fixed-point set $X^{\mathfrak{g}}$ of $\mathfrak{g}$. To do this, we first express $X^{\mathfrak{g}}$ as a disjoint union of isolated fixed points x_j and compact surfaces $\hat{\Sigma}_k$; at each $\hat{\Sigma}_k$, $\mathfrak{g}$ then acts by rotating the normal bundle through some angle ϑ_k, whereas at each isolated fixed point x_j, $\mathfrak{g}$ acts on $T_{x_j}X$ by rotating through angles α_j and β_j in a pair of orthogonal 2-planes. With these conventions, the 4-dimensional case of the relevant fixed-point formula [**5**, proposition (6·12)] becomes

$$\tau(\mathfrak{g}, X) = -\sum_j \cot\frac{\alpha_j}{2}\cot\frac{\beta_j}{2} + \sum_k \left(\csc^2\frac{\theta_k}{2}\right)[\hat{\Sigma}_k]^2 \tag{3·1}$$

where j and k respectively run over the 0- and 2-dimensional components of the fixed-point set $X^{\mathfrak{g}}$.

Let M denote the orbifold $X/\mathbf{G}$, and notice that the $\mathbf{G}$-invariant subspace $H^2(X,\mathbb{R})^{\mathbf{G}}$ of $H^2(X,\mathbb{R})$ can be identified with $H^2(M,\mathbb{R})$ via pull-back. Since

$$\frac{1}{|\mathbf{G}|}\sum_{\mathfrak{g}\in\mathbf{G}}\mathfrak{g}_* : H^2(X,\mathbb{R}) \longrightarrow H^2(X,\mathbb{R})^{\mathbf{G}}$$

is the $\mathbf{G}$-invariant projection, and since the cup product commutes with pull-backs, we therefore have

$$\tau(M) = \frac{1}{|\mathbf{G}|}\left[\tau(X) + \sum_{\mathfrak{g}\neq 1}\tau(\mathfrak{g}, X)\right]. \tag{3·2}$$

We now specialize our discussion by assuming that the action has a fixed point, but that no fixed point is isolated. Thus, if $\mathbf{G} \neq \{1\}$, there must be at least one fixed surface $\hat{\Sigma}_k$, and the induced action on the normal bundle of each such $\hat{\Sigma}_k$ must be effective. Hence $\mathbf{G} = \mathbb{Z}_p$ for some positive integer p. Moreover, for each k, the exponentials $e^{i\vartheta_k}$ of the rotation angles

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

ϑ_k appearing in (3·1) must sweep through all the pth roots of unity as g runs through $\mathbb{Z}_p$. We can therefore rewrite (3·2) as

$$\tau(M) = \frac{1}{p}\left[\tau(X) + \left(\sum_{k=1}^{p-1} \csc^2\left[\frac{k\pi}{p}\right]\right)[\hat{\Sigma}]^2\right] \tag{3·3}$$

where $\hat{\Sigma} = X^{\mathbf{G}} = \bigcup_k \hat{\Sigma}_k$. On the other hand, as pointed out by Hirzebruch [**24**, **25**], the trigonometric sum in (3·3) has an algebraic simplification

$$\sum_{k=1}^{p-1} \csc^2\left(\frac{k\pi}{p}\right) = \frac{p^2-1}{3}\ , \tag{3·4}$$

as can be proved using the the Cauchy residue theorem.

Now since $X^{\mathbb{Z}_p} = \hat{\Sigma}$ has been assumed to be purely of codimension 2, $M = X/\mathbb{Z}_p$ is a manifold, and comes equipped with a surface $\Sigma \subset M$ which is the image of $\hat{\Sigma}$. Observe, moreover, that $[\Sigma]^2 = p[\hat{\Sigma}]^2$. Substituting (3·4) into (3·3) therefore yields

$$\tau(M) - \frac{1}{3}(1-p^{-2})[\Sigma]^2 = \frac{1}{p}\tau(X)\ . \tag{3·5}$$

However, the usual signature theorem tells us that $\tau(X) = \int_X p_1(TX)/3$, and this allows us to rewrite the right-hand side of (3·5) as

$$\frac{1}{p}\cdot\frac{1}{12\pi^2}\int_X \left(|W_+|^2 - |W_-|^2\right)_{\hat{g}} d\mu_{\hat{g}} = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right)_g d\mu_g$$

because $(X - \hat{\Sigma}, \hat{g})$ is a p-sheeted cover of $(M - \Sigma, g)$. Since g is an edge-cone metric on (M, Σ) with $\beta = 1/p$, we have therefore obtained a quite different proof of (2·2) in this special case.

The global quotients with $\beta = 1/p$ we have just analyzed constitute a special class of orbifolds. Without recourse to the results in the previous section, one can similarly show that Theorem 2·2 also applies to arbitrary orbifolds with singular set of pure codimension 2 and cone angle $2\pi/p$, even when the spaces in question are not global quotients; cf. [**31**, **34**, **44**]. As is explained in Appendix A, this may be deduced from the Index Theorem for transversely elliptic operators, and this method also yields interesting results in higher dimensions. We will now assume this more general fact, and see how it leads to a different proof of Theorem 2·2.

To do so, we revisit the global quotients discussed above, but now turn the picture upside down by letting M play the role previously assigned to X. That is, we now assume that there is an effective action of $\mathbb{Z}_q$ on M with fixed point set Σ, and set $Y = M/\mathbb{Z}_q$. Let $\varpi : M \to Y$ be the quotient map, and let $\check{\Sigma} = \varpi(\Sigma)$. Choose an orbifold metric $\check{g}$ of cone angle $2\pi/p$ on $(Y, \check{\Sigma})$, and assume, per the above discussion, that (2·2) is already known to hold for orbifolds. Our previous argument tells us that

$$\tau(Y) = \frac{1}{q}\tau(M) + \frac{1}{3}(1-q^{-2})\ [\check{\Sigma}]^2$$

while we also have the formula

$$\tau(Y) = \frac{1}{3}(1-p^{-2})[\check{\Sigma}]^2 + \frac{1}{12\pi^2}\int_Y \left(|W_+|^2 - |W_-|^2\right)_{\check{g}} d\mu_{\check{g}}$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

by assumption. Remembering that $[\check{\Sigma}]^2 = q[\Sigma]^2$, we therefore obtain

$$\tau(Y) - \frac{1}{3}\left(1 - \left[\frac{q}{p}\right]^2\right)[\Sigma]^2 = q \cdot \frac{1}{12\pi^2}\int_Y \left(|W_+|^2 - |W_-|^2\right)_{\check{g}}\, d\mu_{\check{g}}$$

by straightforward algebraic manipulation. Reinterpreting the right-hand side as a curvature integral on M for the edge cone metric $g = \varpi^*\check{g}$, we thus deduce (2·2) for this large class of examples where $\beta = q/p$ is an arbitrary positive rational number.

We now consider the general case of Theorem 2·2 with $\beta = q/p$ rational. First, notice that the same elementary trick used in Section 2 shows that the the β-dependent correction term in (2·2) can be localized to a neighbourhood of Σ. For the same reason, this correction term is additive under disjoint unions, and multiplicative under covers. Moreover, Lemma 2·1 shows that the correction is independent of the particular choice of metric, so we may assume that the given edge-cone metric of cone angle $2\pi q/p$ is rotationally invariant on a tubular neighbourhood of Σ. Cut out such a tubular neighbourhood $\mathcal{U}$ of Σ, and consider the isometric $\mathbb{Z}_q$-action corresponding to rotation in the normal bundle of Σ though an angle of $2\pi/q$. We then have an induced *free* action on the oriented 3-manifold $\partial\mathcal{U}$. However, the cobordism group for free $\mathbb{Z}_q$-actions is of finite order in any odd dimension [**18**]. Thus, there is an oriented 4-manifold-with-boundary Z with free $\mathbb{Z}_q$-action, where ∂Z is disjoint union of, say, ℓ copies of $\partial\mathcal{U}$, each equipped with the original $\mathbb{Z}_q$-action. Let $\widetilde{M}$ then be obtained from the reverse-oriented manifold $\overline{Z}$ by capping off each of its ℓ boundary components with a copy of $\mathcal{U}$, and notice that, by construction, $\widetilde{M}$ comes equipped with a $\mathbb{Z}_q$-action whose fixed-point set consists of ℓ copies of Σ. We extend ℓ copies of the given edge-cone metric g on $\mathcal{U}$ to a $\mathbb{Z}_q$-invariant metric $\tilde{g}$ on $\widetilde{M}$. However, $Y = \widetilde{M}/\mathbb{Z}_q$ is now a manifold, and $\tilde{g}$ pushes down to Y as an orbifold metric $\check{g}$ of cone angle $2\pi/p$. Our previous argument for global quotients then shows that (2·1) and (2·2) hold for $(Z, \coprod_1^\ell \Sigma)$, and additivity therefore shows that the correction terms are as promised for each of the ℓ identical copies of $\mathcal{U}$. This shows that (2·2) holds for any edge-cone manifold of cone angle $2\pi\beta$, provided that β is a positive rational number q/p. Multiplicativity of the correction under covers similarly allows one to drop the assumption that Σ is orientable.

Finally, an elementary continuity argument allows us to extend our formula from rational to real β. Consider a smooth family of edge-cone metrics on (M, Σ) with $\mathbf{SO}(2)$ symmetry about Σ, but with cone angle varying over the entire positive reals $\mathbb{R}^+$. The left- and right-hand sides of (2·2) then vary continuously as β varies, and their difference vanishes for $\beta \in \mathbb{R}^+ \cap \mathbb{Q}$. By continuity, the two sides therefore agree for all $\beta > 0$. With the help of Lemma 2·1, this gives us alternative proofs of Theorem 2·2 and Proposition 2·2.

4. *Edges and Einstein metrics*

As we saw in Section 2, Theorems 2·1 and 2·2 are equivalent to the fact that every edge-cone metric g of cone angle $2\pi\beta$ on (M, Σ) satisfies

$$(2\chi \pm 3\tau)(M) - \mathrm{def}_\pm(\Sigma, \beta) = \frac{1}{4\pi^2}\int_M \left[\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\mathring{r}|^2}{2}\right]_g d\mu_g \tag{4·1}$$

for both choices of the $\pm$ sign, where

$$\mathrm{def}_\pm(\Sigma, \beta) = 2(1-\beta)\chi(\Sigma) \pm (1-\beta^2)[\Sigma]^2\,.$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

However, if the edge-cone metric g is Einstein, it then satisfies $\mathring{r} \equiv 0$, and the integrand on the right-hand-side of (4·1) is consequently non-negative. The existence of an Einstein edge-cone metric of cone angle $2\pi\beta$ on (M, Σ) therefore implies the topological constraints that

$$(2\chi \pm 3\tau)(M) \geqslant \mathrm{def}_{\pm}(\Sigma, \beta)$$

and equality can occur for a given choice of sign only if the Einstein metric satisfies $s \equiv 0$ and $W_{\pm} \equiv 0$. Since the metric in question also satisfies $\mathring{r} \equiv 0$ by hypothesis, equality only occurs if $\Lambda^{\pm}$ is flat, which is to say that the metric in question is locally hyper-Kähler, in a manner compatible with the $\pm$-orientation of M. This proves a more refined version of Theorem A:

THEOREM 4·1. *Let M be a smooth compact 4-manifold, and let $\Sigma \subset M$ be a compact orientable embedded surface. If (M, Σ) admits an Einstein edge-cone metric of cone angle $2\pi\beta$, then (M, Σ) must satisfy the inequalities*

$$(2\chi + 3\tau)(M) \geqslant (1-\beta)\left[2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right] \tag{4·2}$$

and

$$(2\chi - 3\tau)(M) \geqslant (1-\beta)\left[2\chi(\Sigma) - (1+\beta)[\Sigma]^2\right]. \tag{4·3}$$

Moreover, equality occurs in (4·2) *if and only if g is locally hyper-Kähler, in a manner compatible with the given orientation of M. Similarly, equality occurs in* (4·3) *if and only if g is locally hyper-Kähler, in a manner compatible with the opposite orientation of M.*

Recall that a Riemannian 4-manifold is locally hyper-Kähler iff it is Ricci-flat and locally Kähler. Brendle's recent construction [**11**] of Ricci-flat Kähler manifolds with edge-cone singularities of small cone angle thus provides an interesting class of examples which saturate inequality (4·2).

As a related illustration of the meaning of Theorem 4·1, let us now consider what happens when the cone angle tends to zero.

COROLLARY 4·2. *If (M, Σ) admits a sequence g_j of Einstein edge-cone metrics with cone angles $2\pi\beta_j \to 0$, then (M, Σ) must satisfy the two inequalities*

$$(2\chi \pm 3\tau)(M) \geqslant 2\chi(\Sigma) \pm [\Sigma]^2$$

with equality for a given sign iff the L^2 norms of both s and $W_{\pm}$ tend to zero as $j \to \infty$.

For example, let $\Sigma \subset \mathbb{CP}_2$ be a smooth cubic curve, and let M be obtained from $\mathbb{CP}_2$ by blowing up k points which do not lie on Σ. Considering Σ as a submanifold of M, we always have $2\chi(\Sigma) + [\Sigma]^2 = 0 + 3^2 = 9$, whereas $(2\chi + 3\tau)(M) = (2\chi + 3\tau)(\mathbb{CP}_2) - k = 9 - k$. Thus, (M, Σ) does not admit Einstein metrics of small cone angle when k is a positive integer. By contrast, [**32**] leads one to believe that $(\mathbb{CP}_2, \Sigma)$ *should* admit Einstein metrics of small cone angle, and that, as the angle tends to zero, these should tend to previously discovered hyper-Kähler metrics [**7**, **46**]. This would nicely illustrate the boundary case of Corollary 4·2.

While many interesting results have recently been obtained about the Kähler case of Einstein edge metrics with $\beta \in (0, 1]$, the large cone-angle regime of the problem seems technically intractable. It is thus interesting to observe Theorem 4·1 gives us strong obstructions to the existence of Einstein edge-cone metrics with large cone angle, even without imposing

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

the Kähler condition. Indeed, first notice that dividing (4·2) and (4·3) by β^2 and taking the limit as $\beta \to \infty$ yields the inequalities

$$0 \geqslant -[\Sigma]^2 \text{ and } 0 \geqslant [\Sigma]^2$$

so that existence is obstructed for large β unless $[\Sigma]^2 = 0$. Similarly, dividing the sum of (4·2) and (4·3) by 4β and letting $\beta \to \infty$ yields

$$0 \geqslant -\chi(\Sigma)$$

so existence for large β is obstructed in most cases:

COROLLARY 4·3. *Suppose that $\Sigma \subset M$ is a connected oriented surface with either non-zero self-intersection or genus $\geqslant 2$. Then there is a real number β_0 such that (M, Σ) does not admit Einstein edge-cone metrics of cone angle $2\pi\beta$ for any $\beta \geqslant \beta_0$.*

Of course, this result does not provide obstructions in all cases, and examples show that this is inevitable. For example, consider an equatorial 2-sphere $S^2 \subset S^4$. If S^4 is thought of as the unit sphere in $\mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$, we make take our S^2 to be the inverse image of the origin under projection to $\mathbb{R}^2$. By taking polar coordinates on this $\mathbb{R}^2$, we can thus identify $S^4 - S^2$ with $B^3 \times S^1$. Rotating in the circle factor then gives us a Killing field on S^4, and there is an interesting conformal rescaling of the standard metric obtained by requiring that this conformal Killing field have unit length in the new metric. What we obtain in this way is a conformal equivalence between $S^4 - S^2$ and the Riemannian product $\mathcal{H}^3 \times S^1$, where $\mathcal{H}^3$ denotes hyperbolic 3-space. More precisely, the standard metric on S^4 now becomes

$$(\operatorname{sech}^2 я)[h + d\theta^2]$$

where h is the curvature -1 metric on $\mathcal{H}^3$, and where $я : \mathcal{H}^3 \to \mathbb{R}$ is the distance in $(\mathcal{H}^3, h)$ from some arbitrary base-point. By a minor alteration, we then obtain the family

$$g = (\operatorname{sech}^2 я)[h + \beta^2 d\theta^2]$$

of edge-cone metrics on S^4 with arbitrary cone angle $2\pi\beta$. By setting $\tilde{\theta} = \beta\theta$, we see that these edge-cone metrics are actually locally isometric to the standard metric on S^4, and so, in particular, are all Einstein; in other words, these edge-cone metrics are simply obtained by passing to the universal cover $B^3 \times \mathbb{R}$ of $S^4 - S^2$, and then dividing out by some arbitrary translation of the $\mathbb{R}$ factor. Since this works for any $\beta > 0$, we see that it is inevitable that Corollary 4·3 does not apply to genus zero surfaces of trivial self-intersection.

The above edge-cone metrics g can be obtained from the family

$$g_0 = \beta^{-1} h + \beta\, d\theta^2$$

by a suitable conformal rescaling. In the next section, we will see that this can be interestingly generalized by replacing the constant function $V = \beta^{-1}$ with a harmonic function, and by replacing the flat circle bundle $\mathcal{H}^3 \times S^1$ with a principal $\mathbf{U}(1)$-bundle over $\mathcal{H}^3$ which is equipped with a connection whose curvature is the closed 2-form $\star dV$.

5. *Edges and instantons*

In this section, we will study families of self-dual edge-cone metrics on 4-manifolds, and observe that these metrics are interestingly related to certain gravitational instantons. For our purposes, a gravitational instanton will mean a complete non-compact Ricci-flat Riemannian

4-manifold which is both simply connected and self-dual, in the sense that $W_- = 0$. Note that such spaces are necessarily hyper-Kähler, but that the orientation we will give them here is opposite the one induced by the hyper-Kähler structure.

As indicated at the end of Section 4, we will begin by considering a construction [**39**], called the *hyperbolic ansatz*, that builds explicit self-dual conformal metrics out of positive harmonic functions on regions of hyperbolic 3-space. To this end, let $\mathcal{U} \subset \mathcal{H}^3$ be an open set in hyperbolic 3-space, and let $V : \mathcal{U} \to \mathbb{R}^+$ be a function which is harmonic with respect to the hyperbolic metric h. The 2-form $\star dV$ is then closed, and we will furthermore suppose that the deRham class $[(\star dV)/2\pi]$ represents an integer class in $H^2(\mathcal{U}, \mathbb{R})$. The theory of Chern classes then guarantees that there is a principal $\mathbf{U}(1)$-bundle $\mathcal{P} \to \mathcal{U}$ which carries connection 1-form Θ of curvature $d\Theta = \star dV$. We may then consider the Riemannian metric

$$g_0 = Vh + V^{-1}\Theta^2 \tag{5·1}$$

on the total space $\mathcal{P}$ of our circle bundle. Remarkably, this metric is automatically self-dual with respect to a standard orientation of $\mathcal{P}$. Since the condition that $W_- = 0$ is conformally invariant, multiplying g_0 by any positive conformal factor will result in another self-dual metric. For shrewd choices of V and the conformal factor, interesting compact self-dual edge-cone manifolds can be constructed in this way. Indeed, one can even sometimes arrange for the resulting edge-cone metric to also be Einstein.

We already considered the case of constant V at the end of Section 4. To obtain something more interesting, we now choose our potential to be

$$V = \beta^{-1} + \sum_{j=1}^{n} G_{p_j} \tag{5·2}$$

where β is an arbitrary positive constant, $p_1, \ldots, p_n \in \mathcal{H}^3$ are distinct points in hyperbolic 3-space, and where G_{p_j} are the corresponding Green's functions. For simplicity, we will use the same conformal rescaling

$$g = \beta(\operatorname{sech}^2 я)g_0$$

that was used in Section 4, where $я$ denotes the distance from some arbitrary base-point in $\mathcal{H}^3$. The metric-space completion $M = \mathcal{P} \cup \Sigma \cup \{\hat{p}_j\}$ of $(\mathcal{P}, g)$ then carries a natural smooth structure making it diffeomorphic to the connected sum

$$n\mathbb{CP}_2 = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{n}$$

and g then extends to M as an edge-cone metric with cone angle $2\pi\beta$ along a surface $\Sigma \approx S^2$ of self-intersection n. Indeed, when $\beta = 1$, this exactly reproduces the self-dual metics on $n\mathbb{CP}_2$ constructed in [**39**]. For general β, the picture is essentially the same; metric-space completion adds one point $\hat{p}_j$ for each of the base-points p_j, and a 2-sphere Σ corresponding to the 2-sphere at infinity of $\mathcal{H}^3$. By the same argument used in [**39**, p. 232], the metric g extends smoothly across the $\hat{p}_j$. By contrast, we obtain an edge-cone metric of cone-angle β along Σ, because our potential V is asymptotic to the constant choice considered in Section 4. Since these edge-cone metrics satisfy $W_- = 0$, Theorem 2·2 tells us that they also satisfy

$$\frac{1}{12\pi^2}\int_{n\mathbb{CP}_2} |W_+|^2 d\mu = \tau(n\mathbb{CP}_2) - \frac{1}{3}(1-\beta^2)[\Sigma]^2 = \frac{n(2+\beta^2)}{3}\,. \tag{5·3}$$

The $n = 1$ case has some special features that make it particularly interesting. The potential becomes

$$V = \beta^{-1} + \frac{1}{e^{2я} - 1} \tag{5·4}$$

and (5·1) can then be written explicitly as

$$g_0 = V\left[dя^2 + (4\sinh^2 я)(\sigma_1^2 + \sigma_2^2)\right] + V^{-1}\sigma_3^2 \tag{5·5}$$

where $я$ represents the hyperbolic distance from the point $p = p_1$, and $\{\sigma_j\}$ is a left-invariant orthonormal co-frame for $S^3 = \mathbf{SU}(2)$. Remarkably, the alternative representative

$$\tilde{g} = 4\beta^{-1}[(2-\beta)\cosh я + \beta\sinh я]^{-2} g_0 \tag{5·6}$$

of the conformal class is then Einstein, with Einstein constant $\lambda = 3\beta^2(2-\beta)/2$, as follows from [**40**, equation (2.1)] or [**29**, section 9]; cf. [**48**]. Demanding that g define a metric for all $я \geqslant 0$ imposes the constraint[1] that $\beta < 2$. The resulting family of edge-cone metrics on $(\mathbb{CP}_2, \mathbb{CP}_1)$ of cone angle $2\pi\beta$, $\beta \in (0, 2)$, coincides with the family constructed by Abreu [**1**, section 5] by an entirely different method. Notice that, as a special case of (5·3), these metrics satisfy

$$\frac{1}{12\pi^2}\int_{\mathbb{CP}_2} |W_+|^2 d\mu = \frac{2+\beta^2}{3} \tag{5·7}$$

whether we represent the conformal class by g or, when $\beta \in (0, 2)$, by $\tilde{g}$.

When $\beta = 1$, $\tilde{g}$ is just the standard Fubini-Study metric on $\mathbb{CP}_2$. On the other hand, the $\beta \to 0$ and $\beta \to 2$ limits of $\tilde{g}$ give us other celebrated metrics. For example, introducing the new radial coordinate $\mathfrak{r} = \sqrt{\coth я}$,

$$\lim_{\beta\to 2}\tilde{g} = \frac{d\mathfrak{r}^2}{1-\mathfrak{r}^{-4}} + \mathfrak{r}^2\left[\sigma_1^2 + \sigma_2^2 + (1-\mathfrak{r}^{-4})\sigma_3^2\right] \tag{5·8}$$

which is the usual formula for the Eguchi–Hanson metric [**21**, **22**], a celebrated complete self-dual Einstein metric on the manifold TS^2; however, (5·8) has arisen here as a metric on $\mathbf{SU}(2) \times (1, \infty)$ rather than on $\mathbf{SO}(3) \times (1, \infty)$, so this version of Eguchi–Hanson is actually actually a branched double cover of the usual one, ramified along the zero section of TS^2. On the other hand, after introducing a new radial coordinate $\mathfrak{r} = \beta^{-1}я$, the pointwise coordinate limit

$$\lim_{\beta\to 0}\tilde{g} = \left(1 + \frac{1}{2\mathfrak{r}}\right)\left[d\mathfrak{r}^2 + 4\mathfrak{r}^2(\sigma_1^2 + \sigma_2^2)\right] + \left(1 + \frac{1}{2\mathfrak{r}}\right)^{-1}\sigma_3^2$$

is the Taub-NUT metric [**22**, **38**], a complete non-flat hyper-Kähler metric on $\mathbb{R}^4$. Similarly, by choosing suitable sequence of centers $\{p_j\}$ and conformal rescalings, there are $\beta \to 0$ limits of our conformal metrics on $n\mathbb{CP}_2$ which converge to

$$g_{\text{multi}} = \tilde{V}\mathbf{dx}^2 + \tilde{V}^{-1}\tilde{\Theta}^2 \tag{5·9}$$

where $\mathbf{dx}^2$ is the Euclidean metric on $\mathbb{R}^3$, the harmonic function

$$\tilde{V} = 1 + \sum_{j=1}^{n}\frac{1}{2\mathfrak{r}_j}$$

[1] By contrast, when $\beta > 2$, restricting (5·6) to the 4-ball $я < \tanh^{-1}(1 - 2/\beta)$ produces a family of complete self-dual Einstein metrics originally discovered by Pedersen [**43**].

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

is expressed in terms of the Euclidean distances $\mathfrak{r}_j$ from the $\tilde{p}_j$, and where $d\tilde{\Theta} = \star d\tilde{V}$. The metric (5·9) is a famous gravitational instanton called the multi-Taub-NUT metric [**22**, **38**, **41**].

If we are cavalier about interchanging integrals and limits, these observations provide some interesting information regarding the above gravitational instantons. For example, the standard Eguchi–Hanson space $\mathbb{EH}$ should satisfy

$$\frac{1}{12\pi^2}\int_{\mathbb{EH}} |W_+|^2 d\mu = \lim_{\beta\to 2^-} \frac{1}{2}\left(\frac{2+\beta^2}{3}\right) = 1$$

with respect to the orientation for which $W_- = 0$; here the factor of $1/2$ stems from the fact that $\mathbb{EH}$ is a $\mathbb{Z}_2$-quotient of $\mathbb{CP}_2 - \{p\}$. Because the s, W_- and $\mathring{r}$ pieces of the curvature tensor $\mathcal{R}$ all vanish for $\mathbb{EH}$, the L^2 norm squared of the curvature of the Eguchi–Hanson instanton should therefore be given by

$$\int_{\mathbb{EH}} |\mathcal{R}|^2 d\mu = \int_{\mathbb{EH}} |W_+|^2 d\mu = 12\pi^2. \tag{5·10}$$

Similarly, the Taub-NUT gravitational instanton $\mathbb{TN}$ should satisfy

$$\frac{1}{12\pi^2}\int_{\mathbb{TN}} |W_+|^2 d\mu = \lim_{\beta\to 0^+} [\tau(\mathbb{CP}_2) - \frac{1}{3}(1-\beta^2)[\mathbb{CP}_1]^2] = \frac{2}{3}$$

when oriented so that $W_- = 0$. In particular, the L^2 norm squared of the curvature tensor of the Taub-NUT space is given by

$$\int_{\mathbb{TN}} |\mathcal{R}|^2 d\mu = \int_{\mathbb{TN}} |W_+|^2 d\mu = 8\pi^2. \tag{5·11}$$

Similar thinking predicts that the n-center multi-Taub-NUT metric will have

$$\int |\mathcal{R}|^2 d\mu = \int |W_+|^2 d\mu = 12\pi^2 \lim_{\beta\to 0^+} \frac{n(2+\beta^2)}{3} = 8\pi^2 n. \tag{5·12}$$

All of these interchanges of integrals and limits can in fact be rigorously justified by a careful application of the the dominated convergence theorem. Rather than presenting all the tedious details, however, we will instead simply double-check these answers later, using a more direct method.

The self-dual Einstein edge-cone metrics on $(\mathbb{CP}_2, \mathbb{CP}_1)$ given by (5·6) are invariant under the action of $\mathbf{SU}(2)$ on $\mathbb{CP}_2 = \mathbb{C}^2 \cup \mathbb{CP}_1$. However, Hitchin [**28**, **29**, **30**] discovered a remarkable family of self-dual Einstein edge-cone metrics on $(\mathbb{CP}_2, \mathbb{RP}^2)$ which are invariant under a different action of $\mathbf{SU}(2)$ on $\mathbb{CP}_2$, namely the action of $\mathbf{SO}(3) = \mathbf{SU}(2)/\mathbb{Z}_2$ on $\mathbb{CP}_2$ gotten by thinking of it as the projective space of $\mathbb{C} \otimes \mathbb{R}^3$. The starting point of Hitchin's investigation was a reduction, due to Tod [**47**], of the self-dual Einstein equations with $\mathbf{SU}(2)$ symmetry to an ordinary differential equation belonging to the Painlevé VI family. He then finds a specific family of solutions depending on an integer $k \geqslant 3$ which have the property that the corresponding twistor spaces are (typically singular) algebraic varieties, and then shows [**30**, proposition 5] that the resulting Einstein manifold compactifies as an edge-cone metric on $(\mathbb{CP}_2, \mathbb{RP}^2)$ of cone angle $4\pi/(k-2)$; he phrases this assertion in terms of the $\mathbb{Z}_2$-quotient of this metric by complex conjugation, which is then an orbifold metric on S^4 with an edge-cone singularity of cone-angle $2\pi/(k-2)$ along a Veronese $\mathbb{RP}^2 \subset S^4$. When $\beta = 1$, Hitchin's metric is just the Fubini–Study metric, while for $\beta = 2$ it is just a branched double cover of the standard metric on S^4. The corresponding solutions of Painlevé VI can be explicitly expressed in terms of elliptic functions, and Hitchin observes in a later paper [**29**, remark 2, p. 79] that, in principle, solutions for non-integer k should also give rise to a self-dual Einstein edge-cone metrics on $(\mathbb{CP}_2, \mathbb{RP}^2)$, although he does not try to determine precisely which cone angles $2\pi\beta$ can actually arise in this way. However, since $\mathbb{RP}^2 \subset \mathbb{CP}_2$ is totally real, Proposition 2·2 is perfectly adapted to the study of Hitchin's self-dual edge-cone metrics, and tells us that they necessarily satisfy

$$\int_{\mathbb{CP}_2} |W_+|^2 d\mu = 12\pi^2[\tau(\mathbb{CP}_2) + \frac{1}{3}(1-\beta^2)\chi(\mathbb{RP}^2)] = 4\pi^2(4-\beta^2).$$

We thus see that the constraint $\beta \leqslant 2$, corresponding to $k \geqslant 3$, is both natural and unavoidable.

The $\mathbf{SU}(2)$-action on $\mathbb{CP}_2$ which preserves Hitchin's metrics has two 2-dimensional orbits, namely the $\mathbb{RP}^2$ where the edge-cone singularity occurs, and a conic $C \subset \mathbb{CP}_2$ where the metric is smooth; in the $S^4 = \mathbb{CP}_2/\mathbb{Z}_2$ model, C projects to the antipodal Veronese $\mathbb{RP}^2 = C/\mathbb{Z}_2$. Hitchin now normalizes his metrics so that C has area π for each β, and asks what happens when $\beta = 2/(k-2) \to 0$. He then shows [**30**, proposition 6] that this $k \to \infty$ limit is precisely the Atiyah–Hitchin gravitational instanton. Here we need to be rather precise, because there are really two different versions of the Atiyah–Hitchin manifold. The better known version, which we shall call $\mathbb{AH}$, was constructed [**4**] as the moduli space M_2^0 of solutions of the $\mathbf{SU}(2)$ Bogomolny monopole equations on $\mathbb{R}^3$ with magnetic charge 2 and fixed center of mass. With this convention, the hyper-Kähler manifold $\mathbb{AH}$ is diffeomorphic to a tubular neighbourhood of the Veronese $\mathbb{RP}^2$ in S^4, and so has fundamental group $\mathbb{Z}_2$. Its universal cover $\widetilde{\mathbb{AH}}$ is therefore a gravitational instanton per our definition, and is diffeomorphic to a tubular neighbourhood of a conic C in $\mathbb{CP}_2$; thus, $\widetilde{\mathbb{AH}}$ is diffeomorphic both to $\mathbb{CP}_2 - \mathbb{RP}^2$ and to the $\mathcal{O}(4)$ line bundle over $\mathbb{CP}_1$. If we regard Hitchin's metrics as edge-cone metrics on $(\mathbb{CP}_2, \mathbb{RP}^2)$, he then shows that they converge to $\widetilde{\mathbb{AH}}$ as $k \to \infty$. Interchanging limits and integration as before thus leads us to expect that

$$\int_{\widetilde{\mathbb{AH}}} |W_+|^2 d\mu = \lim_{\beta_k \to 0^+} 4\pi^2(4-\beta_k^2) = 16\pi^2$$

so that

$$\int_{\widetilde{\mathbb{AH}}} |\mathcal{R}|^2 d\mu = 16\pi^2 , \tag{5·13}$$

and hence that

$$\int_{\mathbb{AH}} |\mathcal{R}|^2 d\mu = \frac{1}{2}\int_{\widetilde{\mathbb{AH}}} |\mathcal{R}|^2 d\mu = 8\pi^2 . \tag{5·14}$$

While this argument can again be made rigorous using the dominated convergence theorem, we will instead simply double-check these answers by a second method which fits into a beautiful general pattern.

So far, we have been using the signature formula (2·2) to compute the L^2-norm of the self-dual Weyl curvature for interesting edge-cone metrics, and then, by a limiting process, have inferred the L^2-norm of the Riemann curvature for various gravitational instantons, as W_+ is the only non-zero piece of the curvature tensor for such spaces. However, we could have instead proceeded by considering the Gauss–Bonnet formula (2·1) for edge-cone metrics. In this case, we have

$$\lim_{\beta\to 0}\int_M \left[|\mathcal{R}|^2 - |\mathring{r}|^2\right] d\mu = 8\pi^2 \lim_{\beta\to 0}[\chi(M) - (1-\beta)\chi(\Sigma)] = 8\pi^2\chi(M-\Sigma).$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

Since the limit metric is Einstein, we thus expect any gravitational instanton (X, g_∞) obtained as a $\beta \to 0$ limit to satisfy

$$\int_X |\mathcal{R}|^2 d\mu = 8\pi^2 \chi(X) \tag{5·15}$$

because the underlying manifold of the instanton is $X = M - \Sigma$. For example, the simply-connected Atiyah–Hitchin space $\widetilde{\mathbb{AH}}$ deform-retracts to S^2, and so has Euler characteristic 2; thus, as previously inferred, its total squared curvature should be $16\pi^2$. Similarly, the Taub-NUT metric lives on the contractible space $\mathbb{R}^4$, which has Euler characteristic 1, and so is expected to have $\int |\mathcal{R}|^2 d\mu = 8\pi^2$, in agreement with our previous inference.

These same answers can be obtained in a direct and rigorous manner by means of the Gauss–Bonnet theorem with boundary. Indeed, if (Y, g) is any compact oriented Riemannian 4-manifold-with-boundary, this result tells us

$$\chi(Y) = \frac{1}{8\pi^2}\int_Y \left(|\mathcal{R}|^2 - |\mathring{r}|^2\right)\mu_g + \frac{1}{4\pi^2}\int_{\partial Y}\left[2\det(\mathrm{I\!I}) + \langle \mathrm{I\!I}, \hat{\mathcal{R}}\rangle\right] da \tag{5·16}$$

where $\mathrm{I\!I}$ and da are the the second fundamental form and volume 3-form of the boundary ∂Y, is the of the boundary, and $\hat{\mathcal{R}}$ is the symmetric tensor field on ∂Y gotten by restricting the ambient curvature tensor $\mathcal{R}$ and then using the 3-dimensional Hodge star operator to identify $\odot^2\Lambda^2$ and $\odot^2\Lambda^1$ on this 3-manifold. One can prove (5·16) simply by following Chern's proof of the generalized Gauss–Bonnet theorem, using Stokes theorem to count the zeroes, with multiplicities, of a generic vector field on Y that is an outward pointing normal field along ∂Y. The proof in [**17**] then goes through without changes, except that there is now a non-trivial contribution due to ∂Y.

The best-known class of gravitational instantons consists of ALE (Asymptotically Locally Euclidean) spaces. For such a space X, there is a compact set K such that $X - K$ is diffeomorphic to $(\mathbb{R}^4 - B)/\Gamma$ for some finite subgroup $\Gamma \subset \mathbf{SU}(2)$, in such a manner that the metric is given by

$$g_{jk} = \delta_{jk} + O(\mathsf{r}^{-4})$$

where r is the Euclidean radius, with coordinate derivatives $\partial^k g$ commensurately falling off like r^{-4-k}. In particular, a ball of radius r has 4-volume $\sim \mathsf{r}^4$, and the Riemannian curvature falls off like $|\mathcal{R}| \sim \mathsf{r}^{-6}$. Moreover, the hypersurface $\mathsf{r} = \text{const}$ has 3-volume $\sim \mathsf{r}^3$ and $|\mathrm{I\!I}| \sim \mathsf{r}^{-1}$. If we let $Y \subset X$ be the region $\mathsf{r} \leqslant C$ and then let $C \to \infty$, we thus see that the only significant boundary contribution in (5·16) comes from the $\det \mathrm{I\!I}$ term. In the limit, it is thus easy to show that the boundary terms just equals $1/|\Gamma|$ times the corresponding integral for the standard 3-sphere $S^3 \subset \mathbb{R}^4$. Thus, any ALE instanton satisfies [**13**, **33**, **42**]

$$\int_X |\mathcal{R}|^2 d\mu = 8\pi^2\left(\chi(X) - \frac{1}{|\Gamma|}\right). \tag{5·17}$$

For example, when X is the Eguchi-Hanson instanton $\mathbb{EH}$, $\chi = 2$ and $|\Gamma| = 2$, so the total squared curvature is $12\pi^2$, as previously predicted by (5·10).

The Taub-NUT space is the prototypical example of an asymptotically locally flat (ALF) gravitational instanton. For such spaces, the volume of a large ball has 4-volume $\sim \mathsf{r}^3$, while curvature falls off like r^{-3}. The hypersurface $\mathsf{r} = \text{const}$ has 3-volume $\sim \mathsf{r}^2$, with $|\mathrm{I\!I}| \sim \mathsf{r}^{-1}$ and $\det \mathrm{I\!I} \sim \mathsf{r}^{-4}$. Thus the boundary contribution in (5·16) tends to zero as $\mathsf{r} \to \infty$, and any

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

ALF instanton therefore satisfies the simpler formula [**19**, **33**]

$$\int_X |\mathcal{R}|^2 d\mu = 8\pi^2 \chi(X) \tag{5·18}$$

previously seen in (5·15). For example, the Taub-NUT instanton has $\chi = 1$, so its total squared curvature is $8\pi^2$, as predicted by (5·11). The Atiyah–Hitchin gravitational instanton is also ALF; in fact, it is asymptotic to Taub-NUT with a negative NUT parameter [**4**]. Thus $\widetilde{\mathbb{AH}}$, with an Euler characteristic of 2, has total squared curvature $16\pi^2$, while its $\mathbb{Z}_2$-quotient $\mathbb{AH}$, with an Euler characteristic of 1, has has total squared curvature $8\pi^2$. Note that these conclusions coincide with the predictions of (5·13) and (5·14).

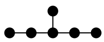
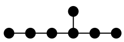
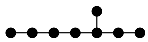
A classification of complete hyper-Kähler ALE 4-manifolds was given by Kronheimer [**36**, **37**]; up to deformation, they are in one-to-one correspondence with the Dynkin diagrams of type A, D and E. For example, the Dynkin diagram A_1 corresponds to the Eguchi-Hanson metric, and, more generally, the Dynkin diagrams A_k correspond to the multi-Eguchi–Hanson metrics independently discovered by Hitchin [**27**] and by Gibbons–Hawking [**22**]. Each Dynkin diagram represents a discrete subgroup Γ of $\mathbf{SU}(2)$, and these groups are then realized as the fundamental group of the 3-dimensional boundary at infinity of the corresponding 4-manifold. The diagram also elegantly encodes the diffeotype of the corresponding 4-manifold, which is obtained by plumbing together copies of TS^2, with one 2-sphere for each node, and with edges of the diagram indicating which pairs of 2-spheres meet. In particular, the number k of nodes in any given diagram is the second Betti number b_2 of the instanton, which consequently has $\chi = k + 1$.

For each diagram of type A or D, there is also an associated ALF instanton. These ALF partners are diffeomorphic to the corresponding ALE instantons, but their geometry at infinity resembles Taub-NUT$/\Gamma$ instead of a Euclidean quotient $\mathbb{R}^4/\Gamma$. In the A_k cases, these spaces are just the multi-Taub-NUT metrics of (5·9), with $n = k + 1$ centers; our heuristic calculation (5·12) of their total squared curvature is thus confirmed by (5·18), since these spaces have $\chi = n$. The D_k metrics were constructed explicitly by Cherkis and Hitchin [**15**], building on earlier existence arguments of Cherkis and Kapustin [**16**]. The fact that these metrics really are ALF follows from results due to Gibbons–Manton [**23**] and Bielawski [**9**]. Table 1 gives a compilation of the total squared curvature of these important spaces.

Of course, several of the gravitational instantons we have discussed do not appear on Table 1; the Taub-NUT space $\mathbb{TN}$ and the Atiyah–Hitchin manifolds $\mathbb{AH}$ and $\widetilde{\mathbb{AH}}$ are nowhere to be found. Of course, one might decree [**15**] that $\mathbb{TN}$ is the ALF entry across from the fictitious diagram A_0, or that $\mathbb{AH}$ is the ALF entry associated with the make-believe diagram D_0; but to us, this is less interesting than the general point that that $\int |\mathcal{R}|^2 d\mu = 8\pi^2\chi$ for complete Ricci-flat 4-manifolds with ALF asymptotics.

There is a large realm of gravitational instantons which have even slower volume growth, but still have finite topological type. Cherkis [**14**] has proposed sorting these into two classes: the ALG spaces, with at least quadratic volume growth, and the ALH instantons, with subquadratic volume growth. For example, Tian and Yau [**46**] constructed hyper-Kähler metrics on the complement of an anti-canonical divisor on any del Pezzo surface; these metrics have volume growth $\sim \mathrm{r}^{4/3}$, and so are of ALH type. For these examples, one has enough control at infinity to see that the boundary term in (5·16) becomes negligeable at large radii, so that the pattern $\int |\mathcal{R}|^2 d\mu = 8\pi^2\chi$ continues to hold. Presumably, this pattern will also turn out to hold for all gravitational instantons of type ALG and ALH.

Table 1. *Total squared curvature of ALE and ALF gravitational instantons*

Group at infinity				$\int \lvert\mathcal{R}\rvert^2 d\mu$	
Dynkin diagram		$\Gamma \subset \mathbf{SU}(2)$	$\lvert\Gamma\rvert$	ALE	ALF
A_k		cyclic	$k+1$	$8\pi^2\left(k+1-\frac{1}{k+1}\right)$	$8\pi^2(k+1)$
D_k		dihedral*	$4k-8$	$8\pi^2\left(k+1-\frac{1}{4k-8}\right)$	$8\pi^2(k+1)$
E_6		tetrahedral*	24	$8\pi^2\left(7-\frac{1}{24}\right)$	—
E_7		octohedral*	48	$8\pi^2\left(8-\frac{1}{48}\right)$	—
E_8		dodecahedral*	120	$8\pi^2\left(9-\frac{1}{120}\right)$	—

Many gravitational instantons do seem to arise as limits of edge-cone Einstein metrics. For instance, the results of [**32**] strongly suggest that the ALH examples of Tian–Yau are $\beta \to 0$ limits of Kähler–Einstein edge-cone metrics; proving this, however, would entail first establishing a suitable lower bound for the K-energy. It would obviously be interesting to prove the existence of sequences of Einstein edge-cone metrics which tend to other known examples of gravitational instantons. On the other hand, one might hope to construct new examples of gravitational instantons as Gromov–Hausdorff limits of suitable sequences of 4-dimensional Einstein edge-cone manifolds. While some features of weak convergence for smooth Einstein metrics [**2**, **12**] may carry over with little change, it seems likely that the introduction of edge-cone singularities may involve some serious technical difficulties. Our hope is that the regime of small β will nonetheless prove to be manageable, and that the theory that emerges will lead to new insights concerning precisely which gravitational instantons arise as $\beta \to 0$ limits of Einstein spaces with edge-cone singularities.

Appendix A.

The proof outlined in Section 3 involves the use of the signature theorem for orbifolds. We will now indicate how this can be deduced from the theory of transversally elliptic operators developed in [**3**]. For further details, see [**34**, **35**].

Let X be a compact manifold equipped with the action of a compact Lie group $\mathbf{G}$. A differential operator D between vector bundles on X is said to be *transversally elliptic* if it is $\mathbf{G}$-invariant, and its restriction to any submanifold transverse to a $\mathbf{G}$-orbit is elliptic. A trivial but important example arises when $\mathbf{G}$ acts freely on X, so that $X/\mathbf{G}$ is itself a manifold; in this case, a transversally elliptic operator is essentially just an elliptic operator on $X/\mathbf{G}$. For us, the case of primary interest occurs when the action of $\mathbf{G}$ has only finite isotropy groups. In this case, $X/\mathbf{G}$ is an orbifold.

The index of such a transversally elliptic operator D is an invariant distribution on $\mathbf{G}$. Equivalently, the index is given by an infinite series

$$\operatorname{ind}(D) = \sum_{\rho} a_\rho \chi_\rho \tag{A 1}$$

where ρ runs over the irreducible representations of $\mathbf{G}$, χ_ρ is the character of ρ and the multiplicities a_ρ do not grow too fast. If D is fully elliptic, then the sum in (A 1) is finite. In

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

general, however, it may be infinite; for example, if $X = \mathbf{G}$ is equipped with the left action of $\mathbf{G}$ on itself, and if D is the zero operator, then $a_\rho = \dim \chi_\rho$ and (A 1) just becomes the Peter-Weyl decomposition of $L^2(\mathbf{G})$.

A general procedure for computing the index of D is described in [**3**]. Some of the key points are:

(i) the distribution $\mathrm{ind}(D)$ only depends [**3**, theorem (2·6)] on the K-theory class of the symbol of D in the group $K_{\mathbf{G}}(T^*_{\mathbf{G}}X)$, where $T^*_{\mathbf{G}}X$ is the subspace of the cotangent bundle T^*X annihilated by the Lie algebra of $\mathbf{G}$ (i.e. “transverse” to the $\mathbf{G}$-orbits);

(ii) the distribution $\mathrm{ind}(D)$ is supported [**3**, theorem (4·6)] by the conjugacy classes of all elements of $\mathbf{G}$ that have fixed points in X;

(iii) furthermore, $\mathrm{ind}(D)$ is covariant [**3**, theorems (4·1) and (4·3)] with respect to embeddings $\mathbf{G} \hookrightarrow \mathbf{H}$ and $X \hookrightarrow Z$;

(iv) for a connected Lie group $\mathbf{G}$, computation of $\mathrm{ind}(D)$ can be reduced [**3**, theorem (4·2)] to the case of a maximal torus $\mathbf{T}$, replacing X by $X \times \mathbf{G}/\mathbf{T}$;

(v) when $\mathbf{G}$ has only finite isotropy groups, the transverse signature operator is transversally elliptic. Its index equals the signature [**3**, theorem (10·3)] of the rational homology manifold $X/\mathbf{G}$;

(vi) if, in ((v)), $\mathbf{G}$ is a torus, then the signature of $X/\mathbf{G}$ can be expressed in cohomological terms involving the (finitely many) fixed point sets X^g for $g \in \mathbf{T}$. This formula coincides with the Lefschetz Theorem formula [**5**, (3·9)];

(vii) using ((iv)), the signature formula ((vi)) extends to all $\mathbf{G}$ by using standard results about the flag manifold $\mathbf{G}/\mathbf{T}$.

In the special case when $\dim X/\mathbf{G} = 4$, the signature formula for an orbifold is just equation (3·1), which is what we need in this paper. However the general theory also applies to the higher-dimensional case, when $\dim X = 4k$.

It remains to explain how the signature theorem for an oriented orbifold M can be derived from this theory of transversally elliptic operators. In fact, we only need the special case when $M = X/\mathbf{G}$ is the quotient of a manifold by a compact group $\mathbf{G}$ acting with only finite isotropy groups. The key observation [**35**] is that the oriented frame bundle of any oriented orbifold is a smooth manifold P; we can thus just take $X = P$ and $\mathbf{G} = \mathbf{SO}(n)$. We just need to to verify that P is a manifold, and that $\mathbf{SO}(n)$ acts on it with only finite isotropy; cf. [**10**, proposition 4·2·17]. This problem is essentially local, we may assume that $M = \Gamma\backslash U$, where the finite group $\Gamma \subset \mathbf{SO}(n)$ acts on $U \approx \mathbb{R}^n$ by left multiplication; moreover, there is a trivialization $P(U) \cong U \times \mathbf{SO}(n)$ such a manner that Γ acts freely on $\mathbf{SO}(n)$ on the left, thus commuting with the natural right action by $\mathbf{SO}(n)$. We now see that $P(M) = \Gamma\backslash P(U)$ is therefore a manifold, and that $M = P(M)/\mathbf{SO}(n)$. Furthermore, every isotropy group of the $\mathbf{SO}(n)$ action on $P(M)$ is now a subgroup of some isotropy group for the right action of $\mathbf{SO}(n)$ on $\Gamma\backslash\mathbf{SO}(n)$, and so is conjugate to a subgroup of Γ.

Much of our discussion generalizes nicely to higher dimensions. The $\mathbf{G}$-signature theorem [**5**, (6·12)] gives an explicit cohomological formula for the signature $\tau(g, X)$, for any $g \in \mathbf{G}$, in terms of its fixed point set X^g. When $Y = X^g$ is of codimension 2 with normal cone angle $\beta = 2\pi/p$, p an integer, the contribution of Y becomes

$$\left\{2^{2k-1}\mathscr{L}(Y)\coth\left(\frac{y+i\beta}{2}\right)\right\}[Y] \tag{A 2}$$

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

where $\mathscr{L}$ is the stable characteristic class given by

$$\mathscr{L} = \sum \mathscr{L}_r(p) = \prod \frac{x_i/2}{\tanh x_i/2} \tag{A 3}$$

which is essentially built from the Hirzebruch L-series in $x/2$. Here, y denotes the first Chern class of the normal bundle of $Y \subset X$.

The "defect" contribution of Y to the signature $\tau(X)$ is given by replacing β by $r\beta$, $1 \leqslant r < p$ in (A 2), summing over r and dividing by p. This gives an explicit polynomial in y and the Pontrjagin classes of Y, depending on β. In the $k = 1$ case, where $\dim X = 4$, we recover the formula (3·1).

Finally, one can extend this formula to all $\beta > 0$, and thereby compute the signature defect due to any edge-cone singularity along Y. To do this, we proceed as before, first obtaining a formula for rational values of $\beta = q/p$, and then extending it to all real values by continuity.

Acknowledgements. The authors would like to express their gratitude to the Simons Center for Geometry and Physics for its hospitality during the Fall semester of 2011, during which time this project began to take shape. They would also like to thank fellow Simons Center visitors Nigel Hitchin and Sergey Cherkis for many useful discussions of gravitational instantons and related topics. The second author would also like to thank Rafe Mazzeo and Xiuxiong Chen for a number of helpful suggestions.

REFERENCES

[**1**] M. ABREU. Kähler metrics on toric orbifolds. *J. Differential Geom.* **58** (2001), 151–187.
[**2**] M. T. ANDERSON. Ricci curvature bounds and Einstein metrics on compact manifolds. *J. Amer. Math. Soc.* **2** (1989), 455–490.
[**3**] M. F. ATIYAH. *Elliptic operators and compact groups*. Lecture Notes in Math., vol. 401 (Springer-Verlag, Berlin, 1974).
[**4**] M. F. ATIYAH and N. J. HITCHIN. *The geometry and dynamics of magnetic monopoles*. M. B. Porter Lectures (Princeton University Press, 1988).
[**5**] M. F. ATIYAH and I. M. SINGER. The index of elliptic operators. III. *Ann. of Math.* (2) **87** (1968), 546–604.
[**6**] M. F. ATIYAH, V. K. PATODI and I. M. SINGER. Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Camb. Phil. Soc.* **79** (1976), 71–99.
[**7**] S. BANDO and R. KOBAYASHI. Ricci-flat Kähler metrics on affine algebraic manifolds. II. *Math. Ann.* **287** (1990), 175–180.
[**8**] A. L. BESSE. *Einstein manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) vol. 10 (Springer-Verlag, 1987).
[**9**] R. BIELAWSKI. Monopoles and the Gibbons–Manton metric. *Comm. Math. Phys.* **194** (1998), 297–321.
[**10**] C. P. BOYER and K. GALICKI. *Sasakian geometry*. Oxford Mathematical Monographs (Oxford University Press, 2008).
[**11**] S. BRENDLE. Ricci flat Kahler metrics with edge singularities. E-print arXiv:1103.5454 [math.DG] (2011).
[**12**] J. CHEEGER and G. TIAN. Curvature and injectivity radius estimates for Einstein 4-manifolds. *J. Amer. Math. Soc.* **19** (2006), 487–525.
[**13**] X. CHEN, C. LEBRUN and B. WEBER. On conformally Kähler, Einstein manifolds. *J. Amer. Math. Soc.* **21** (2008), 1137–1168.
[**14**] S. A. CHERKIS. Instantons on gravitons. *Comm. Math. Phys.* **306** (2011), 449–483.
[**15**] S. A. CHERKIS and N. J. HITCHIN. Gravitational instantons of type D_k. *Comm. Math. Phys.* **260** (2005), 299–317.
[**16**] S. A. CHERKIS and A. KAPUSTIN. D_k gravitational instantons and Nahm equations. *Adv. Theor. Math. Phys.* **2** (1998), 1287–1306 (1999).
[**17**] S.-S. CHERN. A simple intrinsic proof of the Gauss–Bonnet formula for closed Riemannian manifolds. *Ann. of Math.* (2) **45** (1944), 747–752.

https://doi.org/10.1017/S0305004113000169 Published online by Cambridge University Press

[**18**] P. E. CONNER and E. E. FLOYD. *Differentiable periodic maps*. Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33 (Academic Press Inc., New York, 1964).

[**19**] X. DAI and G. WEI. Hitchin–Thorpe inequality for noncompact Einstein 4-manifolds. *Adv. Math.* **214** (2007), 551–570.

[**20**] S. K. DONALDSON. Kähler metrics with cone singularities along a divisor. In *Essays in Mathematics and its Applications* (Springer, Heidelberg, 2012), pp. 49–79.

[**21**] T. EGUCHI and A. J. HANSON. Self-dual solutions to Euclidean gravity. *Ann. Physics* **120** (1979), 82–106.

[**22**] G. W. GIBBONS and S. W. HAWKING. Classification of gravitational instanton symmetries. *Comm. Math. Phys.* **66** (1979), 291–310.

[**23**] G. W. GIBBONS and N. S. MANTON. The moduli space metric for well-separated BPS monopoles. *Phys. Lett. B* **356** (1995), 32–38.

[**24**] F. HIRZEBRUCH. The signature of ramified coverings. In *Global Analysis (Papers in Honor of K. Kodaira)*, pp. 253–265 (Tokyo Press, 1969).

[**25**] F. HIRZEBRUCH. The signature theorem: reminiscences and recreation. In *Prospects in Mathematics*, (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), pp. 3–31. Ann. of Math. Studies, no. 70 (Princeton University Press, 1971).

[**26**] N. J. HITCHIN. Compact four-dimensional Einstein manifolds. *J. Differential Geometry* **9** (1974), 435–441.

[**27**] N. J. HITCHIN. Polygons and gravitons. *Math. Proc. Camb. Phil. Soc.* **85** (1979), 465–476.

[**28**] N. J. HITCHIN. Poncelet polygons and the Painlevé equations. In *Geometry and Analysis* (Bombay, 1992), pp. 151–185 (Tata Inst. Fund. Res., Bombay, 1995).

[**29**] N. J. HITCHIN. Twistor spaces, Einstein metrics and isomonodromic deformations. *J. Differential Geom.* **42** (1995), 30–112.

[**30**] N. J. HITCHIN. A new family of Einstein metrics. In *Manifolds and Geometry* (Pisa, 1993). Sympos. Math., XXXVI, pp. 190–222 (Cambridge Univ. Press, Cambridge, 1996).

[**31**] T. IZAWA. Note on the Riemann–Hurwitz type formula for multiplicative sequences. *Proc. Amer. Math. Soc.* **131** (2003), 3583–3588.

[**32**] T. JEFFRES, R. MAZZEO and Y. RUBINSTEIN. Kähler–Einstein metrics with edge singularities. E-print arXiv:1105.5216 [math.DG] (2011).

[**33**] A. KASUE. A convergence theorem for Riemannian manifolds and some applications. *Nagoya Math. J.* **114** (1989), 21–51.

[**34**] T. KAWASAKI. The signature theorem for V-manifolds. *Topology* **17** (1978), 75–83.

[**35**] T. KAWASAKI. The index of elliptic operators over V-manifolds. *Nagoya Math. J.* **84** (1981), 135–157.

[**36**] P. B. KRONHEIMER. Instantons gravitationnels et singularités de Klein. *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 53–55.

[**37**] P. B. KRONHEIMER. A Torelli-type theorem for gravitational instantons. *J. Differential Geom.* **29** (1989), 685–697.

[**38**] C. LEBRUN. Complete Ricci-flat Kähler metrics on $\mathbf{C}^n$ need not be flat. In *Several complex variables and complex geometry, part* 2 (Santa Cruz, CA, 1989), volume 52 of *Proc. Sympos. Pure Math.* (Amer. Math. Soc., Providence, RI, 1991), pp. 297–304.

[**39**] C. LEBRUN. Explicit self-dual metrics on $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$. *J. Differential Geom.* **34** (1991), 223–253.

[**40**] C. LEBRUN, S. NAYATANI and T. NITTA. Self-dual manifolds with positive Ricci curvature. *Math. Z.* **224** (1997), 49–63.

[**41**] V. MINERBE. Rigidity for multi-Taub-NUT metrics. *J. Reine Angew. Math.* **656** (2011), 47–58.

[**42**] H. NAKAJIMA. Self-duality of ALE Ricci-flat 4-manifolds and positive mass theorem. In *Recent topics in differential and analytic geometry*, volume 18 of *Adv. Stud. Pure Math.* (Academic Press, Boston, MA, 1990), pp. 385–396.

[**43**] H. PEDERSEN. Einstein metrics, spinning top motions and monopoles. *Math. Ann.* **274** (1986), 35–59.

[**44**] I. SATAKE. The Gauss-Bonnet theorem for V-manifolds. *J. Math. Soc. Japan* **9** (1957), 464–492.

[**45**] J. A. THORPE. Some remarks on the Gauss–Bonnet formula. *J. Math. Mech.* **18** (1969), 779–786.

[**46**] G. TIAN and S.-T. YAU. Complete Kähler manifolds with zero Ricci curvature. I. *J. Amer. Math. Soc.* **3** (1990), 579–609.

[**47**] K. P. TOD. Self-dual Einstein metrics from the Painlevé VI equation. *Phys. Lett. A* **190** (1994), 221–224.

[**48**] K. P. TOD. The $SU(\infty)$-Toda field equation and special four-dimensional metrics. In *Geometry and physics* (Aarhus, 1995), vol. 184 of *Lecture Notes in Pure and Appl. Math.* (Dekker, New York, 1997), pp. 307–312.