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ON THE *i*th LATENT ROOT OF A COMPLEX MATRIX(1)

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1. **Introduction and summary.** Goodman [1] has pointed out the applications of the distributional results of the complex multivariate normal statistical analysis. Khatri [4], has suggested the maximum latent root statistic for testing the reality of a covariance matrix. The joint distribution of the latent roots under certain null hypotheses can be written as, [2], [3],

$$c_1\left\{\prod_{j=1}^{q} w_j^m(1-w_j)^n\right\}\prod_{i>j}(w_i-w_j)^2 \tag{1}$$

where

$$c_1 = \prod_{j=1}^{q}\{\Gamma(n+m+q+j)/\{\Gamma(n+j)\Gamma(m+j)\Gamma(j)\}\}$$

and

$$0 \le w_1 \le w_2 \le \cdots \le w_q \le 1.$$

We may also note that when n is large, the joint distribution of $nw_j=f_j$, $j=1,\ldots,q$, $0\le f_1\le\cdots\le f_q\le\infty$, can be written as

$$c_2\prod_{j=1}^{q} f_j^m \exp\left(-\sum_{j=1}^{q} f_j\right)\left\{\prod_{i>j}(f_i-f_j)^2\right\} \tag{2}$$

where

$$c_2 = 1\bigg/\left\{\prod_{j=1}^{q}[\Gamma(m+j)\Gamma(j)]\right\}.$$

Khatri [2], has derived the distribution of w_q (or w_1) and f_q in a determinant form. In this paper we first derive the distribution of w_{q-1} and f_{q-1} and then the distribution of w_i and f_i. In this connection a lemma has been proved.

2. **Preliminary results.** In this section, we first state two lemmas, and prove a third lemma.

LEMMA 1.

$$\sum\int_{\mathscr{D}}\prod_{j=1}^{s}[x_j^{m_j'}(1-x_j)^{n_j'}\,dx_j] = \prod_{j=1}^{s}\left[\int_0^x x_j^{m_j}(1-x_j)^{n_j}\,dx_j\right]$$

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where $\mathscr{D}':(0\leq x_1\leq\cdots\leq x_s\leq x)$, $(x\leq 1)$; *and on the left-hand side* $(m'_s, n'_s), \ldots, (m'_1, n'_1)$ *is any permutation of* $(m_s, n_s), \ldots, (m_1, n_1)$ *and the summation is taken over all such permutations.*

For proof, see Roy [6, p. 203, A. 9.3].

LEMMA 2.

$$\prod_{i>j}(w_i-w_j)^2=\sum\begin{vmatrix} w_{j_1}^{2q-2} & w_{j_2}^{2q-3} & w_{jq}^{q-1} \\ w_{j_1}^{2q-3} & w_{j_2}^{2q-4} & w_{jq}^{q-2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ w_{j_1}^{q-1} & w_{j_2}^{q-2} & w_{jq}^{0} \end{vmatrix},$$

where $\sum$ *means summation over all permutations* $(j_1, j_2, \ldots, j_q)$ *of* $(1, 2, \ldots, q)$, *and* $|A|$ *means the determinant of* A.

For proof, see Khatri [2].

LEMMA 3.

$$\sum\int_{\mathscr{D}'}\prod_{j=1}^{s}[x_j^{m_j'}(1-x_j)^{n_j'}\,dx_j]=\prod_{j=1}^{s}\left[\int_x^1 x_j^{m_j}(1-x_j)^{n_j}\,dx_j\right],$$

where $\mathscr{D}':(x\leq x_1\leq x_2\leq\cdots\leq x_s\leq 1)$, *and on the left-hand side* $(m'_s, n'_s), \ldots, (m'_1, m'_1)$ *is any permutation of* $(m_s, n_s), \ldots, (m_1, n_1)$ *and the summation is taken over all such permutations.*

Proof is similar to Lemma 1.

3. **The distribution of** w_{q-1}. In this section we obtain first the cdf's of w_{q-1} and f_{q-1} and in the next those of w_i and f_i. Note that

$$\Pr\{w_{q-1}\leq x\}=\Pr\{w_q\leq x\}+\Pr\{w_{q-1}\leq x<w_q\leq 1\} \tag{3}$$

Khatri [1], showed that

$$\Pr\{w_q\leq x\}=c_1\,|(\beta_{i+j-2})|=c_1\begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{q-1} \\ \beta_1 & \beta_2 & & \beta_q \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \beta_{q-1} & \beta_q & & \beta_{2q-2} \end{vmatrix}, \tag{4}$$

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where c_1 is defined in (1), $\beta_{i+j-2}=\int_0^x w^{m+i+j-2}(1-w)^n\,dw$ for $i,j=1,2,\ldots,q$ and (β_{i+j-2}) is a $q\times q$ matrix. Now the determinant in Lemma 2, can be written as

$$\sum_1 \operatorname{sign}(t_1,\ldots,t_q)w_{j1}^{q-1+t_1}w_{j2}^{q-2+t_2}\ldots w_{jq}^{t_q}, \tag{5}$$

where $(t_1,\ldots,t_q)$ is a permutation of $(0,1,\ldots,q-1)$, $\operatorname{sign}(t_1,\ldots,t_q)$ is positive if the permutation is even and negative if the permutation is odd, and $\sum_1$ means the summation over all such permutations. Then (1) can be written as

$$\begin{aligned} c_1\Big\{\prod_{j=1}^q w_j^m(1-w_j)^n \sum_{j_1,\ldots,j_{q-1}}\sum_1 \operatorname{sign}(t_1,\ldots,t_q)\\ \times[w_q^{q-1+t_1}w_{j1}^{q-2+t_2}w_{j2}^{q-3+t_3}\cdots w_{jq-1}^{t_q}+w_q^{q-2+t_2}w_{j1}^{q-1+t_1}w_{j2}^{q-3+t_3}\cdots w_{jq-1}^{t_q}+\cdots\\ +w_q^{t_q}w_{j1}^{q-1+t_1}w_{j2}^{q-2+t_2}\cdots w_{jq-1}^{1+t_q}]. \end{aligned} \tag{6}$$

First taking summation over $(j_1,\ldots,j_{q-1})$, the permutation of $(1,2,\ldots,q-1)$ and integrate w_q over $x<w_q<1$, and apply lemma, we get

$$\begin{aligned} \Pr(w_{q-1}\le x\le w_q<1) = c_1\sum_1 \operatorname{sign}(t_1,\ldots,t_q)[\beta'_{q-1+t_1}\beta_{q-2+t_2}\cdots\beta_{t_q}\\ +\beta_{q-1+t_1}\beta'_{q-2+t_2}\cdots\beta_{t_q}+\cdots\beta_{q-1+t_1}\beta_{q-2+t_2}\cdots\beta'_{t_q}, \end{aligned} \tag{7}$$

where

$$\beta'_{i+j-2}=\int_x^1 w^{m+i+j-2}(1-w)^n\,dw,$$

then (7) can be written as

$$c_1\sum_{k=1}^q|(\beta_{i+j-2}^{(k)})|, \tag{8}$$

where $|(\beta_{i+j-2}^{(k)})|$ is the determinant obtained from $|(\beta_{i+j-2})|$ by replacing, the kth column of $|(\beta_{i+j-2})|$, β_α, by the corresponding β'_α's. So we proved the following theorem.

THEOREM 1. *If the joint distribution of $w_1,\ldots,w_q$ is given by* (1), *then*

$$\Pr\{w_{q-1}\le x\} = c_1\sum_{k=0}^q|(\beta_{i+j-2}^{(k)})| \tag{9}$$

where $|(\beta_{i+j-2}^{(0)})|=|(\beta_{i+j-2})|$, and $|(\beta_{i+j-2}^{(k)})|$ is defined in (8), *and c_1 is defined in* (1).

THEOREM 2. *If the distribution of $f_1,\ldots,f_q$ is given by* (2) *then*

$$\Pr\{f_{q-1}\le x\} = c_2\sum_{k=0}^q|(\gamma_{i+j-2}^{(k)})|, \tag{10}$$

where $\gamma_{i+j-2}=\int_0^x w^{m+i+j-2}\exp(-w)\,dw$, (γ_{i+j-2}) is a $q\times q$ matrix and $(\gamma_{i+j-2}^{(k)})$ is defined similar to that of (9), *and c_2 is defined in* (2).

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Proof is similar to that of Theorem 1.

4. **The distribution of** w_i. It may be noted here that

(11) $\Pr\{w_i \le x\} = \Pr\{w_{i+1} \le x\} + \Pr\{w_i \le x < w_{i+1}\}, \qquad i = 1, \ldots, q-1.$

To evaluate the second term of (11), we may write

(12)
$$\prod_{i>j} (w_i - w_j)^2 = \sum_1 \operatorname{sign}(t_1, \ldots, t_q) \sum_2 \sum_{i_1, \ldots, i_{q-i}} w_{i1}^{\alpha_1} w_{i2}^{\alpha_2} \cdots w_{i_{q-i}}^{\alpha_{q-1}} \sum_{j_1, \ldots, j_i} w_{j1}^{\alpha_{q-i+1}} w_{j2}^{\alpha_{q-i+2}} \cdots w_{j_q}^{\alpha_i}$$

where $(i_1, \ldots, i_{q-1})$ is permutation of $(i+1, \ldots, q)$ and $\sum_{i_1, \ldots, i_{q-i}}$ runs over all such permutations; $(j_1, \ldots, j_i)$ is a permutation of $(1, \ldots, i)$ and $\sum_{j1, \ldots, ji}$ runs over all such permutations; $\sum_2$ is the summation over the terms $\binom{q}{q-i}$ terms of obtained by taking $q-i$, $(\alpha_1, \ldots, \alpha_{q-i})$, at a time of $q-1+t_1, q-2+t_2, \ldots, t_q$.

Substituting (12) in (1) and using Lemma 1 and Lemma 3, and as in § (3), we get

(13)
$$\Pr(w_i \le x < w_{i+1}) = c_1 \sum_2 |(\beta_{i+j-2}^{(i_0)})|,$$

where $(\beta_{i+j-2}^{(i_0)})$ is a $q \times q$ matrix obtained from (β_{i+j-2}) by replacing i columns of (β_{i+j-2}) by the corresponding β'_α's. Therefore by (10), (14) and Theorem 1 and reduction process, we can get the distribution of w_i.

It may be pointed out that, [5],

(13)′
$$\Pr\{w_i \le x; m, n) = 1 - \Pr(w_{q-i+1} \le 1-x; n, m)$$

where on the right side of (13) the parameters m and n are interchanged, hence the distribution of w_1, [2], can be written as

(14)
$$\Pr\{w_1 \le x\} = 1 - c_1 |(\delta_{i+j-2})|,$$

where $\delta_{i+j-2} = \int_0^{1-x} z^{n+i+j-2}(1-z)^m \, dz$, and (δ_{i+j-2}) is a $q \times q$ matrix, similarly, if we define $\delta'_{i+j-2} = \int_{1-x}^1 z^{n+i+j-2}(1-z)^m \, dz$, the distribution of w_2 can be written as

(15)
$$\Pr\{w_2 \le x\} = 1 - c_1 \sum_{k=0}^{q} |(\delta_{i+j-2}^{(k)})|,$$

where, as before, $|(\delta_{i+j-2}^{(k)})|$ is the determinant obtained from $|(\delta_{i+j-2})|$ by replacing the kth column of $|(\delta_{i+j-2})|$ by the corresponding δ'_α's, and $(\delta_{i+j-2}^{(0)}) = (\delta_{i+j-2})$. A similar method gives

(16) $\Pr\{f_i \le x\} = \Pr\{f_{i+1} \le x\} + \Pr\{f_i \le x < f_{i+1}\}, \qquad i = 1, 2, \ldots, q-1,$

and

(17)
$$\Pr\{f_i \le x < f_{i+1}\} = c_2 \sum_2 |(\gamma_{i+j-2}^{(i_0)})|,$$

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where c_2 is defined in (2), and also $(\gamma_{i+j-2}^{(i_0)})$ is a $q \times q$ matrix obtained from (γ_{i+j-2}) by replacing i columns of (γ_{i+j-2}) by the corresponding γ_α''s.

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