Can. J. Math. Vol. **46** (4), 1994 pp. 808–817

THE EXTENSIONS OF AN INVARIANT MEAN AND THE SET LIM $\sim$ TLIM

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ABSTRACT. Let $\mathcal{F} = \{\varphi \in \ell^\infty(\mathbb{N})^* : \varphi \geq 0, \|\varphi\| = 1$ and $\varphi(f) = 0$ if $f \in \ell^\infty(\mathbb{N})$ with $\lim_n f(n) = 0\}$. If G is a nondiscrete locally compact group which is amenable as a discrete group and $m \in \mathrm{LIM}(\mathrm{CB}(G))$, then we can embed $\mathcal{F}$ into the set of all extensions of m to left invariant means on $L^\infty(G)$ which are mutually singular to every element of $\mathrm{TLIM}(L^\infty(G))$, where $\mathrm{LIM}(S)$ and $\mathrm{TLIM}(S)$ are the sets of left invariant means and topologically left invariant means on S with $S = \mathrm{CB}(G)$ or $L^\infty(G)$. It follows that the cardinalities of $\mathrm{LIM}(L^\infty(G)) \sim \mathrm{TLIM}(L^\infty(G))$ and $\mathrm{LIM}(L^\infty(G))$ are equal. Note that $\mathcal{F}$ which contains $\beta\mathbb{N}$ is a very big set. We also embed $\mathcal{F}$ into the set of all left invariant means on $\mathrm{CB}(G)$ which are mutually singular to every element of $\mathrm{TLIM}(\mathrm{CB}(G))$ for $G = G_1 \times G_2$, where G_1 is nondiscrete, non-compact, σ-compact and amenable as a discrete group and G_2 is any amenable locally compact group. The extension of any left invariant mean on $\mathrm{UCB}(G)$ to $\mathrm{CB}(G)$ is discussed. We also provide an answer to a problem raised by Rosenblatt.

1. **Introduction and notations.** Let G be a locally compact group with a fixed left Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated real Lebesgue spaces ($1 \leq p \leq \infty$). For $f \in L^\infty(G)$ and $x \in G$, the left translation of f by x is defined by ${}_xf(y) = f(xy)$, $y \in G$. Let $\mathrm{CB}(G)$ be the subspace of $L^\infty(G)$ consisting of all continuous bounded functions on G. A function $f \in \mathrm{CB}(G)$ is called *right uniformly continuous* if the map $x \longrightarrow {}_xf$ from G to $\big(\mathrm{CB}(G), \|\cdot\|_\infty\big)$ is continuous. Let $\mathrm{UCB}_r(G)$ denote the subspace of $\mathrm{CB}(G)$ consisting of all right uniformly continuous bounded functions. The space $\mathrm{UCB}_\ell(G)$ of left uniformly continuous bounded functions on G is defined analogously, and the space of uniformly continuous bounded functions on G is $\mathrm{UCB}(G) = \mathrm{UCB}_r(G) \cap \mathrm{UCB}_\ell(G)$.

Let $P(G)$ denote the set of all $\varphi \in L^1(G)$ with $\varphi \geq 0$ and $\|\varphi\|_1 = \int_G |f(x)|\,dx = 1$. For $f \in L^\infty(G)$ and $\varphi \in P(G)$, the convolution $\varphi * f \in L^\infty(G)$ of φ and f on G is defined by

$$\varphi * f(x) = \int_G \varphi(t) f(t^{-1}x)\,dt \quad (x \in G).$$

Then $\varphi * f \in \mathrm{UCB}_r(G)$ (see [4] p. 24). We say that a subspace S of $L^\infty(G)$ is *admissible* if $1 \in S$ and S is invariant under both the left translation by G and the convolution by $P(G)$. A mean on S is a positive functional on S with $m(1) = 1$. A left invariant mean is

This research is supported by an NSERC grant.

Received by the editors October 19, 1992.

AMS subject classification: 43A07.

Key words and phrases: locally compact groups, amenable groups, invariant means, the set of left invariant means.

a mean with $m({}_xf) = m(f)$ for any $x \in G$ and $f \in S$. The set of left invariant means on S is denoted by $\mathrm{LIM}(S)$. A mean on S is topologically left invariant if $m(\varphi * f) = m(f)$ for any $\varphi \in P(G)$ and $f \in S$. Let $\mathrm{TLIM}(S)$ denote the set of topologically left invariant means on S. Then $\mathrm{TLIM}(S) \subseteq \mathrm{LIM}(S)$ (see [4], p. 25).

If $\mathrm{LIM}\big(L^\infty(G)\big) \neq \emptyset$, we say that G is *amenable*. Let G_d be the same algebraic group as G with a discrete topological structure. Then G is amenable if G_d is amenable. Properties of amenable groups and left invariant means can be found in Greenleaf [4], Pier [10] and Paterson [9].

Let G be nondiscrete and let G_d be amenable. Granirer [3] and Rudin [16] showed that $\mathrm{LIM}\big(L^\infty(G)\big) \neq \mathrm{TLIM}\big(L^\infty(G)\big)$. In addition, if G is noncompact, Liu and van Rooij [6] proved that $\mathrm{LIM}\big(\mathrm{CB}(G)\big) \neq \mathrm{TLIM}\big(\mathrm{CB}(G)\big)$. For any set A, the cardinality of A is denoted by $|A|$ and let $H = \mathrm{span}\{{}_xf - f : x \in G, f \in L^\infty(G)\}$. We say that $m_1, m_2 \in \mathrm{LIM}(S)$ are *mutually singular* if there exists $f \in S$ with $0 \leq f \leq 1$ such that $m_1(f) = 1$ and $m_2(f) = 0$ for an admissible subspace S.

In this paper, we will be primarily concerned with the number of the extensions of a left invariant mean to a larger space such that they are not topologically left invariant. The main technique that will be used is to embed $\mathcal{F}$ into the set of all the extensions (see [1] and [8]), where $\mathcal{F} = \{\varphi \in \ell^\infty(\mathbb{N})^* : \varphi \geq 0, \|\varphi\| = 1$ and $\varphi(f) = 0$ if $f \in \ell^\infty(\mathbb{N})$ with $\lim_n f(n) = 0\}$.

Suppose that G is nondiscrete and G_d is amenable. In Section 2, we show that for any $m \in \mathrm{LIM}\big(\mathrm{CB}(G)\big)$, there is a linear isometry $\pi^* \colon \ell^\infty(\mathbb{N})^* \longrightarrow L^\infty(G)^*$ such that for any $\varphi \in \mathcal{F}$, $\pi^*\varphi \in \mathrm{LIM}\big(L^\infty(G)\big)$ is an extension of m and is singular to $\mathrm{TLIM}\big(L^\infty(G)\big)$. This shows that m has at least 2^c extensions to left invariant means on $L^\infty(G)$ which are singular to $\mathrm{TLIM}\big(L^\infty(G)\big)$. Hence one of the main results of Rosenblatt [15] is true without the conditions of σ-compact and metrization (see [15]). Consequently, we have that $\Big|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\Big| = \Big|\mathrm{LIM}\big(L^\infty(G)\big)\Big|$. Hence the two cardinality questions are the same in this case (see Paterson [9] Chapter 7, Question 1 and 3).

Let G_1 be nondiscrete noncompact σ-compact and amenable as a discrete group and let G_2 be any amenable group. In Section 3, we prove that if $G = G_1 \times G_2$, then there exists an embedding π^* of $\mathcal{F}$ into $\mathrm{LIM}\big(\mathrm{CB}(G)\big)$ such that for any $\varphi \in \mathcal{F}$, $\pi^*\varphi$ is mutually singular to every element of $\mathrm{TLIM}\big(\mathrm{CB}(G)\big)$. This will improve Theorem 2.5 of Rosenblatt [14] and answers the problem raised by [14] whether discrete amenability is necessary in this theorem.

In Section 4, we discuss the extension of an $m \in \mathrm{LIM}\big(\mathrm{UCB}(G)\big)$ to $\mathrm{CB}(G)$. We have that if G is nondiscrete, noncompact, σ-compact and amenable as a discrete group, then there exists an $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ such that $\psi(f) = 0$ for any $\psi \in \mathrm{TLIM}\big(\mathrm{CB}(G)\big)$ and any $m \in \mathrm{LIM}\big(\mathrm{UCB}(G)\big)$ has an extension to a left invariant functional $\bar{m}$ on $\mathrm{CB}(G)$ with $\bar{m}(f) = 1$ and $\|\bar{m}\| \leq 3$.

A portion of this paper is contained in the author's Ph.D. thesis. He wishes to thank Professor A. Lau for his advice and encouragement.

2. **The number of extensions of an invariant mean.** The study of the number of extensions of an invariant mean on $CB(G)$ to $L^\infty(G)$ was initiated by Rosenblatt in [15]. By a Baire category argument, he showed in [15] that there is a continuum of measurable sets $\{A_r : f \in \Gamma\}$ independent on the open sets for a nondiscrete σ-compact locally compact metric group G. The metrization and σ-compact are important in this Baire category argument. Consequently, if G is amenable as a discrete group, then any left invariant mean m on $CB(G)$ has at least 2^c mutually singular extensions to a left invariant mean on $L^\infty(G)$ each of which is singular to $\mathrm{TLIM}\big(L^\infty(G)\big)$.

By a technique used in Chou [1] (also see [2] and [8]), we will improve this result by removing the conditions of σ-compact and metrization.

A function $f \in L^\infty(G)$ is called *topologically null* if $m(f) = 0$ for any $m \in \mathrm{TLIM}\big(L^\infty(G)\big)$. A subset U in G is called *strictly positive* (s. p.) in G if $(\bigcap_{i=1}^n x_i U) \cap O$ is not locally null for any open set O in G and $x_1, x_2, \ldots, x_n \in G$. The following lemma is motivated by Theorem 1.2 of Rosenblatt [15].

LEMMA 2.1. *Let G be a locally compact group which is amenable as a discrete group. If U is a* s. p. *set in G, then any $m \in \mathrm{LIM}\big(CB(G)\big)$ can be extended to a left invariant mean on $L^\infty(G)$ such that $m(1_U) = 1$.*

PROOF. Let S be the smallest left invariant subspace containing $CB(G)$ and 1_U. For any $f \in S$, there exists $f_1 \in CB(G)$, $x_1, x_2, \ldots, x_n \in G$ and constants $a_1, a_2, \ldots, a_n$ such that $f = f_1 + \sum_{i=1}^n a_{ix_i} 1_U$. Define an extension $\bar{m}$ of m to S by

$$\bar{m}(f) = m(f_1) + \sum_{i=1}^n a_i.$$

Then $\bar{m}$ is well defined. Indeed, let $f_1 + \sum_{i=1}^n a_{ix_i} 1_U = 0$ and $m(f_1) + \sum_{i=1}^n a_i \neq 0$. So $f_1 + \sum_{i=1}^n a_i \neq 0$. Since $f_1 + \sum_{i=1}^n a_i \in CB(G)$, there exists an open set V in G such that $f_1 + \sum_{i=1}^n a_i \neq 0$ on V. But $(\bigcap_{i=1}^n x_i^{-1} U) \cap V$ is not locally null and $f_1 + \sum a_{ix_i} 1_U = f_1 + \sum a_i \neq 0$ on $(\bigcap_{i=1}^n x_i^{-1} U) \cap V$. This is impossible. Similarly, we can show that $\bar{m}$ is positive on S. It is clear that $\bar{m}$ is left invariant and $\bar{m}(1_U) = 1$. By the discrete amenability of G, we can extend $\bar{m}$ to an invariant mean on $L^\infty(G)$ with $\bar{m}(1_U) = 1$. ∎

As in Chou [2], let

$$\mathcal{F} = \Big\{\theta \in \ell^\infty(\mathbb{N})^* : \theta \geq 0, \|\theta\| = 1 \text{ and } \theta(f) = 0 \text{ if } f \in \ell^\infty(\mathbb{N}) \text{ with } \lim_n f(n) = 0\Big\}.$$

Then $\beta\mathbb{N} \subseteq \mathcal{F}$ and $|\mathcal{F}| = 2^c$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of the set of positive integers $\mathbb{N}$ with discrete topology. We are going to embed $\mathcal{F}$ into the set of all the left invariant means on $L^\infty(G)$ which are extensions of a left invariant mean on $CB(G)$.

THEOREM 2.2. *Let G be a nondiscrete locally compact group which is amenable as a discrete group. If $m \in \mathrm{LIM}\big(CB(G)\big)$, then there exists a positive mapping $\pi\colon L^\infty(G) \longrightarrow \ell^\infty(\mathbb{N})$ such that π is onto, $\|\pi\| = 1$ and its conjugate $\pi^*\colon \ell^\infty(\mathbb{N})^* \longrightarrow L^\infty(G)^*$ is a linear* *isometry with $\pi^*\mathcal{F} \subseteq \mathrm{LIM}\big(L^\infty(G)\big)$ and every $\pi^*\theta$ is an extension of m. Moreover, $\pi^*\theta$ is singular to $\mathrm{TLIM}\big(L^\infty(G)\big)$ and elements of $\pi^*(\beta\mathbb{N})$ are mutually singular.*

PROOF. Let $\{E_n : n = 1, 2, \ldots\}$ be a set of disjoint strictly positive subsets of G such that $1_{\bigcup_{i=1}^\infty E_n}$ is topologically null when G is noncompact. If G is compact, we can take this sequence of subsets such that $\lambda(\bigcup_{n=1}^\infty E_n) < 1$ (see [8] Lemma 2.4). By Lemma 2.1, there is an $m_n \in \mathrm{LIM}\big(L^\infty(G)\big)$ such that m_n is an extension of m and $m_n(1_{E_n}) = 1$ for $n = 1, 2, \ldots$. Define $\pi \colon L^\infty(G) \longrightarrow \ell^\infty(\mathbb{N})$ by $\pi(f)(n) = m_n(f)$ for $f \in L^\infty(G)$ and $n \in \mathbb{N}$. Then π is positive, onto and $\|\pi\| = 1$. Also, π^* is a linear isometry with $\pi^*\mathcal{F} \subseteq \mathrm{LIM}\big(L^\infty(G)\big)$ and $\pi^*\theta$ is mutually singular to $\mathrm{TLIM}\big(L^\infty(G)\big)$ for any $\theta \in \mathcal{F}$ (see [8] Theorem 2.5).

For any $\theta \in \mathcal{F}$ and $f \in \mathrm{CB}(G)$, $\pi^*\theta(f) = \theta(\pi f)$. Since $\pi f(n) = m_n(f) = m(f)$ $(n \in \mathbb{N})$, $\pi f = m(f)$ and $\pi^*\theta(f) = \theta\big(\pi(f)\big) = m(f)$. Hence $\pi^*\theta$ is an extension of m.

Let $\theta_1, \theta_2 \in \beta\mathbb{N}$ such that $\theta_1 \neq \theta_2$. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$ which is a subspace of $\ell^{\infty *}(\mathbb{N})$ with w^*-topology, there are nets $\{n_\alpha\}$ and $\{n_\beta\}$ in $\mathbb{N}$ such that $n_\alpha \longrightarrow \theta_1$ and $n_\beta \longrightarrow \theta_2$ in the w^*-topology. Since $\beta\mathbb{N}$ is a Hausdorff space, we can assume that $\{n_\alpha\} \cap \{n_\beta\} = \phi$. Let $f = 1_{\bigcup_\alpha E_{n_\alpha}}$. Then $f \in L^\infty(G)$ with $0 \leq f \leq 1$. Since $\pi(f)(n_\alpha) = m_{n_\alpha}(f) = m_{n_\alpha}(1_{E_{n_\alpha}}) = 1$, $\pi^*\theta_1(f) = \theta_1(\pi f) = \lim_\alpha \pi f(n_\alpha) = 1$. Similarly, since $\pi(f)(n_\beta) = m_{n_\beta}(f) = 0$, $\pi^*\theta_2(f) = \theta_2(\pi f) = \lim_\beta \pi(f)(n_\beta) = 0$.

Therefore $\pi^*\theta_1$ and $\pi^*\theta_2$ are mutually singular. ∎

Since $|\beta\mathbb{N}| = 2^c$, we have the following:

COROLLARY 2.3. *Let G be nondiscrete and let G_d be amenable. Then any $m \in \mathrm{LIM}\big(\mathrm{CB}(G)\big)$ has at least 2^c mutually singular extensions to a left invariant mean on $L^\infty(G)$ each of which is mutually singular to any element of $\mathrm{TLIM}\big(L^\infty(G)\big)$.*

REMARK. This removes the condition of σ-compact and metrization for Corollary 1.3 of Rosenblatt [15].

Two of the cardinality questions that arise in connection with left invariant means are: what are $\big|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\big|$ and $\big|\mathrm{LIM}\big(L^\infty(G)\big)\big|$? The following corollary shows that these two questions are the same in some case (see [9] Chapter 7 Question 1 and 3). It has been showed that when G is nondiscrete and G_d is amenable, then $\big|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\big| \geq 2^c$ (see [8], [9] and [13]). Since, by Theorem 2.2,

$$\big|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\big| \geq \big|\mathrm{TLIM}\big(\mathrm{CB}(G)\big)\big| = \big|\mathrm{TLIM}\big(L^\infty(G)\big)\big|$$

and $\big|\mathrm{LIM}\big(L^\infty(G)\big)\big| \geq \big|\mathrm{TLIM}\big(L^\infty(G)\big)\big|$ always holds, we have

$$\begin{aligned}\big|\mathrm{LIM}\big(L^\infty(G)\big)\big| &= \big|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\big| + \big|\mathrm{TLIM}\big(L^\infty(G)\big)\big| \\ &= \big|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\big|\end{aligned}$$

which is the following:

https://doi.org/10.4153/CJM-1994-046-x Published online by Cambridge University Press

COROLLARY 2.4. *If G is nondiscrete and G_d is amenable, then*

$$\left|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\right| = \left|\mathrm{LIM}\big(L^\infty(G)\big)\right|.$$

It is proved in [8] that $\left|\mathrm{LIM}\big(L^\infty(G)\big)\right| = \left|\mathrm{TLIM}\big(L^\infty(G)\big)\right|$ for a metrizable noncompact group, and Lau and Paterson showed in [5] that $\left|\mathrm{TLIM}\big(L^\infty(G)\big)\right| = 2^{2^{d(G)}}$, where $d(G) = \mathit{inf}\{|C| : C$ is a cover of G by compact subsets in $G\}$. Hence we have the following.

COROLLARY 2.5. *If G is nondiscrete noncompact and metrizable, and G_d is amenable, then*

$$\left|\mathrm{LIM}\big(L^\infty(G)\big) \sim \mathrm{TLIM}\big(L^\infty(G)\big)\right| = 2^{2^{d(G)}}.$$

REMARK. There are locally compact groups such that $\left|\mathrm{LIM}\big(L^\infty(G)\big)\right| > \left|\mathrm{TLIM}\big(L^\infty(G)\big)\right|$ (see [8]).

3. **Discrete Amenability and the Set** LIM $\sim$ TLIM. Now we come to discuss the size of $\mathrm{LIM}\big(\mathrm{CB}(G)\big) \sim \mathrm{TLIM}\big(\mathrm{CB}(G)\big)$.

Rosenblatt in [14] showed that if G is a noncompact σ-compact and nondiscrete group which is amenable as a discrete group, then there are at least 2^c mutually singular elements of $\mathrm{LIM}\big(\mathrm{CB}(G)\big)$ all of which are singular to $\mathrm{TLIM}\big(\mathrm{CB}(G)\big)$.

He asked whether the condition of discrete amenability is necessary. Our Corollary 3.4 improves this theorem of Rosenblatt's and answers this problem. The technique that we are going to apply is to embed $\mathcal{F}$ into $\mathrm{LIM}\big(\mathrm{CB}(G)\big) \sim \mathrm{TLIM}\big(\mathrm{CB}(G)\big)$. It is different from Rosenblatt's. He used the Baire category argument.

DEFINITION 3.1. (a) An array $\{K_{nm}, m, n \geq 1\}$ of pairwise disjoint subsets in G is scattered if for all compact sets $C \subseteq G$, there exists $M, N \geq 1$ such that for each $x \in G$ at most one $Cx \cap K_{nm} \neq \phi$ with $m \geq M$ or $n \geq N$.

(b) A function $f \in \mathrm{CB}(G)$ is said to be *permanently near* one if for any $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n \in G$, there is an $x_0 \in G$ such that

$$|1 - {}_{x_i}f(x_0)| < \varepsilon \quad (i = 1, 2, \ldots, n).$$

Actually, Proposition 1.5 of Rosenblatt [14] proved the following. We show this result in details.

PROPOSITION 3.2. *Assume G is a noncompact σ-compact nondiscrete group. Then there exist $f_i \in \mathrm{CB}(G)$ with $0 \leq f_i \leq 1$ $(i = 1, 2, \ldots)$ such that each f_i is permanently near one. Moreover, for any bounded sequence $\{a_n\}$, $a_1f_1 + a_2f_2 + \cdots \in \mathrm{CB}(G)$, $\|a_1f_1 + a_2f_2 + \cdots\|_\infty = \sup\{|a_n|\}$ and $f_0 = f_1 + f_2 + \cdots$ is topologically null.*

PROOF. We follow the proof of Proposition 1.5 of Rosenblatt [14]. Let $\{U_m\}$ be an increasing sequence of nonempty open sets with compact closures $\bar{U}_m$ such that $G =$

$\bigcup_{m=1}^{\infty} U_m$ and let V be an open dense set in G with $\lambda(V^{-1}) < 1$. For each $m \geq 1$, let $\{f(m,n) : n \geq 1\}$ be a sequence of continuous functions with compact support contained in V_m such that (1) $0 \leq f(m,n) \leq 1$ for all $m, n \geq 1$; and (2) $f(m,n) \xrightarrow{n} 1_{V_m}$ pointwise a.e. $[\lambda]$, where $V_m = U_m \cap V$. Let $g_{mn} \in G$ with $\{\bar{U}_m g_{mn}\}$ scattered (see [14] Lemma 1.2) and define $f_0 \in \mathrm{CB}(G)$ by $f_0(x) = \sum\{f(m,n)(xg_{mn}^{-1}) : m,n \geq 1\}$ for all $x \in G$. Then $\{x \in G : f(m,n)(xg_{mn}^{-1}) \neq 0\} \subseteq V_m g_{mn}$ and since $\{V_m g_{mn}\}$ is scattered, $f_0 \in \mathrm{CB}(G)$. Let $m_1(k) = 2k$, $m_1^{(1)}(k) = 2(k-1)$ $(k = 1,2,\ldots)$ and $f_1(x) = \sum\big\{f\big(m_1(k),n\big)(xg_{m_1(k),n}^{-1}) : k,n \geq 1\big\}$ then $f_1 \in \mathrm{CB}(G)$. Let $m_2(k) = 2m_1^{(1)}(k)$ and $m_2^{(2)}(k) = 2m_1^{(1)}(k) - 1, \ldots, m_{i+1}(k) = 2m_i^{(i)}(k)$ and $m_{i+1}^{(i+1)} = 2m_i^{(i)} - 1$ $(i = 1,2,\ldots)$ and define $f_i(x) = \sum\big\{f\big(m_i(k),n\big)(xg_{m_i(k),n}^{-1}) : k,n \geq 1\big\}$. Then $f_i \in \mathrm{CB}(G)$ is topologically null and permanently near one for $i = 0,1,2,\ldots$ (see the proof of Proposition 1.5 of [14]). For any bounded sequence $\{a_n\}$, since $\big\{x \in G : f\big(m_i(k),n\big)(xg_{m_i(k),n}^{-1}) \neq 0\big\} \subseteq (V_{m_i(k)n})(g_{m_i(k),n}) \subseteq (\bar{U}_{m_i(k),n})(g_{m_i(k),n})$ and $\{\bar{U}_m g_{mn}\}$ is scatter, $a_1 f_1 + a_2 f_2 + \cdots \in \mathrm{CB}(G)$ and $\|a_1 f_1 + a_2 f_2 + \cdots\|_\infty = \sup\{|a_n|\}$. It is clear that $f_0 = f_1 + f_2 + \cdots$. ∎

By using this sequence of functions, we can embed $\mathcal{F}$ into

$$\mathrm{LIM}\big(\mathrm{CB}(G)\big) \sim \mathrm{TLIM}\big(\mathrm{CB}(G)\big)$$

as the following:

THEOREM 3.3. *Let G_1 be a noncompact σ-compact nondiscrete locally compact group and let G_2 be any amenable group. If G_1 is amenable as a discrete group and $G = G_1 \times G_2$, then there is a positive onto mapping $\pi\colon \mathrm{CB}(G) \longrightarrow \ell^\infty(\mathbb{N})$ such that $\pi^*\colon \ell^\infty(\mathbb{N})^* \longrightarrow \mathrm{CB}(G)^*$ is a linear isometry with $\pi^*\mathcal{F} \subseteq \mathrm{LIM}\big(\mathrm{CB}(G)\big)$. Furthermore, the elements of $\pi^*\beta\mathbb{N}$ are mutually singular and $\pi^*\theta$ is mutually singular to every elements of $\mathrm{TLIM}\big(\mathrm{CB}(G)\big)$ for any $\theta \in \mathcal{F}$.*

PROOF. Let $\{f_i\}$ be a sequence of continuous functions on G_1 with the properties in Proposition 3.2. For any $i \geq 1$, define $F_i \in \mathrm{CB}(G)$ by $F_i(x,y) = f_i(x)$ for $x \in G_1$ and $y \in G_2$. As in the proof of Theorem 3.8 in [8], each F_i is topologically null and there exists $m_i \in \mathrm{LIM}\big(\mathrm{CB}(G)\big)$ such that $m_i(F_i) = 1$. It is easy to see that $\{F_i\}$ also has the properties in Proposition 3.2. Define $\pi\colon \mathrm{CB}(G) \longrightarrow \ell^\infty(\mathbb{N})$ by $\pi(f)(n) = m_n(f)$ for $f \in \mathrm{CB}(G)$ and $n \in \mathbb{N}$. Then it is clear that π is positive, $\pi(1) = 1$ and $\|\pi\| = 1$. For any $\ell \in \ell^\infty(\mathbb{N})$, let $f = \ell(1)F_1 + \ell(2)F_2 + \cdots$. Then $f \in \mathrm{CB}(G)$ and $\pi(f)(n) = m_n(f) = m_n\big(\ell(n)F_n\big) = \ell(n)$, *i.e.* $\pi(f) = \ell$, and $\|\pi(f)\|_\infty = \sup_{n\geq 1}|\ell(n)| = \|\ell\|_\infty$. Hence π is onto and π^* is a linear isometry. It is clear that $\pi^*\mathcal{F} \subseteq \mathrm{LIM}\big(\mathrm{CB}(G)\big)$ (see [8] Theorem 2.5). Let $\theta \in \mathcal{F}$ and let $F = F_1 + F_2 + \cdots$. Since $\pi(F)(n) = m_n(F) = 1$ $(n \in \mathbb{N})$, $(\pi^*\theta)(F) = \theta(\pi F) = 1$. Since F is also topologically null, $\pi^*\theta$ is mutually singular to every elements of $\mathrm{TLIM}\big(\mathrm{CB}(G)\big)$. Let $\theta_1, \theta_2 \in \beta\mathbb{N}$ with $\theta_1 \neq \theta_2$. Then there exist nets $\{n_\alpha\}$ and $\{n_\beta\}$ in $\mathbb{N}$ such that $n_\alpha \longrightarrow \theta_1$ and $n_\beta \longrightarrow \theta_2$ in w^*-topology of $\ell^\infty(\mathbb{N})^*$. Since $\beta\mathbb{N}$ is a Hausdorff topological space, we can assume that $\{n_\alpha\} \cap \{n_\beta\} = \phi$. Let $F = \sum_\alpha F_{n_\alpha}$. Then by the structure of $\{F_i\}$ (see Proposition 3.2) $F \in \mathrm{CB}(G)$ with $0 \leq F \leq 1$ and $\pi^*\theta_1(F) = \theta_1(\pi F) = \lim_\alpha(\pi F)(n_\alpha) = \lim_\alpha m_{n_\alpha}(F) = 1$. Also, $\pi^*\theta_2(F) = \lim_\alpha(\pi F)(n_\beta) = \lim_\beta m_{n_\beta}(F) = 0$. Hence $\pi^*\theta_1$ and $\pi^*\theta_2$ are mutually singular. ∎

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Since $|\beta\mathbb{N}| = 2^c$, we have the following:

COROLLARY 3.4. *Assume G is as in Theorem 3.3. Then there are at least 2^c mutually singular elements of* $\mathrm{LIM}\big(\mathrm{CB}(G)\big)$ *all of which are singular to* $\mathrm{TLIM}\big(\mathrm{CB}(G)\big)$.

Let $\mathcal{C}(G)$ = closed span of $\{\varphi * f - f : \varphi \in P(G), f \in \mathrm{CB}(G)\}$ and let $H_c(G)$ = closed span of $\{{}_xf - f : x \in G \text{ and } f \in \mathrm{CB}(G)\}$.

COROLLARY 3.5. *Assume G is as in Theorem 3.3. Then the space $\mathcal{C}(G)/H_c(G)$ is not separable.*

REMARK 1. These improve Theorem 2.5 and Corollary 2.6 in [14] since we can take $G_2 = \{e\}$, the identity. If we choose an amenable group but not amenable as a discrete group as G_2, then G is not amenable as a discrete group. Hence the condition of discrete amenability in Theorem 2.5 and Corollary 2.6 of [14] is not necessary. This answers the problem asked by Rosenblatt in [14].

Let G_1 be noncompact nondiscrete and amenable as a discrete group. We can embed $\mathcal{F}$ into $\mathrm{LIM}\big(L^\infty(G_1)\big) \sim \mathrm{TLIM}\big(L^\infty(G_1)\big)$ as in Miao [8]. For a locally compact group $G = G_1 \times G_2$, we also have this result, where G_2 is any amenable group.

THEOREM 3.6. *Let $G = G_1 \times G_2$ be as above, then there exists a linear isometry $\pi^*: \ell^\infty(\mathbb{N})^* \longrightarrow L^\infty(G)^*$ with $\pi^*\mathcal{F} \subseteq \mathrm{LIM}\big(L^\infty(G)\big)$. Moreover, $\pi^*\theta$ is mutually singular to every element of* $\mathrm{TLIM}\big(L^\infty(G)\big)$ *and elements of $\pi^*\beta\mathbb{N}$ are mutually singular.*

PROOF. Take disjoint s. p. sets E_n $(n = 1, 2, \ldots)$ in G_1 such that for any $\varepsilon > 0$, there is an $\varphi_1 \in P(G_1)$ with $\|\varphi_1 * 1_{\cup E_n}\|_\infty < \varepsilon$ (see [8] Lemma 2.4). Since G_1 is amenable as a discrete group, there exist $m_n \in \mathrm{LIM}\big(L^\infty(G_1)\big)$ with $m_n(1_{E_n}) = 1$, $(n = 1, 2, \ldots)$. Note that G_2 is a closed normal subgroup of G and $G_1 = G/G_2$. There exists a linear mapping $f \longrightarrow \dot{f}$ from $L^\infty(G)$ onto $L^\infty(G/G_2)$ with the properties

(i) $\dot{1}_{E_n \times G_2} = 1_{E_n}$ and if $f \geq 0$ then $\dot{f} \geq 0$.

(ii) $({}_xf)^{\cdot} = {}_{\pi_{G_2}(x)}\dot{f}$, where π_{G_2} is the natural projection of G onto G/G_2 (see [11] Chapter 8, §5.5(ii)).

Let $M_n \in \mathrm{LIM}\,\big(L^\infty(G)\big)$ be defined by $M_n(f) = m_n(\dot{f})$. Then $M_n(1_{E_n \times G_2}) = m_n(1_{E_n}) = 1$. If $\varepsilon > 0$ is given, then there exists an $\varphi_1 \in P(G_1)$ such that $\|\varphi_1 * 1_{\cup E_n}\|_\infty < \varepsilon$. Let $\varphi_2 \in P(G_2)$ and $\varphi(x, y) = \varphi_1(x)\varphi_2(y)$, then $\varphi \in P(G)$. $\|\varphi * 1_{\cup E_n \times G_2}\|_\infty = \|\varphi_1 \times 1_{\cup E_n}\|_\infty < \varepsilon$. Hence $1_{\cup E_n \times G_2}$ is topologically null. For $f \in L^\infty(G)$, $n \in \mathbb{N}$, let $\pi(f)(n) = M_n(f)$. Then $\pi: L^\infty(G) \longrightarrow \ell^\infty(\mathbb{N})$ and $\pi^*: \ell^{\infty *}(\mathbb{N}) \longrightarrow L^\infty(G)^*$ is the mapping we want (see [8] Theorem 2.5). ∎

REMARK. This improves Theorem 2.5 of [8] and shows that the discrete amenability is not necessary in that theorem.

4. Extension of an invariant mean on UCB(G).

In this section, we will discuss the extension of any left invariant mean on $\mathrm{UCB}(G)$ to a left invariant functional on an admissible subspace S containing $\mathrm{UCB}(G)$ which is not "topologically left invariant".

PROPOSITION 4.1. *Assume that G is amenable as a discrete group and S is an admissible subspace of $L^\infty(G)$. If $f_0 \notin S$, $f_0 \in L^\infty(G)$ with $0 \le f_0 \le 1$, then any $m \in \mathrm{LIM}(S)$ has an extension to a left invariant mean on $L^\infty(G)$ with $m(f_0) > 0$ if and only if $\ell > 0$, where*

$$\ell = \inf\Big\{m(f) : f \ge \sum_{i=1}^{n} \lambda_{i x_i} f_0, f \in S, x_i \in G \text{ and } \lambda_i > 0 (i = 1, 2, \ldots, n) \text{ with } \sum_{i=1}^{n} \lambda_i = 1\Big\}.$$

PROOF. Let $m \in \mathrm{LIM}\big(L^\infty(G)\big)$ be an extension of m on S with $m(f_0) > 0$. For any $f \in S$, $x_i \in G$ and $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and

$$f \ge \sum_{i=1}^{n} \lambda_{i x_i} f, \quad m(f) \ge m\Big(\sum \lambda_{i x_i} f_0\Big) = m(f_0) > 0.$$

Hence $\ell \ge m(f_0) > 0$.

Conversely, let $\ell > 0$. Since G is amenable as a discrete group, at first we extend m to a left invariant mean on $S + H$. For any $g \in L^\infty(G)$, let

$$P(g) = \inf\{m(f) : f \ge g, f \in S + H\}.$$

Then P is subadditive, $P(\alpha g) = \alpha P(g)$ for $g \in S + H$ and $\alpha \ge 0$. Since m is a mean on $S + H$, $|m(g)| \le p(g)$ $(g \in S + H)$. By the Hahn-Banach extension theorem (see [18] p. 102), m can be extended to S_{f_0} such that $|m(g)| \le P(g)$ for any $g \in S_{f_0}$ and $m(f_0) = c$, where S_{f_0} is the smallest subspace of $L^\infty(G)$ containing $S + H$ and f_0 and c is any number between $\sup_{g \in S+H}[m(g) - P(g - f_0)]$ and $\inf_{g \in S+H}[P(g + f_0) - m(g)]$. We claim that $\inf_{g \in S+H}[P(g + f_0) - m(g)] \ge \ell$. Indeed, let $g \in S + H$. $P(g + f_0) - m(g) = \inf\{m(f) : f \ge g + f_0, f \in S + H\} - m(g) = \inf\{m(f - g) : f - g \ge f_0, f \in S + H\} = P(f_0)$. We only need to show that $P(f_0) \ge \ell$. For any $f \in S + H$ with $f \ge f_0$, there exist $f_1 \in S$ and $h \in H$ such that $f = f_1 + h$. Let $\varepsilon > 0$, since h left averages to 0 (see [7]), there exist $x_i \in G$ and $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $\|\sum \lambda_{i x_i} h\|_\infty < \varepsilon$. Hence $\varepsilon + \sum \lambda_{i x_i} f_1 \ge \sum \lambda_{i x_i} f_0$ and $m(\varepsilon + \sum \lambda_{i x_i} f_1) = \varepsilon + m(f_1) = \varepsilon + m(f)$. But $\varepsilon + \sum \lambda_{i x_i} f_1 \in S$, so $P(f_0) = \inf\{m(f) : f \ge f_0, f \in S + H\} \ge \ell - \varepsilon$. Since ε is arbitrary, $P(f_0) \ge \ell$. Take $c = \inf_{g \in S+H}\{p(g + f_0) - m(g)\}$. Then $c \ge \ell > 0$ and $m(f_0) = c$. Since $|m(g)| \le P(g) \le \|g\|_\infty$ for any $g \in S_{f_0}$, $m \in \mathrm{LIM}(S_{f_0})$. We can extend m to a left invariant mean on $L^\infty(G)$ by the discrete amenability. ∎

Let $m \in \mathrm{LIM}\big(\mathrm{UCB}(G)\big)$ and let $f \in \mathrm{CB}(G)$ be a topologically null function.

If we can extend m to a left invariant mean on $\mathrm{CB}(G)$ such that $m(f) \ne 0$, then this extension is not topologically left invariant.

PROPOSITION 4.2. *Let G be a locally compact group and let $f \in \mathrm{CB}(G)$ with $0 \le f \le 1$ be a permanently near one function. If f is topologically null and G_d is amenable, then*

$$d\big(f, \mathrm{UCB}(G) + H\big) = \inf\{\|f - u - h\|_\infty : u \in \mathrm{UCB}(G), h \in H\} = \frac{1}{2}.$$

https://doi.org/10.4153/CJM-1994-046-x Published online by Cambridge University Press

PROOF. Suppose $d\big(f, \mathrm{UCB}(G) + H\big) < \frac{1}{2}$. Then there are $u \in \mathrm{UCB}(G)$ and $h \in H$ such that $\|f - u - h\|_\infty < \frac{1}{2}$. By Theorem 1.6 of [14], there is an $m \in \mathrm{LIM}\big(\mathrm{CB}(G)\big)$ with $m(f) = 1$. Therefore

$$m(u) = m(u - f - h) + m(f) \geq 1 - \|u - f - h\|_\infty > \frac{1}{2}.$$

By Lemma 2.2 of [4], (p. 27), $\mathrm{LIM}\big(\mathrm{UCB}(G)\big) = \mathrm{TLIM}\big(\mathrm{UCB}(G)\big)$. Since G_d is amenable, we can extend m to an element of $\mathrm{LIM}\big(L^\infty(G)\big)$. Note that $m = 0$ on H and $\mathrm{LIM}\big(\mathrm{UCB}(G) + H\big) = \mathrm{TLIM}\big(\mathrm{UCB}(G) + H\big)$. Hence

$$\begin{aligned} m(u - h) &= m\big(\varphi * (u - h)\big) \\ &= m\big(\varphi * (u - h - f) + m(\varphi * f)\big) \\ &\leq \|u - h - f\|_\infty + m(\varphi * f) \\ &< \frac{1}{2} + m(\varphi * f). \end{aligned}$$

Since f is topologically null, $m(\varphi * f) = 0$ (see [4], p. 27). Therefore, $m(u) = m(u - h) < \frac{1}{2}$ which is impossible. Thus, $d\big(f, \mathrm{UCB}(G) + H\big) \geq \frac{1}{2}$. Let $u = \frac{1}{2}$, then $\|f - \frac{1}{2}\|_\infty = \frac{1}{2}$. So $d(f, \mathrm{UCB}(G) + H) = \frac{1}{2}$. ∎

Rosenblatt showed in [14] (also see Proposition 3.3) that there always exists an $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ which is topologically null and permanently near one.

THEOREM 4.3. *Let G be nondiscrete noncompact locally compact group. If G_d is amenable and $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ is topologically null and permanently near one, then any $m \in \mathrm{LIM}\big(\mathrm{UCB}(G)\big)$ can be extended to a left invariant functional on $\mathrm{CB}(G)$ such that $m(f) = 1$ and $\|m\| \leq 3$.*

REMARK. Recall that a mean is a functional m on $\mathrm{CB}(G)$ with $m(1) = 1$ and $\|m\| = 1$. We do not know whether the norm of the extension above can be 1.

PROOF OF THEOREM 4.3. Since G_d is amenable, by Day's fixed point theorem, at first we can extend m to a left invariant mean on $\mathrm{UCB}_r(G)$ such that $\|m\| = 1$. Then we take an $\varphi \in P(G)$. For any $g \in \mathrm{CB}(G)$, define $m(g) = m(\varphi * g)$. Then it is well-defined since $\varphi * g \in \mathrm{UCB}_r(G)$ and also $m \in \mathrm{TLIM}\big(\mathrm{CB}(G)\big)$ (see [4], p. 24). Thus $m(f) = 0$. By Proposition 4.2 and Hahn-Banach theorem, there exists an $m_f \in \mathrm{CB}(G)^*$ such that $m_f(f) = 1$, $m_f = 0$ on $\mathrm{UCB}(G) + H$ and $\|m_f\| = 2$. Since $m_f = 0$ on H, m_f is left invariant. Also $m_f = 0$ on $\mathrm{UCB}(G)$, so $m + m_f$ is an extension of m and $\|m + m_f\| \leq \|m\| + \|m_f\| = 3$. ∎

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