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Patterson–Sullivan theory for groups with a strongly contracting element

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Abstract. Using Patterson–Sullivan measures, we investigate growth problems for groups acting on a metric space with a strongly contracting element.

Key words: Patterson–Sullivan measures, critical exponent, growth rate, contracting element, amenability
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Contents

1. *Introduction*

Let G be a group acting properly by isometries on a proper geodesic space (X, d). In particular, G is countable. It will always be endowed with the discrete topology. Its *exponential growth rate* measures the size of its orbits and is defined as

$$\omega(G, X) = \limsup_{\ell\to\infty} \frac{1}{\ell} \ln |\{g \in G \ : \ d(o, go) \leq \ell\}|.$$

This number does depend on the space X. Nevertheless, if the context is clear, we simply write ω_G instead of $\omega(G, X)$. It is also the critical exponent of the *Poincaré series* of G defined by

$$\mathcal{P}_G(s) = \sum_{g\in G} e^{-sd(o,go)},$$

that is, $\mathcal{P}_G(s)$ diverges (respectively converges) whenever $s < \omega_G$ (respectively $s > \omega_G$). If G is the fundamental group of a hyperbolic manifold M acting on the universal cover $X = \tilde{M}$, then ω_G has numerous interpretations: it is the entropy of the geodesic flow, the Hausdorff dimension of the radial limit set of G, etc. In this context, the exponential growth rate is a central object connecting geometry, group theory, dynamical systems, etc.

1.1. *Growth spectrum.*

In this article, we are interested in the (*normal*) *subgroup growth spectrum* of G, that is, the set

$$\text{Spec}(G, X) = \{\omega(N, X) \ : \ N \lhd G\}.$$

Note that $\text{Spec}(G, X)$ is contained in $[0\,, \omega_G]$. In particular, $\omega_N = 0$ (respectively $\omega_N = \omega_G$) if N is finite (respectively has finite index in G). A natural question, which has received much attention, is to understand more precisely the extremal values of this set. This problem is rather well understood if G is a group acting properly, co-compactly by isometries on a Gromov hyperbolic metric space X. For instance, any infinite normal subgroup $N \lhd G$ satisfies

$$\tfrac{1}{2}\omega(G, X) < \omega(N, X) \leq \omega(G, X),$$

see [36]. Moreover, the second inequality is an equality if and only if G/N is amenable. Similarly, if G has Kazhdan property (T), then ω_N cannot be arbitrarily close to ω_G, unless N has finite index in G. See [18] and the references therein.

1.2. *Contracting elements.* In the past decades, there have been many efforts to investigate the marks of negative curvature in groups, beyond the context of hyperbolic spaces. One notion which has emerged is the one of contracting element. Roughly speaking, a subset $Y \subset X$ is contracting if any ball disjoint from Y has a projection onto Y, whose diameter is uniformly bounded [6].

Remark. In the literature, this property is sometimes called *strong* contraction to distinguish it from a weaker version involving a certain system of non-geodesic paths, see for instance [1]. Since we will work with this single notion, we simply call it contraction.

An element $g \in G$ is *contracting* if the orbit map $\mathbb{Z} \to G$ sending n to $g^n x$ is a quasi-isometric embedding with contracting image. Contracting elements can be thought of as hyperbolic directions in the space X. Here are a few examples of group actions with contracting elements.

- If X is a hyperbolic space (in the sense of Gromov) endowed with a proper action of G, then every loxodromic element in G is contracting [26]. This is typically the case if G is the fundamental group of a manifold M whose sectional curvature is negative and bounded away from zero, and X the universal cover of M. Any metric quasi-isometric to this one will also work.
- Assume that G is hyperbolic relative to $\{P_1, \dots, P_m\}$. Suppose that G acts properly, co-compactly on X (e.g. X is the Cayley graph of G with respect to a finite generating set of G). Any infinite order element of G which is not conjugated in some P_i is contracting [23, 46].
- If X is a CAT(0) space endowed with a proper, co-compact action of G, then any rank one element of G is contracting [6]. Recall that the universal cover of any closed, compact manifold with non-positive sectional curvature is CAT(0) [7].
- Let Σ be a closed, compact surface. Assume that G is the mapping class group of Σ and X the Teichmüller space of Σ endowed with the Teichmüller metric. Then any pseudo-Anosov element is contracting [37].

Groups with a contracting element are known to be acylindrically hyperbolic, see Sisto [47]. Acylindrical hyperbolicity is a powerful tool for studying the structure of a given group. Nevertheless, it is rather useless here as the exponential growth rate $\omega(G, X)$ heavily depends on the metric space X. The typical spaces X we are interested in are indeed not hyperbolic.

1.3. *Main results.* The goal of this article is to investigate the extremal values of the subgroup growth spectrum in the context of group actions admitting a contracting element. Some of our results refine existing statements in the literature. In particular, we answer most of the questions raised by Arzhantseva and Cashen in [2]. Our main

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contribution though is the method that we use: we extend to this context the construction of Patterson–Sullivan measures (see below).

When it comes to counting problems, the behavior of the Poincaré series of G at the critical exponent plays a major role. This motivates the following definition. The action of G on X is *divergent* (respectively *convergent*) if the Poincaré series $\mathcal{P}_G(s)$ of G diverges (respectively converges) at $s = \omega_G$. Our first statement deals with the bottom of the subgroup growth spectrum.

THEOREM 1.1. (See Corollary 4.29 and Proposition 5.23) *Let X be a proper geodesic metric space. Let G be a group acting properly, by isometries on X with a contracting element. Let N be an infinite normal subgroup of G. Then*

$$\omega(N, X) + \tfrac{1}{2}\omega(G/N, X/N) \geq \omega(G, X).$$

Assume in addition that G is not virtually cyclic and the action of G is divergent. Then

$$\omega(N, X) > \tfrac{1}{2}\omega(G, X).$$

Remark. The first inequality was proved by Matsuzaki and Jaerisch when G is a finitely generated free group acting on its Cayley graph with respect to a free basis [**28**]. Their method involves fine estimates of the Cheeger constant and the spectral radius of the random walk in G/N. To the best of our knowledge, this result is new, even if G is a hyperbolic group.

The second inequality is well known in the context of hyperbolic spaces, see [**42**] and [**36**]. For groups acting with a contracting element, it was proved by Arzhantseva and Cashen under the stronger assumption that G has pure exponential growth, that is, when the map

$$\ell \mapsto |\{g \in G \ : \ d(o, go) \leq \ell\}| e^{-\omega_G \ell}$$

is bounded from above and away from zero [**2**]. Note that even if X is Gromov hyperbolic, there are groups G acting on X, which are divergent but do not have pure exponential growth.

The next two results focus on the top of the subgroup growth spectrum. Let Q be a discrete group. The left action of Q on itself induces an action of Q on $\ell^\infty(Q)$. The group Q is *amenable* if there exists a Q-invariant mean $\ell^\infty(Q) \to \mathbb{R}$.

THEOREM 1.2. (See Corollary 4.27) *Let X be a proper, geodesic, metric space. Let G be a group acting properly, by isometries on X with a contracting element. Let N be a normal subgroup of G. If G/N is amenable, then $\omega(N, X) = \omega(G, X)$.*

Remark. This type of result has a long history. Assume that G is the fundamental group of a compact hyperbolic manifold and X the universal cover of M. Let N be a normal subgroup of G. Brooks proved that $\omega_N = \omega_G$ if and only if G/N is amenable [**8**]—Brooks' result is actually stated in terms of the bottom spectra of certain Laplace operators, but they can be related to the growth rates of the groups via Sullivan's formula [**48**]. A similar statement was obtained independently by Grigorchuk and Cohen when G is a free group acting on

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its Cayley graph X with respect to a free basis [12, 25]. The 'easy direction' stated above was generalized by Roblin to the settings of CAT(-1) spaces [42].

Recall that two subgroups H_1 and H_2 of G are *commensurable* if $H_1 \cap H_2$ has finite index in both H_1 and H_2. A subgroup $H \subset G$ is *commensurated* if H and gHg^{-1} are commensurable, for every $g \in G$. The class of commensurated subgroups contains all normal subgroups and finite index subgroups of G. More generally, any subgroup of G that is commensurable with a normal subgroup of G is commensurated. However, there are numerous other examples, see Remark 5.26.

THEOREM 1.3. (See Theorem 5.25) *Let X be a proper, geodesic, metric space. Let G be a group acting properly, by isometries on X with a contracting element. Let H be a commensurated subgroup of G. If the action of H on X is divergent, then $\omega(H, X) = \omega(G, X)$ and the action of G on X is divergent.*

Remark. To the best of our knowledge, the statements in the literature only cover the case where H is normal. With this stronger assumption, it was proved by Matsuzaki and Yabuki if G is a Kleinian group, and generalized by Matsuzaki, Yabuki, and Jaerisch when X is Gromov hyperbolic [35, 36].

1.4. *Patterson–Sullivan theory.* Assume that G is the fundamental group of a closed Riemannian manifold M with negative sectional curvature. In this context dynamical systems—first and foremost, the study of the geodesic flow on the unit tangent bundle of M—provide efficient tools to tackle counting problems. For instance, using the dynamics of the geodesic flow, Margulis proved that the number $c(\ell)$ of simple closed geodesics on M of length at most ℓ behaves like

$$c(\ell) \underset{\ell \to \infty}{\sim} \frac{e^{\omega_G \ell}}{\omega_G \ell}.$$

See [34]. Fix a base point $o \in X$. Denote by $X = \tilde{M}$ the universal cover of M and ∂X its visual boundary. In this topic, measures on the boundary play a prominent role. Recall that a G-invariant, ω_G-conformal density is a collection $\nu = (\nu_x)_{x \in X}$ of non-zero finite measures on $X \cup \partial X$, all in the same measure class, satisfying the following properties: for all $g \in G$, for all $x, y \in X$, we have:

- $g_* \nu_x = \nu_{gx}$ (invariance);
- $d\nu_x / d\nu_y(\xi) = e^{-\omega_G b_\xi(x,y)}$ ν-almost everywhere (a.e.) (conformality),

where b_ξ stands for the Buseman cocycle at $\xi \in \partial X$. In particular, ω_G can be interpreted as the dimension of the measure ν_o. Patterson's construction provides examples of such densities which are supported on ∂X. These measures are designed so that the action of G on $(\partial X, \nu_o)$ captures many properties of the geodesic flow on M. The theory can be generalized for groups acting on a Gromov hyperbolic space, see for instance [13] and [4]. For such groups, Theorems 1.1, 1.2, and 1.3 can be proved using invariant conformal densities.

In the past years, growth problems in groups with a contracting element have been investigated by various people, see for instance [1, 2, 20, 32, 49–51]. Since no

Patterson–Sullivan theory existed in this context, each time the authors developed *ad hoc* methods. Actually, they often make a point of avoiding 'fairly sophisticated' results about Patterson–Sullivan 'machinery'. We adopt here an opposite point of view. For us, these results witnessed the fact that a Patterson–Sullivan theory should exist. Building this 'missing' theory is the purpose of this work. If the ambient space X is CAT(0), this task has been achieved by Link [33] extending the work of Knieper [29, 30]. Our approach does not require any CAT(0) assumption though. Our goal is to stress that this construction is particularly robust and requires very little hypotheses, beside the existence of a contracting element. Once the basic properties of invariant conformal densities have been established, they provide a unified framework for solving various growth problems. We believe that these tools can be used for many other applications inspired by non-positive curvature.

1.5. *Strategy.* We would like to understand the behavior of certain densities supported on the 'boundary at infinity' of X. Thus, the first task is to build an appropriate compactification of X to carry these measures. There have been many attempts to build an analogue of the Gromov boundary for groups with a contracting element: the contracting and Morse boundaries [11, 15, 38], the sublinearly Morse boundary [40, 41], etc. However these boundaries are sometimes 'too small' (for instance, the Morse boundary cannot be used as a topological model of the Poisson boundary) and often not compact. This can be a difficulty to build Patterson–Sullivan measures. Instead, we choose to work with the horocompactification. In short, it is the 'smallest' compactification $\bar{X}$ of X such that the map

$$\begin{array}{rcl} X \times X \times X & \to & \mathbb{R} \\ (x, y, z) & \mapsto & d(x, z) - d(y, z) \end{array}$$

extends continuously to a map $X \times X \times \bar{X} \to \mathbb{R}$. The horoboundary of X is $\partial X = \bar{X} \setminus X$. A point in the horoboundary is a cocycle $c\colon X \times X \to \mathbb{R}$, playing the role of a Buseman cocycle. Hence, this choice is natural to give a rigorous sense to conformal densities. If X is CAT(0), then the horoboundary coincides with the visual boundary. In general, this boundary is slightly too large though for invariant conformal densities to behave as expected. Let us illustrate this fact with an example.

Example. Consider a group G acting properly, co-compactly, by isometries on a CAT(0) space X_0. Build a new space $X = X_0 \times [0\,,1]$ endowed with the L^1-metric. Let G act trivially on $[0\,,1]$ and consider the diagonal action of G on X. This action is still proper and co-compact and $\omega(G, X) = \omega(G, X_0)$. The horoboundary of X is homeomorphic to $\partial X = \partial X_0 \times [0\,,1]$. To carry the analogy with negatively curved manifold, we would like that if $\mu = (\mu_x)$ is a G-invariant, ω_G-conformal density supported on ∂X, then the action of G on $(\partial X, \mu_o)$ is ergodic. However, in this example, we can choose a G-invariant, ω_G-conformal density $\nu = (\nu_x)$ on ∂X_0 and form the average $\mu = (\nu^0 + \nu^1)/2$, where ν^i is a copy of ν supported on $\partial X_0 \times \{i\}$. Then the action of G on $(\partial X, \mu_o)$ is not ergodic.

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This issue already arises if X is Gromov hyperbolic. In this context, it can be fixed by passing to the reduced horoboundary. Endow the horoboundary ∂X with the equivalence relation $\sim$ defined as follows: two cocycles $c, c' \in \partial X$ are equivalent if $\|c - c'\|_\infty < \infty$. The reduced horoboundary is the quotient $\partial X/\sim$. If X is hyperbolic, then it coincides with the Gromov boundary. Moreover, the projection $\pi : \partial X \twoheadrightarrow \partial X/\sim$ is very well understood, see [**14**]. Pushing forward in $\partial X/\sim$, the densities built in ∂X provides well-behaved measures.

However, in general, the reduced horoboundary $\partial X/\sim$ is a rather nasty topological space. For instance, if $X = \mathbb{R}^2$ is endowed with the taxicab metric, then $\partial X/\sim$ is not even Hausdorff. To bypass this difficulty, we adopt a measure theoretic point of view. Denote by $\mathfrak{R}$ the σ-algebra that consists of all Borel sets which are saturated for the equivalence relation $\sim$. We make an abuse of vocabulary and call the measurable space $(\partial X, \mathfrak{R})$ the *reduced horoboundary*. When restricted to the reduced horoboundary, the invariant, conformal densities are well behaved. For instance, we prove the following partial form of the Hopf–Tsuji–Sullivan dichotomy (we refer the reader to §4.6 for the definition of the radial limit set).

THEOREM 1.4. (See Corollaries 4.25 and 5.19) *Let X be a proper geodesic metric space and $o \in X$. Let G be a group acting properly, by isometries on X with a contracting element. Suppose that G is not virtually cyclic. Let $\omega \in \mathbb{R}_+$. Let $\mu = (\mu_x)$ be the restriction to the reduced horoboundary $(\partial X, \mathfrak{R})$ of a G-invariant, ω-conformal density. The following are equivalent.*

- (i) *The action of G on X is divergent (and thus $\omega = \omega_G$).*
- (ii) *μ_o gives positive measure to the radial limit set.*
- (iii) *μ_o gives full measure to the radial limit set.*

Remark. In a forthcoming work, see [**17**], we plan to complete the Hopf–Tsuji–Sullivan dichotomy by investigating the ergodicity of the geodesic flow in this context and its consequences for growth problems.

If the action of G on X is divergent, we prove that invariant, conformal densities are ergodic and essentially unique, when restricted to the reduced horoboundary.

THEOREM 1.5. (See Proposition 5.22) *Let X be a proper, geodesic, metric space and $o \in X$. Let G be a non-virtually cyclic group acting properly, by isometries on X with a contracting element. Assume that the action of G on X is divergent. Let $\mu = (\mu_x)$ be the restriction to the reduced horoboundary $(\partial X, \mathfrak{R})$ of a G-invariant, ω_G-conformal density. Then:*

- (i) *μ_o is ergodic;*
- (ii) *μ_o is non-atomic;*
- (iii) *μ is almost unique in the following sense: there is $C \in \mathbb{R}_+^*$, such that if $\mu' = (\mu'_x)$ is the restriction to the reduced horoboundary of another G-invariant, ω_G-conformal density, then for every $x \in X$, we have $\mu'_x \leq C\mu_x$.*

Finally, we complete Theorem 1.3 as follows.

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THEOREM 1.6. (See Theorem 5.25) *Let X be a proper, geodesic, metric space. Let G be a group acting properly, by isometries on X with a contracting element. Suppose that G is not virtually cyclic. Let H be a commensurated subgroup of G. If the action of H on X is divergent, then any H-invariant, ω_H-conformal density is G-almost invariant when restricted to the reduced horoboundary* $(\partial X, \mathfrak{R})$.

Remark. Other applications can be found in §§4.6 and 5.5. In this article, we focused on growth problems. Nevertheless, we believe that the tools we introduced can be used for other purposes, e.g. to generalize the 'no proper conjugation' property of divergent subgroups exhibited by Matsuzaki, Yabuki, and Jaerisch [36].

1.6. *Strongly positively recurrent actions.* As we mentioned before, divergent actions play an important role in counting problems. Any proper and co-compact action is divergent. In particular, if G acts properly on X with a quasi-convex orbit, then its action is divergent. This framework has been generalized independently by Schapira and Tapie [44] and Yang [50] under the names *strongly positively recurrent* action (SPR) and *statistically convex co-compact* action (SCC), respectively—the idea also implicitly appears in the work of Arzhantseva, Cashen, and Tao [1]. The notion has an independent dynamical origin as well, see for instance [27, 43]. Roughly speaking, the idea is to ask that the elements of $g \in G$ which 'violate' the quasi-convexity of G are statistically very rare. It was proved by Yang that such actions are divergent. In Appendix A, we provide an alternative proof of this fact in the spirit of Schapira and Tapie [44].

Remark. Since obtaining the results in this article, we have learned that Wenyuan Yang independently investigated conformal measures in the same context [52]. The techniques used by Wenyuan Yang are slightly different. For instance, his proof of Theorem 1.5 (partial form of the Hopf–Tsuji–Sullivan dichotomy) relies on projection complexes introduced by Bestvina, Bromberg, and Fujiwara [5]. In contrast, we tried to use, whenever possible, low-tech arguments (both of geometric and measure theoretic nature).

2. *Groups with a contracting element*

2.1. *Notation and vocabulary.* In this article, (X, d) is a proper, metric space. A *geodesic* is a path $\gamma : I \to X$ (where $I \subset \mathbb{R}$ is an interval) such that

$$d(\gamma(t), \gamma(t')) = |t' - t| \quad \text{for all } t, t' \in I.$$

From now on, we assume that X is *geodesic*, that is, any two points are joined by a (non-necessarily unique) geodesic. Note that we do require X to be geodesically complete.

For every $x \in X$ and $r \in \mathbb{R}_+$, we denote by $B(x, r)$ the open ball of radius r centered at x. Let Y be a closed subset of X. Given $x \in X$, a point $y \in Y$ is a (*nearest point*) *projection* of x on Y if $d(x, y) = d(x, Y)$. The *projection* of a subset $Z \subset X$ onto Y is

$$\pi_Y(Z) = \{y \in Y \ : \ y \text{ is the projection of a point } z \in Z\}.$$

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Let $I \subset \mathbb{R}$ be a closed interval and $\gamma : I \to X$ a continuous path intersecting Y. The *entry point* and *exit point* of γ in Y are the points $\gamma(t)$ and $\gamma(t')$, where

$$t = \inf\{s \in I \ : \ \gamma(s) \in Y\} \quad \text{and} \quad t' = \sup\{s \in I \ : \ \gamma(s) \in Y\}.$$

If I is bounded, such points always exist (the subset Y is closed). Given $d \in \mathbb{R}_+$, we denote by $\mathcal{N}_d(Y)$ the *d-neighborhood* of Y, that is, the set of points $x \in X$ such that $d(x, Y) \leq d$. The distance between two subsets Y, Y' of X is

$$d(Y, Y') = \inf_{(y,y') \in Y \times Y'} d(y, y').$$

2.2. *Contracting set*

Definition 2.1. (Contracting set) Let $\alpha \in \mathbb{R}_+^*$. A closed subset $Y \subset X$ is *α-contracting* if for any geodesic γ with $d(\gamma, Y) \geq \alpha$, we have $\operatorname{diam}(\pi_Y(\gamma)) \leq \alpha$. The set Y is *contracting* if Y is α-contracting for some $\alpha \in \mathbb{R}_+^*$.

The next statements are direct consequences of the definition. Their proofs are left to the reader.

LEMMA 2.2. (Quasi-convexity) *Let Y be an α-contracting subset. If γ is a geodesic joining two points of $\mathcal{N}_\alpha(Y)$, then γ lies in the $5\alpha/2$-neighborhood of Y.*

LEMMA 2.3. (Projections) *Let Y be an α-contracting subset. Let $x, y \in X$ and γ be a geodesic from x to y. Let p and q be respective projections of x and y onto Y. If $d(x, Y) < \alpha$ or $d(p, q) > \alpha$, then the following hold:*

- (i) $d(\gamma, Y) < \alpha$;
- (ii) *the entry point (respectively exit point) of γ in $\mathcal{N}_\alpha(Y)$ is 2α-closed to p (respectively q);*
- (iii) $d(x, y) \geq d(x, p) + d(p, q) + d(q, y) - 8\alpha$.

Remark 2.4. It follows from the above statement that the nearest point projection onto Y is large-scale 1-Lipschitz. Actually, refining the above argument, one can prove that for every subset $Z \subset X$, we have

$$\operatorname{diam}(\pi_Y(Z)) \leq \operatorname{diam}(Z) + 4\alpha.$$

LEMMA 2.5. *For every $\alpha, d \in \mathbb{R}_+^*$, there exists $\beta \in \mathbb{R}_+^*$ with the following property. Let Y and Z be two closed subsets of X. Assume that the Hausdorff distance between them is at most d. If Y is α-contracting, then Z is β-contracting.*

2.3. *Contracting element.*

Consider now a group G acting properly, by isometries on X.

Definition 2.6. (Contracting element) Let $y \in X$. An element $g \in G$ is *contracting*, for its action on X, if the orbit map $\mathbb{Z} \to G$ sending n to $g^n y$ is a quasi-isometric embedding with contracting image.

Note that the definition does not depend on the point y (see Lemma 2.5). The next statement is a reformulation of [51, Lemma 3.3 (and its proof)].

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LEMMA 2.7. *Let $g \in G$ be a contracting element. For every $d \in \mathbb{R}_+$ and $z \in X$, there is $\alpha \in \mathbb{R}_+^*$ with the following property. Let $p, q \in \mathbb{Z}$ with $p \leq q$. Let $x, y \in X$ such that $d(x, g^p z) \leq d$ and $d(y, g^q z) \leq d$. Let γ be a geodesic joining x to y. Then any subpath of γ is α-contracting. Moreover, for every integer $n \in [\![p, q]\!]$, the point $g^n z$ is α-close to γ.*

Let $g \in G$ be a contracting element and A be the $\langle g \rangle$-orbit of a point $y \in X$. Define $E(g)$ as the set of elements $u \in G$ such that the Hausdorff distance between A and uA is finite. It follows from the definition that $E(g)$ is a subgroup of G that does not depend on y. It is the maximal virtually cyclic subgroup of G containing $\langle g \rangle$. Moreover, $E(g)$ is almost-malnormal, that is, $uE(g)u^{-1} \cap E(g)$ is finite for every $u \in G \setminus E(g)$, see [50, Lemma 2.11].

LEMMA 2.8. *Assume that G is not virtually cyclic and contains a contracting element. Let $H \subset G$ be a commensurated subgroup. If H is infinite, then H is not virtually cyclic and contains a contracting element.*

Proof. We only sketch the proof. For more details, we refer the reader to [1, §3] where similar arguments are given. We first claim that for every contracting element $g \in G$, the group H is not contained in $E(g)$. Assume on the contrary that $H \subset E(g)$. Since G is not virtually cyclic, there is $u \in G \setminus E(g)$. By malnormality, $uHu^{-1} \cap H$ is finite and cannot have finite index in H, which contradicts the fact that H is commensurated.

We now fix once and for all a contracting element $g \in G$. We denote by A an orbit of $\langle g \rangle$. It is α-contracting for some $\alpha \in \mathbb{R}_+^*$. Moreover, there is $C \in \mathbb{R}_+$ such that for every $u \in G \setminus E(g)$, we have $\operatorname{diam}(\pi_A(uA)) \leq C$, see [50, Lemma 2.11]. We choose $n \in \mathbb{N}$ such that $d(o, g^n o)$ is very large compared to both α and C.

We claim that there is an element $h \in H \setminus E(g)$ such that both $g^n h g^{-n}$ and $g^{-n} h g^n$ belong to H. Since H is commensurated, both $g^{-n} H g^n \cap H$ and $g^n H g^{-n} \cap H$ have finite index in H. Thus,

$$H_0 = (g^{-n} H g^n) \cap (g^n H g^{-n}) \cap H$$

has finite index in H. It follows that H_0 is not contained in $E(g)$. Indeed, otherwise, $E(g)$ should also contain H, contradicting our previous claim. Any element $h \in H_0 \setminus E(g)$ satisfies the conclusions of our second claim.

Consider now the element

$$f = (g^n h g^{-n})(g^{-n} h g^n)^{-1} = g^n (h g^{-2n} h^{-1}) g^n.$$

Note that f belongs to H. We claim that f is contracting. For simplicity, we let $g_0 = g^n$ and $g_1 = h g^{-2n} h^{-1}$ so that $f = g_0 g_1 g_0$. They respectively act 'by translation' on the α-contracting sets $A_0 = A$ and $A_1 = hA$. Fix a point $x \in \pi_A(hA)$. Since $\pi_A(hA)$ has diameter at most C, any geodesic $\gamma : [0, T] \to X$ from x to fx fellow-travels for a long time with $A_0 = A$, $g_0 A_1 = g_0 h A$, and $g_0 g_1 A_0 = fA$ (see Figure 1).

Note that the construction has been designed so that the end of γ and the beginning of $f\gamma$ both fellow-travel with fA in the 'same direction'. Consequently, the concatenation of γ and $f\gamma$ cannot backtrack much. We extend γ to a bi-infinite $\langle f \rangle$-invariant path, still denoted by $\gamma : \mathbb{R} \to X$, which is characterized as follows: $\gamma(t + kT) = f^k \gamma(t)$ for every

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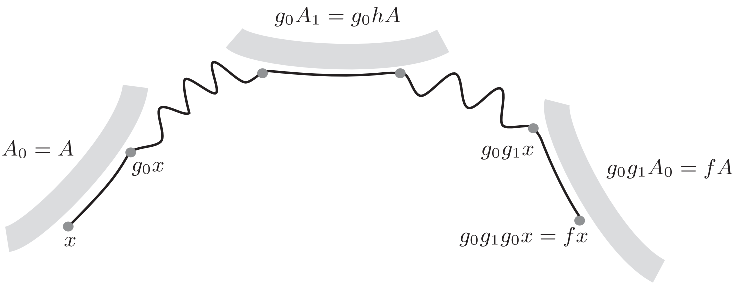

FIGURE 1. The geodesic from x to fx. The gray shapes are 'axis' of conjugates of g.

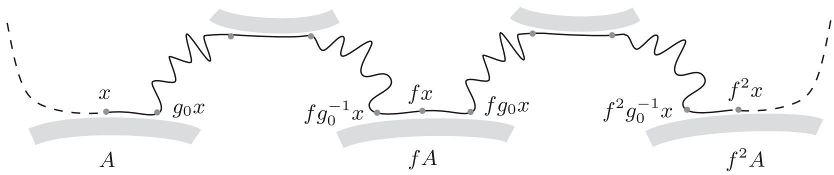

FIGURE 2. The 'axis' of f.

$t \in \mathbb{R}$ and $k \in \mathbb{Z}$. One proves that γ is a quasi-geodesic which fellow-travels for a long time with f^kA for every $k \in \mathbb{Z}$ (see Figure 2). This suffices to show that f is contracting.

We are left to prove that H is not virtually cyclic. If H was virtually cyclic, it would be contained in $E(f)$. This contradicts our first claim. □

3. *Compactification of X*

3.1. *Horocompactification.* Let $C(X)$ be the set of all real valued, continuous functions on X endowed with the topology of uniform convergence on every compact subset. We denote by $C^*(X)$ the quotient of $C(X)$ by the subspace consisting of all constant functions and endowed with the quotient topology. Given a base point $o \in X$, one can think of $C^*(X)$ as the set of all continuous functions that vanish at o. Alternatively, $C^*(X)$ is the set of continuous cocycles $c\colon X \times X \to \mathbb{R}$. By *cocycle*, we mean that

$$c(x, z) = c(x, y) + c(y, z) \quad \text{for all } x, y, z \in X.$$

For example, given $z \in X$, we define the cocycle $b_z\colon X \times X \to \mathbb{R}$ by

$$b_z(x, y) = d(x, z) - d(y, z) \quad \text{for all } x, y \in X.$$

Since X is geodesic, the map

$$\begin{aligned} \iota\colon X &\to C^*(X) \\ z &\mapsto b_z \end{aligned}$$

is a homeomorphism from X onto its image.

Definition 3.1. (Horoboundary) The *horocompactification* $\bar{X}$ of X is the closure of $\iota(X)$ in $C^*(X)$. The *horoboundary* of X is the set $\partial X = \bar{X} \setminus \iota(X)$.

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From now on, we identify X with its image under the map $\iota\colon X \to C^*(X)$. By construction, every cocycle $c \in \bar{X}$ is 1-Lipschitz, or equivalently $|c(x, x')| \leq d(x, x')$, for every $x, x' \in X$. It is a consequence of the Azerla–Ascoli theorem that the horocompactification $\bar{X}$ is indeed a compact set. We denote by $\mathfrak{B}$ the Borel σ-algebra on $\bar{X}$. In the remainder of the article, we make an abuse of notation and write $(\partial X, \mathfrak{B})$ to denote the horoboundary endowed with the σ-algebra $\mathfrak{B}$ restricted to ∂X.

Definition 3.2. Let $c \in \bar{X}$ and $\epsilon \in \mathbb{R}_+$. An *$\epsilon$-quasi-gradient arc* for c is a path $\gamma\colon I \to X$ parameterized by arc length such that

$$(t - s) - \epsilon \leq c(\gamma(s), \gamma(t)) \leq t - s \quad \text{for all } s, t \in I.$$

A *gradient arc for c* is a 0-quasi-gradient arc for c. If $I = \mathbb{R}_+$, we call γ a *(quasi-)gradient ray*.

Remark 3.3. The following observations follow from the definition and/or the triangle inequality.

- Since cocycles in $\bar{X}$ are 1-Lipschitz, a gradient arc is always geodesic.
- Conversely, let $x, y \in X$ and $\epsilon \in \mathbb{R}_+$ such that $c(x, y) \geq d(x, y) - \epsilon$. Any geodesic from x to y is an ϵ-quasi-gradient arc for c.
- Let $\gamma_1\colon [a_1, b_1] \to X$ and $\gamma_2\colon [a_2, b_2] \to X$ be two paths such that γ_i is an ϵ_i-quasi-gradient arc for c. If $\gamma_1(b_1) = \gamma_2(a_2)$, then the concatenation of γ_1 and γ_2 is an $(\epsilon_1 + \epsilon_2)$-quasi-gradient arc for c.

The existence of gradient rays is given by the next statement.

LEMMA 3.4. *Let $c \in \partial X$. For every $x \in X$, there exists a gradient ray $\gamma\colon \mathbb{R}_+ \to X$ for c such that $\gamma(0) = x$.*

Proof. Let (z_n) be a sequence of points of X converging to c. For every $n \in \mathbb{N}$, we let $b_n = \iota(z_n)$ and denote by $\gamma_n\colon [0, \ell_n] \to X$ a geodesic from x to z_n. Since X is proper, (γ_n) converges, up to passing to a subsequence, to a geodesic ray $\gamma\colon \mathbb{R}_+ \to X$ starting at x. As (b_n) converges uniformly on every compact subset to c, we check that γ is a gradient ray for c. □

Given $c \in \partial X$ and a gradient ray $\gamma\colon \mathbb{R}_+ \to X$ for c, we think of γ as a geodesic from $\gamma(0)$ to c. The next definition is designed to handle simultaneously the cocycles corresponding to points in X and the ones in ∂X.

Definition 3.5. Let $x \in X$ and $c \in \bar{X}$. A *gradient arc from x to c* is:

- any geodesic from x to z if $c = \iota(z)$ for some $z \in X$,
- any gradient ray for c starting at x if c belongs to ∂X.

3.2. *Reduced horoboundary.*

Given a cocycle $c \in C^*(X)$, we write $\|c\|_\infty$ for its uniform norm, that is,

$$\|c\|_\infty = \sup_{x, x' \in X} |c(x, x')|.$$

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Note that $\|c\|_\infty$ can be infinite. If $K \subset X$ is compact, then $\|c\|_K$ is the uniform norm of c restricted to $K \times K$. We endow $\bar{X}$ with a binary relation: two cocycles $c, c' \in \bar{X}$ are equivalent, and we write $c \sim c'$ if one of the following holds:

- either c and c' lie in the image of $\iota \colon X \to C^*(X)$ and $c = c'$;
- or $c, c' \in \partial X$ and $\|c - c'\|_\infty < \infty$.

Given a subset $B \subset \bar{X}$, the *saturation* of B, denoted by B^+, is the union of all equivalence classes intersecting B. We say that B is *saturated* if it is a union of equivalence classes, or equivalently if $B^+ = B$. Note that the collection of saturated subsets is closed under complement as well as (uncountable) union and intersection. The *reduced algebra*, denoted by $\mathfrak{R}$, is the sub-σ-algebra of $\mathfrak{B}$ which consists of all saturated Borel subsets.

Definition 3.6. The *reduced horocompactification* and *reduced horoboundary* of X are respectively the measurable spaces $(\bar{X}, \mathfrak{R})$ and $(\partial X, \mathfrak{R})$.

LEMMA 3.7. *If $F \subset \bar{X}$ is closed, then F^+ belongs to $\mathfrak{R}$.*

Proof. It suffices to prove that F^+ is a Borel subset. Without loss of generality, we can assume that F is contained in ∂X. Given $D \in \mathbb{R}_+$ and a compact subset $K \subset X$, we write

$$F_{K,D} = \{c \in \partial X \ : \ \text{there exists } b \in F,\ \|c - b\|_K \leq D\}.$$

Since F is compact, $F_{K,D}$ is closed. Using again the fact that F is compact, we observe that

$$F^+ = \bigcup_{D \in \mathbb{R}_+} \bigcap_{K \subset X} F_{K,D},$$

where K runs over all compact subsets of X. Hence the result. □

3.3. *Boundary at infinity of a contracting set.* The goal of this section is to understand how cocycles at infinity interact with contracting subsets of X.

Definition 3.8. Let Y be a closed subset of X. Let $c \in \bar{X}$. A *projection* of c on Y is a point $q \in Y$ such that for every $y \in Y$, we have $c(q, y) \leq 0$.

Given $z \in X$, the projection of $b = \iota(z)$ on Y coincides with the definition of the nearest point projection.

LEMMA 3.9. *Let Y be a closed subset of X. Let $c \in \bar{X}$. Assume that c admits a projection q on Y and denote by $\gamma : I \to \mathbb{R}_+$ a gradient arc for c starting at q. Then for every $t \in I$, the point q is a projection of $\gamma(t)$ on Y.*

Proof. Let $t \in I$. Note that

$$d(q, \gamma(t)) \leq c(q, \gamma(t)) \leq c(q, y) + c(y, \gamma(t)) \leq c(q, y) + d(y, \gamma(t)).$$

The first inequality holds since γ is a gradient line for c, while the last one follows from the fact that c is 1-Lipschitz. However, q being a projection of c on Y, we have $c(q, y) \leq 0$. Consequently, $d(q, \gamma(t)) \leq d(y, \gamma(t))$ for every $y \in Y$, which completes the proof. □

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If c is a point in ∂X, a projection of c on Y may exist or not. This leads to the following definition.

Definition 3.10. Let Y be a closed subset of X. The *boundary at infinity* of Y, denoted by $\partial^+ Y$, is the set of all cocycles $c \in \partial X$ for which there is no projection of c on Y.

We give several equivalent characterizations of the boundary at infinity of a contracting subset.

PROPOSITION 3.11. *Let $\alpha \in \mathbb{R}_+^*$. Let Y be an α-contracting subset of X. Let (z_n) be a sequence of points in X which converges to a cocycle $c \in \partial X$. For every $n \in \mathbb{N}$, denote by q_n a projection of z_n onto Y. Let $\gamma \colon \mathbb{R}_+ \to X$ be a gradient ray for c. Define $T \in \mathbb{R}_+ \cup \{\infty\}$ by*

$$T = \sup \{t \in \mathbb{R}_+ \ : \ d(\gamma(t), Y) \leq \alpha\}$$

with the convention that $T = 0$, whenever γ does not intersect $\mathcal{N}_\alpha(Y)$. The following are equivalent.

(i) $c \notin \partial^+ Y$.
(ii) *For every $x \in X$, the map $Y \to \mathbb{R}$ sending y to $c(x, y)$ is bounded from above.*
(iii) *The ray γ does not stay in a neighborhood of Y.*
(iv) *The projection $\pi_Y(\gamma)$ is bounded.*
(v) $T < \infty$.
(vi) *The sequence (q_n) is bounded.*

Moreover, in this situation:

- *the diameter of the set $Q = \pi_Y(\gamma_{[T,\infty)})$ is at most α;*
- *any accumulation point q^* of (q_n) is a projection of c on Y which lies in the α-neighborhood of Q.*

Remark. It follows from item (ii) that $\partial^+ Y$ is saturated (as the notation suggested).

Proof. The equivalences (iii) $\Longleftrightarrow$ (iv) and (iv) $\Longleftrightarrow$ (v) are standard properties of contracting sets, which only use the fact that γ is a geodesic. Note that $c(x, y) = c(x, x') + c(x', y)$ for every $x, x' \in X$ and $y \in Y$. The implication (i) $\Rightarrow$ (ii) follows from this observation. The proof of (ii) $\Rightarrow$ (iii) is by contraposition. Suppose that there exists $d \in \mathbb{R}_+$ such that γ lies in $\mathcal{N}_d(Y)$. Let $x = \gamma(0)$. Let $t \in \mathbb{R}_+$. We denote by q_t a projection of $\gamma(t)$ onto Y. Using the fact that c is 1-Lipschitz, we get

$$c(x, q_t) \geq c(\gamma(0), \gamma(t)) - d(\gamma(t), q_t) \geq t - d.$$

This inequality holds for every $t \in \mathbb{R}_+$, hence the map $Y \to \mathbb{R}$, sending y to $c(x, y)$, is not bounded from above.

We now focus on implication (v) $\Rightarrow$ (vi). For every $n \in \mathbb{N}$, we let $b_n = \iota(z_n)$. Assume that $T < \infty$. Let Q be the projection onto Y of γ restricted to $[T, \infty)$. Since Y is contracting, the diameter of Q is at most α. Let $q \in Q$ be a projection of $\gamma(T)$ on Y. We claim that there is $N \in \mathbb{N}$, such that for every $n \geq N$, the point q_n stays at a distance at most α of Q. Assume on the contrary that it is not the case. Up to passing to a subsequence,

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$d(q_n, Q) > \alpha$ for every $n \in \mathbb{N}$. Using Lemma 2.3, we observe that for every $t \geq T$, for every $n \in \mathbb{N}$,

$$b_n(\gamma(t), q) \geq d(q, \gamma(t)) - 4\alpha.$$

After passing to the limit, we get $c(\gamma(t), q) \geq d(q, \gamma(t)) - 4\alpha$ for every $t \geq T$. In particular, $t \mapsto c(\gamma(t), q)$ diverges to infinity as t tends to infinity, which contradicts the fact that γ is a gradient line for c. This completes the proof of our claim and thus implies item (vi).

We finish the proof with implication (vi) $\Rightarrow$ (i). Assume now that (q_n) is bounded. Let q^* be an accumulation point of (q_n). As Y is closed, q^* belongs to Y. Observe that for every $y \in Y$, for every $n \in \mathbb{N}$, we have

$$b_n(q^*, y) \leq b_n(q_n, y) + d(q^*, q_n) \leq d(q^*, q_n).$$

By construction, b_n converges to c on every compact subset, and hence $c(q^*, y) \leq 0$ for every $y \in Y$. Thus, q^* is a projection of c onto Y. In particular, $c \notin \partial^+ Y$. □

The next statement is a variation on the previous one, sharpening the estimates.

LEMMA 3.12. *Let $\alpha \in \mathbb{R}_+^*$. Let Y be an α-contracting set. Let $c \in \bar{X} \setminus \partial^+ Y$ and q be a projection of c on Y. Fix a sequence (z_n) of points in X converging to c. Denote by q^* an accumulation point of (q_n) where q_n stands for a projection of z_n on Y. Then $d(q, q^*) \leq \alpha$.*

Proof. Consider a gradient line $\gamma \colon \mathbb{R}_+ \to X$ from q to c. According to Lemma 3.9, q is a projection of any point $\gamma(t)$ on Y. Assume that, in contrast to our claim, $d(q, q^*) > \alpha$. Reasoning as in the proof of Proposition 3.11, we observe that $c(\gamma(t), q) \geq d(q, \gamma(t)) - 4\alpha$ for every $t \in \mathbb{R}_+$, which contradicts the fact that γ is a gradient line for c. □

The next two statements extend Lemma 2.3 for a gradient ray joining a point $x \in X$ to a cocycle $c \in \partial X$. We distinguish two cases depending on whether c belongs to $\partial^+ Y$ or not.

COROLLARY 3.13. *Let $\alpha \in \mathbb{R}_+^*$. Let Y be an α-contracting set. Let $x \in X$ and $c \in \bar{X} \setminus \partial^+ Y$. Let γ be a gradient arc from x to c. Let p and q be respective projections of x and c on Y. If $d(x, Y) < \alpha$ or $d(p, q) > 4\alpha$, then the following hold:*

- *$d(\gamma, Y) < \alpha$;*
- *the entry point (respectively exit point) of γ in $\mathcal{N}_\alpha(Y)$ is 2α-closed (respectively 5α-closed) to p (respectively q);*
- *$c(x, q) \geq d(x, p) + d(p, q) - 14\alpha$.*

Proof. If c is the cocycle associated to a point $z \in X$, then the statement is just a particular case of Lemma 2.3. Assume now that c belongs to $\partial X \setminus \partial^+ Y$. Fix a sequence (z_n) of points in X converging to c. Denote by q^* an accumulation point of (q_n), where q_n stands for a projection of z_n on Y. Denote by $\gamma(T)$ the exit point of γ from $\mathcal{N}_\alpha(Y)$, and Q the projection onto Y of γ restricted to $[T, \infty)$. According to Proposition 3.11, such an exit point exists. Moreover, Q has diameter at most α and q^* is a projection of c on Y lying in the α-neighborhood of Q.

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Suppose that $d(x, Y) < \alpha$ or $d(p, q) > 4\alpha$. According to Lemma 3.12, $d(p, q^*) > 3\alpha$, and hence, $d(p, Q) > \alpha$. It follows from the definition of contracting sets that $d(\gamma, Y) < \alpha$. Let p' (respectively q') be a projection of the entry (respectively exit) point of γ in $\mathcal{N}_\alpha(Y)$—note that if $d(x, Y) < \alpha$, then x is the entry point of γ in $\mathcal{N}_\alpha(Y)$, so we can choose $p' = p$. Using again the contraction of Y, we see that $d(p, p') \leq \alpha$. Moreover, q' belongs to Q, thus $d(q', q^*) \leq 2\alpha$. In addition,

$$d(x, \gamma(T)) \geq d(x, p') + d(p', q') + d(q', \gamma(T)) - 4\alpha.$$

The path γ is a gradient line, thus $c(x, \gamma(T)) = d(x, \gamma(T))$, and hence

$$c(x, \gamma(T)) \geq d(x, p') + d(p', q') + d(q', \gamma(T)) - 4\alpha.$$

Combined with the fact that c is 1-Lipschitz, we get

$$c(x, q') \geq c(x, \gamma(T)) - d(q', \gamma(T)) \geq d(x, p') + d(p', q') - 4\alpha. \tag{1}$$

We observed that $d(p, p') \leq \alpha$, while $d(q', q) \leq d(q', q^*) + d(q^*, q) \leq 4\alpha$. The conclusion follows from equation (1) and the triangle inequality. □

COROLLARY 3.14. *Let $\alpha \in \mathbb{R}_+^*$. Let Y be an α-contracting set and $c \in \partial^+ Y$. Let $x \in X$ and p be a projection of x onto Y. Let $\gamma : \mathbb{R}_+ \to X$ be a gradient ray from x to c. Then the following hold:*

- *$d(\gamma, Y) < \alpha$;*
- *the entry point of γ in $\mathcal{N}_\alpha(Y)$ is 2α-closed to p;*
- *$c(x, p) \geq d(x, p) - 4\alpha$.*

Proof. According to Proposition 3.11(iv), the set $\pi_Y(\gamma)$ is unbounded. Thus there exists $s \in \mathbb{R}_+$ and a projection q of $\gamma(s)$ onto Y such that $d(p, q) > \alpha$. It follows that $d(\gamma, Y) < \alpha$. Let $\gamma(t)$ be the entry point of γ in $\mathcal{N}_\alpha(Y)$. Using again the contraction of Y, we get $d(p, \gamma(t)) \leq 2\alpha$. Since γ is a gradient line, we have $c(x, \gamma(t)) = d(x, \gamma(t))$. Combined with the triangle inequality and the fact that cocycles are 1-Lipschitz, we get $c(x, p) \geq d(x, p) - 4\alpha$. □

PROPOSITION 3.15. *Assume that G is not virtually cyclic. Let g be a contracting element and A an orbit of $\langle g \rangle$. There exists $u \in G$ such that $\partial^+ A \cap \partial^+(uA) = \emptyset$.*

Proof. Consider an element $u \in G$ such that $\partial^+ A \cap \partial^+(uA)$ is not empty. Let c be a cocycle in this intersection and $\gamma : \mathbb{R}_+ \to X$ a gradient ray for c. By Proposition 3.11(iii), there exist $d, T \in \mathbb{R}_+$ such that γ restricted to $[T, \infty)$ is contained in $\mathcal{N}_d(A) \cap \mathcal{N}_d(uA)$. In particular, the diameter of this intersection is infinite. It follows that $u \in E(g)$, see [**50**, Lemma 2.12]. Recall that $E(g)$, unlike G, is virtually cyclic. Thus there exists $u \in G \setminus E(g)$. It follows from the above discussion that $\partial^+ A \cap \partial^+(uA) = \emptyset$. □

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4. *Conformal densities*

As previously, (X, d) is a proper, geodesic, metric space, while G is a group acting properly, by isometries on X.

4.1. *Definition and existence.*

If μ is a finite measure on $\bar{X}$, we denote by $\|\mu\|$ its total mass.

Definition 4.1. (Density) Let $\omega \in \mathbb{R}_+$. Let $\mathfrak{A}$ be a G-invariant sub-σ-algebra of the Borel σ-algebra $\mathfrak{B}$. A *density* on $(\bar{X}, \mathfrak{A})$ is a collection $\nu = (\nu_x)$ of positive finite measures on $(\bar{X}, \mathfrak{A})$ indexed by X such that $\nu_x \ll \nu_y$ for every $x, y \in X$, and normalized by $\|\nu_o\| = 1$. Such a density is:

(i) *G-invariant* if $g_* \nu_x = \nu_{gx}$, for every $g \in G$ and $x \in X$;

(ii) *ω-conformal* if for every $x, y \in X$,

$$\frac{d\nu_x}{d\nu_y}(c) = e^{-\omega c(x,y)}, \quad \nu_y\text{-a.e.}$$

(iii) *ω-quasi-conformal*, if there is $C \in \mathbb{R}_+^*$ such that for every $x, y \in X$,

$$\frac{1}{C} e^{-\omega c(x,y)} \leq \frac{d\nu_x}{d\nu_y}(c) \leq C e^{-\omega c(x,y)}, \quad \nu_y\text{-a.e.}$$

Vocabulary. Let $\nu = (\nu_x)$ be a density on $(\bar{X}, \mathfrak{A})$. We make an abuse of vocabulary and say that a property on $(\bar{X}, \mathfrak{A})$ holds ν-*a.e.* if it holds ν_x-a.e. for some (hence every) $x \in X$.

Remark 4.2. Let $\nu = (\nu_x)$ be an ω-conformal density on $(\bar{X}, \mathfrak{A})$. Recall that every cocycle in $\bar{X}$ is 1-Lipschitz. It follows that

$$\mu_x(A) \leq e^{\omega d(x,y)} \mu_y(A) \quad \text{for all } x, y \in X, \text{ for all } A \in \mathfrak{A}. \tag{2}$$

This observation will be useful many times later.

In practice, we will consider only two σ-algebras on $\bar{X}$: the Borel σ-algebra $\mathfrak{B}$ and the reduced σ-algebra $\mathfrak{R}$ (see §3.2). If ν is a conformal density on $(\bar{X}, \mathfrak{B})$, then its restriction to the reduced σ-algebra $\mathfrak{R}$ is not necessarily quasi-conformal. We will see later that this pathology can be avoided if the action of G on X is divergent.

4.1.1. *Topology.*

We denote by $\mathcal{D}(\omega)$ the set of all ω-conformal densities on the horocompactification $(\bar{X}, \mathfrak{B})$. We endow $\mathcal{D}(\omega)$ with the following topology: a sequence $\nu^n = (\nu_x^n)$ of densities converges to $\nu = (\nu_x)$ if for every $x \in X$, the measure ν_x^n converges to ν_x for the weak-* topology. Let $\mathcal{P}(\bar{X})$ be the set of all Borel probability measures on $\bar{X}$ (endowed with the weak-* topology). An ω-conformal density $\nu \in \mathcal{D}(\omega)$ is entirely determined by the measure ν_o. More precisely, the map

$$\begin{aligned} \mathcal{D}(\omega) &\to \mathcal{P}(\bar{X}) \\ \nu &\mapsto \nu_o \end{aligned} \tag{3}$$

is a homeomorphism. We denote by $\mathcal{D}(G, \omega)$ the convex closed subspace of $\mathcal{D}(\omega)$ consisting of all G-invariant, ω-conformal densities on $(\bar{X}, \mathfrak{B})$. The densities $\nu = (\nu_x)$

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in $\mathcal{D}(G, \omega)$ for which the action of G on $(\bar{X}, \mathfrak{B}, \nu_o)$ is ergodic are exactly the extremal points of $\mathcal{D}(G, \omega)$.

4.1.2. *Patterson's construction.* We now prove the existence of invariant conformal densities supported on the horoboundary. Actually, we focus on a slightly more general setting that will be useful for our applications. A map $\chi : G \to \mathbb{R}$ is a *quasi-morphism* if there exists $C \in \mathbb{R}_+$ such that for every $g, g' \in G$, we have

$$|\chi(g) + \chi(g') - \chi(gg')| \leq C. \tag{4}$$

To such a quasi-morphism χ, we associate a *twisted Poincaré series* defined as

$$\mathcal{P}_\chi(s) = \sum_{g \in G} e^{\chi(g)} e^{-sd(o,go)},$$

and write ω_χ for its critical exponent. It follows from equation (4) that

$$e^{-2C}\mathcal{P}_\chi(s) \leq \mathcal{P}_{-\chi}(s) \leq e^{2C}\mathcal{P}_\chi(s) \quad \text{for all } s \in \mathbb{R}_+.$$

Hence, $\omega_{-\chi} = \omega_\chi$. Note also that

$$\frac{1}{2}[\mathcal{P}_\chi(s) + \mathcal{P}_{-\chi}(s)] \geq \sum_{g \in G} \operatorname{ch}(\chi(g)) e^{-sd(o,go)} \geq \mathcal{P}_G(s).$$

Thus, $\omega_\chi \geq \omega_G$.

PROPOSITION 4.3. *Let H be a subgroup of G. Let $\chi : G \to \mathbb{R}$ be a quasi-morphism such that $\chi(hg) = \chi(g)$ for all $h \in H$ and $g \in G$. There is an H-invariant, ω_χ-conformal density $\nu = (\nu_x)$ on $(\bar{X}, \mathfrak{B})$ with the following properties:*

- *ν is supported on ∂X;*
- *there is $C \in \mathbb{R}_+^*$ such that for every $g \in G$ and $x \in X$, we have*

$$\frac{1}{C}\nu_x \leq e^{-\chi(g)} {g^{-1}}_* \nu_{gx} \leq C\nu_x.$$

Remark. If $H = G$ and χ is the trivial morphism, then the proposition says that there exists a G-invariant, ω_G-conformal density supported on ∂X.

Proof. The proof follows Patterson's strategy [39]. As Burger and Mozes already observed, this construction can be carried without difficulty in the horocompactification of X [9]. We only review here its main steps. Note that the twisted Poincaré series $\mathcal{P}_\chi(s)$ may converge at the critical exponent $s = \omega_\chi$. Using Patterson's idea, one produces a 'slowly growing' function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties—see [42, Lemme 2.1.1].

(P1) For every $\varepsilon > 0$, there exists $t_0 \geq 0$, such that for every $t \geq t_0$ and $u \geq 0$, we have $\theta(t + u) \leq e^{\varepsilon u}\theta(t)$.

(P2) The weighted twisted Poincaré series, defined by

$$\mathcal{Q}(s) = \sum_{g \in G} \theta(d(o, go)) e^{\chi(g)} e^{-sd(go,o)} \tag{5}$$

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is divergent whenever $s \leq \omega_\chi$ and convergent otherwise. In particular, $Q(s)$ diverges to infinity as s approaches ω_χ (from above).

For every $x \in X$ and $s > \omega_\chi$, we define a measure on $\bar{X}$ by

$$\nu_x^s = \frac{1}{Q(s)} \sum_{g \in G} \theta(d(x, go)) e^{\chi(g)} e^{-sd(x,go)} \mathrm{Dirac}(go). \tag{6}$$

Since $\bar{X}$ is compact, the space of probability measures on $\bar{X}$ is compact for the weak-* topology. Consequently, there exists a sequence of real numbers (s_n) converging to ω_χ from above and such that for every $x \in X$, the measure $\nu_x^{s_n}$ converges to a measure on $\bar{X}$ that we denote by ν_x. Note that $\nu^s = (\nu_x^s)$ is H-invariant for every $s > \omega_\chi$. Moreover, since χ is a quasi-morphism, there exists $C \in \mathbb{R}_+^*$ such that for every $s > \omega_\chi$, $g \in G$, and $x \in X$, we have

$$\frac{1}{C} \nu_x^s \leq e^{-\chi(g)} {g^{-1}}_* \nu_{gx}^s \leq C \nu_x^s.$$

Hence, the same properties hold for ν. The horocompactification is precisely designed so that the map

$$\begin{aligned} X \times X \times X &\to \mathbb{R} \\ (x, y, z) &\mapsto d(x, z) - d(y, z) \end{aligned}$$

extends continuously to a map $X \times X \times \bar{X} \to \mathbb{R}$. Taking advantage of this fact, one checks that ν is ω_χ-conformal. Since $Q(s)$ diverges when s approaches ω_χ, the density ν is supported on ∂X. □

4.2. *Group action on the space of density*

4.2.1. *Group action.* The action of G on X induces a right action of G on the set of densities. Let $\mathfrak{A}$ be a G-invariant sub-σ-algebra of $\mathfrak{B}$. Given a density $\nu = (\nu_x)$ on $(\bar{X}, \mathfrak{A})$ and $g \in G$, we define a new density ν^g as follows:

$$\nu_x^g = \frac{1}{\|\nu_{go}\|} {g^{-1}}_* \nu_{gx} \quad \text{for all } x \in X.$$

We make the following observations.

(i) If ν is ω-conformal, then the same holds for ν^g.

(ii) If ν is H-invariant for some subgroup $H \subset G$, then ν^g is H^g-invariant, where $H^g = g^{-1}Hg$.

In particular, if N is a normal subgroup of G, then the map

$$\begin{aligned} \mathcal{D}(N, \omega) \times G &\to \mathcal{D}(N, \omega) \\ (\nu, g) &\mapsto \nu^g \end{aligned}$$

defines a right action of G on $\mathcal{D}(N, \omega)$ which is trivial when restricted to N.

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4.3. *Fixed point properties.* For our study, we need to distinguish several fixed point properties for the action of G on the space of densities. Recall that a density $\nu = (\nu_x)$ is:

(i) *G-invariant* if $g_* \nu_x = \nu_{gx}$ for every $g \in G$ and $x \in X$.

We say that ν is:

(ii) *G-almost invariant* if there exists $C \in \mathbb{R}_+^*$ such that for every $g \in G$ and $x \in X$,

$$\frac{1}{C} \nu_{gx} \leq g_* \nu_x \leq C \nu_{gx};$$

(iii) *fixed by G* if $\nu^g = \nu$ for every $g \in G$;

(iv) *almost-fixed by G* if there exists $C \in \mathbb{R}_+^*$ such that for every $g \in G$ and $x \in X$,

$$\frac{1}{C} \nu_x \leq \nu_x^g \leq C \nu_x;$$

(v) *G-quasi-invariant* if $g_* \nu_o \ll \nu_o$ for every $g \in G$.

If we want to emphasize the constant C in item (iv), we will say that ν is *C-almost-fixed by G*. These properties are related as follows:

$$\begin{array}{ccccc} & & \text{(ii)} & & \\ & \nearrow & & \searrow & \\ \text{(i)} & & & & \text{(iv)} \Longrightarrow \text{(v)}. \\ & \searrow & & \nearrow & \\ & & \text{(iii)} & & \end{array}$$

The reverse implications do not hold in general. The next statement highlights the role of quasi-morphisms in our study.

LEMMA 4.4. *Let $\nu = (\nu_x)$ be a density. Assume that ν is fixed (respectively almost-fixed) by G. Then the map $\chi \colon G \to \mathbb{R}$ sending g to $\ln \|\nu_{go}\|$ is a morphism (respectively quasi-morphism).*

Proof. Assume that ν is almost-fixed by G (if ν is fixed by G, the proof works in the exact same way). There exists $C \in \mathbb{R}_+^*$ such that for every $g \in G$ and $x \in X$, we have

$$\frac{1}{C} \nu_x \leq \nu_x^g \leq C \nu_x.$$

Let $g, g' \in G$. Comparing the total masses of the above measures for $x = g'o$, we get

$$\frac{1}{C} \|\nu_{g'o}\| \leq \frac{\|\nu_{gg'o}\|}{\|\nu_{go}\|} \leq C \|\nu_{g'o}\|. \qquad \square$$

LEMMA 4.5. *Let H be a subgroup of G. Let $\mu = (\mu_x)$ be an H-invariant, ω-quasi-conformal density on the reduced horocompactification $(\bar{X}, \mathfrak{R})$. Assume that for every $g \in G$, the action of $H^g \cap H$ on $(\bar{X}, \mathfrak{R}, \mu_o)$ is ergodic. If μ is G-quasi-invariant, then μ is almost-fixed by G.*

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Proof. Since μ is G-quasi-invariant, we can define the following map:

$$\begin{aligned} F\colon \quad G \times \bar{X} &\to \qquad \mathbb{R}_+ \\ (g, c) &\mapsto \frac{d(g^{-1}{}_*\mu_{go})}{d\mu_o}(c) \end{aligned}$$

Claim 4.6. There exists $C \in \mathbb{R}_+^*$ such that for every $g_1, g_2 \in G$, we have

$$\frac{1}{C}\frac{F(g_1g_2, c)}{F(g_2, c)} \leq F(g_1, g_2c) \leq C\frac{F(g_1g_2, c)}{F(g_2, c)}, \quad \mu\text{-a.e.}$$

Let $g_1, g_2 \in G$. The computation gives

$$F(g_1g_2, c) = \frac{d(g_1^{-1}{}_*\mu_{g_1g_2o})}{d\mu_{g_2o}}(g_2c)\,\frac{d(g_2^{-1}{}_*\mu_{g_2o})}{d\mu_o}(c), \quad \mu\text{-a.e.} \tag{7}$$

Since μ is ω-quasi-conformal, there exists $C \in \mathbb{R}_+$ such that for every $x, y \in X$, for every $g \in G$, we have

$$\frac{1}{C}\frac{d\mu_x}{d\mu_y} \leq \frac{d\mu_{gx}}{d\mu_{gy}} \circ g \leq C\frac{d\mu_x}{d\mu_y}.$$

Hence,

$$\frac{1}{C}\frac{d(g^{-1}{}_*\mu_{gy})}{d\mu_y} \leq \frac{d(g^{-1}{}_*\mu_{gx})}{d\mu_x} \leq C\frac{d(g^{-1}{}_*\mu_{gy})}{d\mu_y}.$$

It follows that the first factor in the right-hand side of (7) is $F(g_1, g_2c)$—up to a multiplicative error that does not depend on g_1 or g_2—while the second factor is exactly $F(g_2, c)$. This completes the proof of our claim.

Let $g \in G$ and set $H_0 = H^g \cap H$. Let $h \in H_0$. According to our claim, we have

$$\frac{1}{C}\frac{F(gh, c)}{F(h, c)} \leq F(g, hc) \leq C\frac{F(gh, c)}{F(h, c)}, \quad \mu\text{-a.e.}$$

Recall that μ is H-invariant, thus

$$F(h, c) = 1 \quad \text{and} \quad F(gh, c) = F(ghg^{-1}g, c) = F(g, c), \quad \mu\text{-a.e.}$$

Our previous inequalities becomes

$$\frac{1}{C}F(g, c) \leq F(g, hc) \leq CF(g, c), \quad \mu\text{-a.e.}$$

We now define an auxiliary function $F_g\colon \bar{X} \to \mathbb{R}_+$ by

$$F_g(c) = \inf_{h \in H_0} F(g, hc).$$

By construction, F_g is H_0-invariant. Since the action of H_0 on $(\bar{X}, \mathfrak{R}, \mu_o)$ is ergodic, F_g is constant. From now on, we denote by F_g its essential value. It follows from our previous observation that

$$F_g \leq F(g, c) \leq CF_g, \quad \mu\text{-a.e.}$$

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Coming back to the definition of F, this means that

$$F_g \leq \frac{d({g^{-1}}_* \mu_{go})}{d\mu_o} \leq CF_g, \quad \mu\text{-a.e.}$$

Integrating these inequalities, we see that $F_g \leq \|\mu_{go}\| \leq CF_g$. Hence,

$$\frac{1}{C}\|\mu_{go}\| \leq \frac{d({g^{-1}}_* \mu_{go})}{d\mu_o} \leq C\|\mu_{go}\|.$$

Recall that C does not depend on g. We have proved that there exists $C \in \mathbb{R}_+^*$ such that for every $g \in G$,

$$\frac{1}{C}\mu_o \leq \mu_o^g \leq C\mu_o.$$

Using the quasi-conformality of μ, we conclude that μ is almost-fixed by G. □

Remark. The same argument shows that if μ is ω-conformal (instead of ω-quasi-conformal), then μ is fixed by G.

4.4. *The shadow principle.* Given $x, y \in X$ and $c \in \bar{X}$, we define the following Gromov product:

$$\langle x, c\rangle_y = \tfrac{1}{2}[d(x, y) + c(y, x)].$$

Remark. If $c = \iota(z)$ for some $z \in X$, then the above formula coincides with the usual definition of the Gromov product.

Since cocycles in $\bar{X}$ are 1-Lipschitz, we always have $0 \leq \langle x, c\rangle_y \leq d(x, y)$. We also observe that

$$|\langle x, c\rangle_y - \langle x', c\rangle_{y'}| \leq d(x, x') + d(y, y') \quad \text{for all } x, x', y, y' \in X. \tag{8}$$

Definition 4.7. Let $x, y \in X$. Let $r \in \mathbb{R}_+$. The *r-shadow* of y seen from x is the set

$$\mathcal{O}_x(y, r) = \{c \in \bar{X} \ : \ \langle x, c\rangle_y \leq r\}.$$

By construction, $\mathcal{O}_x(y, r)$ is a closed subset of $\bar{X}$. It follows from equation (8) that for every $x, x', y, y' \in X$ and $r \in \mathbb{R}_+$,

$$\mathcal{O}_x(y, r) \subset \mathcal{O}_{x'}(y', r') \quad \text{where } r' = r + d(x, x') + d(y, y'). \tag{9}$$

Remark. A more intuitive definition of shadows could have been the following: a cocycle $c \in \bar{X}$ belongs to $\mathcal{O}_x(y, r)$ if some gradient arc from x to c passes at a distance at most r from y. Nevertheless, unlike our approach, this definition is very sensitive to the change of point x.

Following Roblin with small variations, we define the shadow principle [42].

Definition 4.8. Let $\omega \in \mathbb{R}_+$ and $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$. Let $\nu = (\nu_x)$ be an ω-conformal density on $(\bar{X}, \mathfrak{B})$. We say that (G, ν) satisfies the *shadow principle with parameters* (ϵ, r_0) if for every $g \in G$ and $r \geq r_0$, we have

$$\nu_o(\mathcal{O}_o(go, r)) \geq \epsilon \|\nu_{go}\| e^{-\omega d(o,go)}. \tag{10}$$

We say that (G, ν) satisfies the *shadow principle* if there are $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ such that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0).

Remark. If the inequality in equation (10) holds for $r = r_0$, then it automatically holds for every $r \geq r_0$. We will see later that a similar upper bound is always satisfied without any additional assumption.

Our next task is to adapt Sullivan's celebrated shadow lemma (Corollary 4.10). It states that (G, ν) satisfies the shadow principle whenever ν is an ω-conformal density which is N-invariant for some infinite normal subgroup $N \lhd G$.

PROPOSITION 4.9. *Assume that G is not virtually cyclic and contains a contracting element. Let $\mathcal{D}_0$ be a closed subset of G-quasi-invariant densities on $\bar{X}$. There exists $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ with the following property. For every $r \geq r_0$, for every density $\nu = (\nu_x)$ in $\mathcal{D}_0$, for every $z \in X$, we have*

$$\nu_o(\mathcal{O}_z(o, r)) \geq \epsilon.$$

Proof. Assume that our claim fails. We can find a sequence (r_n) diverging to infinity, a sequence (z_n) of points in X, and a sequence $\nu^n = (\nu_x^n)$ of densities in $\mathcal{D}_0$ such that

$$\nu_o^n(\mathcal{O}_{z_n}(o, r_n))$$

converges to zero. Since $\mathcal{D}_0$ is closed, up to passing to a subsequence, we may assume that ν^n converges to a density ν in $\mathcal{D}_0$. Note that for every $x, y \in X$ and $r \in \mathbb{R}_+$, we have $\mathcal{O}_x(y, r) = \bar{X}$, whenever $d(x, y) \leq r$. Hence, $d(o, z_n)$ necessarily diverges to infinity. Up to passing again to a subsequence, we can assume that z_n converges to $b \in \partial X$.

By assumption G is not virtually cyclic and contains a contracting element. According to Proposition 3.15, there exists a contracting element $h \in G$ such that $b \notin \partial^+ A$, where $A = \langle h \rangle o$ is α-contracting for some $\alpha \in \mathbb{R}_+^*$. For every $n \in \mathbb{N}$, we write p_n for a projection of z_n onto A. Up to passing to a subsequence, (p_n) converges to a point $p \in A$ which is a projection of b onto A (Proposition 3.11). We introduce the following closed subset of $\bar{X}$:

$$F = \{c \in \bar{X} \ : \ c(p, y) \leq 4\alpha \text{ for all } y \in Y\}.$$

We are going to prove that $\nu_o(F) = 1$.

Let $c \in \bar{X} \setminus F$. Suppose first that $c \notin \partial^+ A$. Let q be a projection of c onto A. Since c is 1-Lipschitz, we have

$$c(p, y) \leq d(p, q) + c(q, y) \leq d(p, q) \quad \text{for all } y \in A.$$

As c does not belong to F, necessarily $d(p, q) > 4\alpha$. In particular, there exists $N_0 \in \mathbb{N}$, such that for every $n \geq N_0$, we have $d(p_n, q) > 4\alpha$. According to Corollary 3.13,

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$\langle z_n, c\rangle_{p_n} \leq 2\alpha$. Note that the conclusion still holds if $c \in \partial^+ A$. This is indeed a consequence of Corollary 3.14. Hence, in all cases, we have

$$\langle z_n, c\rangle_o \leq \langle z_n, c\rangle_{p_n} + d(o, p_n) \leq 2\alpha + d(o, p_n).$$

Recall that (p_n) is bounded. Consequently, there is $N_1 \geq N_0$ such that for every $n \geq N_1$, the set $\bar{X} \setminus F$ is contained in $\mathcal{O}_{z_n}(o, r_n)$. In particular, $\nu_o^n(\bar{X} \setminus F)$ converges to zero. Since $\bar{X} \setminus F$ is an open subset of $\bar{X}$, we deduce that $\nu_o(\bar{X} \setminus F) = 0$, that is, $\nu_o(F) = 1$.

The group $\langle h\rangle$ has unbounded orbits. Thus, there is $g \in \langle h\rangle$ such that $d(gp, p) > 36\alpha$. We claim that $F \cap gF$ is empty. Let $c \in F$. Remember that c does not belong to $\partial^+ A$ by Proposition 3.11(ii). Choose a projection q of c onto A. It follows from Corollary 3.13 that

$$4\alpha \geq c(p, q) \geq d(p, q) - 14\alpha.$$

Hence, $d(p, q) \leq 18\alpha$. Suppose now that, in contrast to our claim, c also belongs to gF. In particular, $g^{-1}q$ is a projection of $g^{-1}c$ onto A. Following the same argument as above, we get $d(p, g^{-1}q) \leq 18\alpha$. Consequently, $d(p, gp) \leq 36\alpha$, which is a contradiction. Observe that

$$\nu_o(gF) = {g^{-1}}_*\nu_o(F) = \|\nu_{go}\|\nu^g_{g^{-1}o}(F).$$

As an element of $\mathcal{D}_0$, the density ν is G-quasi-invariant, and hence $\nu_o \ll \nu^g_{g^{-1}o}$. We proved that $\nu_o(F) > 0$, thus $\nu_o(gF) > 0$. Consequently,

$$\nu_o(F \cup gF) = \nu_o(F) + \nu_o(gF) > 1.$$

This contradicts the fact that ν_o is a probability measure. □

COROLLARY 4.10. *Assume that G is not virtually cyclic and has a contracting element. Let N be an infinite normal subgroup of G. Let $\omega \in \mathbb{R}_+$. There exists $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ such that for every $r \geq r_0$, for every N-invariant, ω-conformal density $\nu = (\nu_x)$, for every $g \in G$, we have*

$$\epsilon\|\nu_{go}\|e^{-\omega d(o,go)} \leq \nu_o(\mathcal{O}_o(go, r)) \leq e^{2\omega r}\|\nu_{go}\|e^{-\omega d(o,go)}. \tag{11}$$

In particular, (G, ν) satisfies the shadow principle with parameters (ϵ, r_0).

Proof. Let $\nu = (\nu_x)$ be an ω-conformal density. A classical computation shows that for every $g \in G$ and $r \in \mathbb{R}_+$, we have

$$\nu_o(\mathcal{O}_o(go, r)) = \|\nu_{go}\| \int \mathbb{1}_{\mathcal{O}_{g^{-1}o}(o,r)}(c)e^{-\omega c(g^{-1}o,o)}\, d\nu_o^g(c).$$

Shadows have been designed so that for every $c \in \mathcal{O}_{g^{-1}o}(o, r)$, we have

$$d(o, go) - 2r \leq c(g^{-1}o, o) \leq d(o, go).$$

Consequently,

$$\nu_o^g(\mathcal{O}_{g^{-1}o}(o, r)) \leq \frac{e^{\omega d(go,o)}}{\|\nu_{go}\|}\nu_o(\mathcal{O}_o(go, r)) \leq e^{2\omega r}\nu_o^g(\mathcal{O}_{g^{-1}o}(o, r)).$$

The upper bound in equation (11) follows from the fact that ν_o^g is a probability measure. Let us focus on the lower bound. To that end, we assume now that ν is N-invariant. As N is a normal subgroup of G, the density ν^g is N-invariant as well. Since G contains a contracting element and is not virtually cyclic, the same holds for N (Lemma 2.8). The result now follows from Proposition 4.9 applied with the group N and the closed set $\mathcal{D}_0 = \mathcal{D}(N, \omega)$. □

Remark 4.11. Note that the upper bound in equation (11) was proved without assuming any invariance for ν. Moreover, it works for any $r \in \mathbb{R}_+$.

Here is another variation of the shadow lemma.

COROLLARY 4.12. *Assume that G is not virtually cyclic and contains a contracting element. Let $\omega \in \mathbb{R}_+$ and $\nu = (\nu_x)$ be an ω-conformal density. If ν is almost fixed by G, then (G, ν) satisfies the shadow principle.*

Proof. Let $g \in G$ and $r \in \mathbb{R}_+$. Reasoning as in the proof of Corollary 4.10, we see that

$$\nu_o(\mathcal{O}_o(go, r)) \geq \|\nu_{go}\| e^{-\omega d(o,go)} \nu_o^g(\mathcal{O}_{g^{-1}o}(o, r)).$$

Since ν is almost-fixed by G, there is $\epsilon \in \mathbb{R}_+^*$, which does not depend on g or r, such that $\nu_o^g \geq \epsilon \nu_o$. Consequently,

$$\nu_o(\mathcal{O}_o(go, r)) \geq \epsilon \|\nu_{go}\| e^{-\omega d(o,go)} \nu_o(\mathcal{O}_{g^{-1}o}(o, r)).$$

The result now follows from Proposition 4.9 applied with $\mathcal{D}_0 = \{\nu\}$. □

4.5. *Contracting tails.* To take full advantage of the shadow lemma, we consider a particular kind of shadow namely shadows of the form $\mathcal{O}_x(y, r)$ where x is joined to y by a 'geodesic' whose tail is contracting. The next definition quantifies the 'contraction strength' of this tail.

Definition 4.13. Let $\alpha \in \mathbb{R}_+^*$ and $L \in \mathbb{R}_+$. Let $x, y \in X$. The pair (x, y) has an *(α, L)-contracting tail* if there exists an α-contracting geodesic τ ending at y and a projection p of x on τ satisfying $d(p, y) \geq L$. The path τ is called a *(contracting) tail* of (x, y)—the contracting strength and the length of the tail should be clear from the context.

Notation 4.1. Given $\alpha \in \mathbb{R}_+^*$ and $L \in \mathbb{R}_+$, we denote by $\mathcal{T}(\alpha, L)$ the set of all elements $g \in G$ such that the pair (o, go) has an (α, L)-contracting tail.

The next statement is essentially a reformulation of Corollary 3.13. It states that if (x, y) has a sufficiently long contracting tail, then the shadow of y seen from x behaves according to our intuition coming from hyperbolic geometry.

LEMMA 4.14. *Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 13\alpha$. Let $x, y \in X$. Assume that (x, y) has an (α, L)-contracting tail, say τ. Let p be a projection of x on τ. Let $c \in \mathcal{O}_x(y, r)$. Let γ be a gradient arc from x to c. Let q be a projection of c onto τ. Then the following hold.*

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(i) $d(y, q) \leq r + 7\alpha$.
(ii) $d(\gamma, \tau) < \alpha$.
(iii) *The entry point of* γ *in* $\mathcal{N}_\alpha(\tau)$ *is* 2α*-close to* p.
(iv) *The exit point of* γ *from* $\mathcal{N}_\alpha(\tau)$ *is* 5α*-close to* q.

Proof. Since τ is a contracting tail, there is a projection p' of x on τ such that $d(p', y) \geq L$. The computation shows that

$$d(y, q) + \langle x, c\rangle_{p'} + \langle p', c\rangle_q = \langle x, c\rangle_y + \langle x, y\rangle_{p'} + \langle y, c\rangle_q + \langle p', y\rangle_q,$$

(it suffices to expand the definition of the Gromov products). On the one hand, we observed that $\langle x, c\rangle_{p'} \geq 0$ and $\langle p', c\rangle_q \geq 0$. On the other hand, since τ is contracting, we have $\langle x, y\rangle_{p'} \leq 2\alpha$ (Lemma 2.3) and $\langle y, c\rangle_q \leq 5\alpha$ (Corollary 3.13). It follows that

$$d(y, q) \leq r + 7\alpha + \langle p', y\rangle_q.$$

We now discuss the relative positions of p', q, and y on τ. If p' lies between q and y, then $\langle p', y\rangle_q = d(p', q)$ while $d(y, q) = d(y, p') + d(p', q)$. It forces $L \leq d(y, p') \leq r + 7\alpha$, which contradicts our assumption. Thus, q lies between p' and y so that $\langle p', y\rangle_q = 0$. Consequently, $d(y, q) \leq r + 7\alpha$, which completes the proof of item (i).

By assumption, p is another projection of x on τ, which is α-contracting, thus $d(p, p') \leq 2\alpha$. Consequently,

$$d(p, q) \geq d(p', y) - d(p, p') - d(y, q) \geq L - (r + 9\alpha) > 4\alpha.$$

According to Corollary 3.13, $d(\gamma, \tau) < \alpha$. Moreover, the entry (respectively exit) point of γ in $\mathcal{N}_\alpha(\tau)$ is 2α-close to p (respectively 5α-closed to q). Hence the result. □

LEMMA 4.15. *Let* $\alpha \in \mathbb{R}_+^*$ *and* $r, L \in \mathbb{R}_+$ *with* $L > r + 13\alpha$. *Let* $x, y_1, y_2 \in X$ *such that* (x, y_i) *has an* (α, L)*-contracting tail, say* τ_i. *Let* p_i *be a projection of* x *on* τ_i. *If* $\mathcal{O}_x(y_1, r) \cap \mathcal{O}_x(y_2, r)$ *is non-empty, then:*
(i) $d(y_1, y_2) \leq |d(x, y_1) - d(x, y_2)| + 4r + 48\alpha$*; and*
(ii) $d(p_1, p_2) \leq |d(x, p_1) - d(x, p_2)| + 8\alpha$.

Proof. Let c be a cocycle in the intersection $\mathcal{O}_x(y_1, r) \cap \mathcal{O}_x(y_2, r)$. Let γ be a gradient arc from x to c. According to Lemma 4.14, the points y_1 and y_2 lie in the $(r + 12\alpha)$-neighborhood of γ, while p_1 and p_2 are 2α-closed to γ. The result follows. □

To state the next lemma, we need a notion of spheres in G (for the metric induced by X). This is the purpose of the following notation. Given $\ell, a \in \mathbb{R}_+$, we let

$$S(\ell, a) = \{g \in G \ :\ \ell - a \leq d(o, go) < \ell + a\}.$$

LEMMA 4.16. *Assume that* G *has a contracting element. There is* $\alpha \in \mathbb{R}_+^*$ *such that for every* $L \in \mathbb{R}_+$, *for every* $g \in G$, *there exist* $u \in S(L, \alpha)$ *and a geodesic* $\tau\colon [0, T] \to X$ *from* o *to* guo *with the following properties.*
(i) *The path* γ *restricted to* $[T - L, T]$ *is* α*-contracting. In particular,* $gu \in \mathcal{T}(\alpha, L)$.
(ii) *The point* go *is* α*-close to* $\tau(T - L)$.

Proof. Let $h \in G$ be a contracting element. Denote by A the $\langle h\rangle$-orbit of o. It is β-contracting for some $\beta \in \mathbb{R}_+^*$. Let $\alpha_0 \in \mathbb{R}_+^*$ be the parameter given by Lemma 2.7 applied with the element h, the point $z = o$, and $d = 2\beta$. Up to increasing the value of α_0, we can assume that $\alpha_0 \geq 2\beta + d(o, ho)$.

Let $L \in \mathbb{R}_+$. Since h is contracting, the orbit map $\mathbb{Z} \to X$ sending n to $h^n o$ is a quasi-isometric embedding. Thus, there is $N \in \mathbb{N}$ such that for every $n \geq N$, we have $d(o, h^n o) \geq L + 2\beta$. We choose N minimal with the above property. In particular,

$$L + 2\beta \leq d(o, h^N o) \leq d(o, h^{N-1} o) + d(o, ho) < L + 2\beta + d(o, ho).$$

Hence, h^N and h^{-N} belong to $S(L, \alpha_0)$.

Let $g \in G$. There is $k \in \mathbb{Z}$ such that $h^k o$ is a projection of $g^{-1} o$ onto A. If $k \leq 0$ (respectively $k \geq 0$), we choose $u = h^N$ (respectively $u = h^{-N}$). We now prove that u satisfies the announced properties. We suppose that $k \geq 0$. The other case works in the exact same way. Let $\gamma \colon [0 , T] \to X$ be a geodesic from $g^{-1} o$ to $h^{-N} o$. According to Lemma 2.3, $h^k o$ is 2β-close to the entry point $\gamma(t)$ of γ in $\mathcal{N}_\beta(A)$. By our choice of N, we have

$$T - t \geq d(\gamma(t), \gamma(T)) \geq d(h^k o, h^{-N} o) - 2\beta \geq d(o, h^{N+k} o) - 2\beta \geq L.$$

In particular, $t \leq T - L \leq T$. It follows from Lemma 2.7 and our choice of α_0 that:

- γ restricted to $[T - L , T]$ is α_0-contracting;
- there is $s \in [T - L , T]$ such that $d(\gamma(s), o) \leq \alpha_0$.

Using the triangle inequality, we observe that

$$|d(\gamma(s), h^{-N} o) - d(o, h^{-N} o)| \leq d(\gamma(s), o) \leq \alpha_0.$$

On the one hand, $d(\gamma(s), h^{-N} o) = T - s$. On the other hand, $h^{-N} \in S(L, \alpha_0)$. Thus, $|(T - L) - s| \leq 2\alpha_0$. Consequently, the triangle inequality yields

$$d(o, \gamma(T - L)) \leq d(o, \gamma(s)) + d(\gamma(s), \gamma(T - L)) \leq 3\alpha_0.$$

Observe now that the path $\tau = g\gamma$ satisfies the conclusion of the lemma with the parameter $\alpha = 3\alpha_0$. □

Given $\alpha, r, L, \ell \in \mathbb{R}_+$, we consider the following set:

$$A_\ell(\alpha, r, L) = \bigcup_{g \in S(\ell, r) \cap \mathcal{T}(\alpha, L)} \mathcal{O}_o(go, r).$$

Observe that for a fixed $\ell \in \mathbb{R}$, the set $A_\ell(\alpha, r, L)$ is a non-decreasing function of r (respectively α) and a non-increasing function of L.

PROPOSITION 4.17. *Assume that G contains a contracting element. There is $\alpha \in \mathbb{R}_+^*$ such that for every $\omega, a \in \mathbb{R}_+$ and $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$, there exist $r_1, C \in \mathbb{R}_+^*$, with the following property. Let $\nu = (\nu_x)$ be an ω-conformal density. If (G, ν) satisfies the shadow principle with parameters (ϵ, r_0), then for every $r \geq r_1$, $L > r + 13\alpha$, and $\ell \in \mathbb{R}_+$,*

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o, go)} \leq C e^{2\omega L} \nu_o(A_{\ell+L}(\alpha, r, L)).$$

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Proof. We choose for α the parameter given by Lemma 4.16. Let $\omega, a \in \mathbb{R}_+$ and $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$. The action of G on X is proper, so there is $M \in \mathbb{N}$ such that

$$|\{g \in G \ : \ d(o, go) \leq 2a + 12\alpha\}| \leq M.$$

In addition, we set

$$r_1 = \max\{r_0, a + 3\alpha\}.$$

Let $\nu = (\nu_x)$ be an ω-conformal density such that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0). Let $r \geq r_1$, $L > r + 13\alpha$ and $\ell \in \mathbb{R}_+$. According to Lemma 4.16, for every $g \in S(\ell, a)$, there is $u_g \in S(L, \alpha)$ such that gu_g belongs to $\mathcal{T}(\alpha, L)$. Moreover, there is a geodesic $\tau_g \colon [0\ , T_g] \to X$ joining o to $gu_g o$, whose restriction to $[T_g - L\ , T_g]$ is α-contracting, and such that go is α-close to $z_g = \tau_g(T_g - L)$. In particular,

$$|d(o, go) + d(o, u_g o) - d(o, gu_g o)| \leq 2\alpha.$$

Since $g \in S(\ell, a)$ and $u_g \in S(L, \alpha)$, we get $gu_g \in S(\ell + L, a + 3\alpha)$. Hence, $\mathcal{O}_o(gu_g o, r)$ is contained in $A_{\ell+L}(\alpha, r, L)$.

Consider now $g, g' \in S(\ell, a)$ such that $\mathcal{O}_o(gu_g o, r) \cap \mathcal{O}_o(g'u_{g'} o, r)$ is non-empty. By construction, go and $g'o$ are α-close to z_g and $z_{g'}$. Note that $z_g = \tau_g(T_g - L)$ is the (unique) projection of o on τ_g restricted to $[T_g - L\ , T_g]$, which is α-contracting. A similar statement holds for $\tau_{g'}$. Using Lemma 4.15 and the triangle inequality, we observe that

$$\begin{aligned} d(go, g'o) \leq d(z_g, z_{g'}) + 2\alpha &\leq |d(o, z_g) - d(o, z_{g'})| + 10\alpha \\ &\leq |d(o, go) - d(o, g'o)| + 12\alpha \\ &\leq 2a + 12\alpha. \end{aligned}$$

It follows from our choice of M that any cocycle $c \in \bar{X}$ belongs to at most M shadows of the form $\mathcal{O}_o(gu_g o, r)$, where $g \in S(\ell, a)$. Consequently,

$$\sum_{g \in S(\ell, a)} \nu_o(\mathcal{O}_o(gu_g o, r)) \leq M \nu_o(A_{\ell+L}(\alpha, r, L)). \tag{12}$$

Recall that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0). Hence, for every $g \in S(\ell, a)$, we have

$$\nu_o(\mathcal{O}_o(gu_g o, r)) \geq \epsilon \|\nu_{gu_g o}\| e^{-\omega d(o, gu_g o)} \geq \epsilon e^{-2\omega(L+\alpha)} \|\nu_{go}\| e^{-\omega d(o, go)}.$$

The second inequality follows from equation (2). Consequently, equation (12) becomes

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o, go)} \leq \left(\frac{M e^{2\omega\alpha}}{\epsilon}\right) e^{2\omega L} \nu_o(A_{\ell+L}(\alpha, r, L)). \qquad \square$$

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COROLLARY 4.18. *Assume that G contains a contracting element. For every $\omega, a \in \mathbb{R}_+$ and $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$, there exists $C \in \mathbb{R}_+^*$ with the following property. Let $\nu = (\nu_x)$ be an ω-conformal density. Assume that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0). For every $\ell \in \mathbb{R}_+$,*

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o, go)} \leq C \nu_o(\bar{X} \setminus B(o, \ell)).$$

Proof. Denote by α, r_1, C the parameters given by Proposition 4.17 applied with ω, a, and (ϵ, r_0). We choose $r \geq r_1$ and $L > \max\{2r, r + 15\alpha\}$. Let $\nu = (\nu_x)$ be an ω-conformal density such that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0). By Proposition 4.17, we have

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o, go)} \leq C e^{2\omega L} \nu_o(A_{\ell + L}(\alpha, r, L)) \quad \text{for all } \ell \in \mathbb{R}_+.$$

Consider now $x, y, z \in X$. By the very definition of shadows, if $z \in \mathcal{O}_x(y, r)$, then

$$d(x, y) - d(x, z) \leq \langle x, z \rangle_y \leq r.$$

Hence, for every $\ell \in \mathbb{R}_+$, the set $A_{\ell+L}(\alpha, r, L)$ lies in $\bar{X} \setminus B(o, \ell + L - 2r)$. The result follows from the fact that $L \geq 2r$. $\square$

4.6. *First applications.* For our first applications, we assume that G is a group acting properly, by isometries on X with a contracting element. In addition, we suppose that G is not virtually cyclic.

PROPOSITION 4.19. *There exists $C \in \mathbb{R}_+$ such that for every $\ell \in \mathbb{R}_+$,*

$$|\{g \in G \ : \ d(o, go) \leq \ell\}| \leq C e^{\omega_G \ell}.$$

Remark. An alternative proof of this fact can be found in [**50**].

Proof. According to Proposition 4.3, there exists a G-invariant, ω_G-conformal density $\nu = (\nu_x)$. By Corollary 4.10, the pair (G, ν) satisfies the shadow principle for some parameters $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$. Fix $a \in \mathbb{R}_+^*$. Applying Corollary 4.18, there exists $C \in \mathbb{R}_+$ such that for every $\ell \in \mathbb{R}_+$,

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega_G d(o, go)} \leq C \nu_o(\bar{X} \setminus B(o, \ell)) \leq C.$$

However, since ν is G-invariant, $\|\nu_{go}\| = \|\nu_o\| = 1$ for every $g \in G$. Hence, we get

$$|S(\ell, a)| \leq C e^{\omega_G a} e^{\omega_G \ell} \quad \text{for all } \ell \in \mathbb{R}_+.$$

The result follows by summing this inequality over $\ell \in a\mathbb{N}$. $\square$

Before stating our next application, we define the radial limit set for the action of G on X.

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Definition 4.20. Let $r \in \mathbb{R}_+$ and $\in X$. The set $\Lambda_{\mathrm{rad}}(G, x, r)$ consists of all cocycles $c \in \partial X$ with the following property: for every $T \geq 0$, there exists $g \in G$ with $d(x, go) \geq T$ such that $c \in \mathcal{O}_x(go, r)$. The *radial limit set* of G is the union

$$\Lambda_{\mathrm{rad}}(G) = \bigcup_{r \geq 0} G\Lambda_{\mathrm{rad}}(G, o, r).$$

Remark 4.21. Note that $\Lambda_{\mathrm{rad}}(G, x, r)$ is a non-decreasing function of r. The radial limit set $\Lambda_{\mathrm{rad}}(G)$ is G-invariant. It follows from equation (9) that

$$\Lambda_{\mathrm{rad}}(G) = \bigcup_{r \geq 0} \Lambda_{\mathrm{rad}}(G, o, r).$$

LEMMA 4.22. *The radial limit set is saturated.*

Proof. Let $x, y \in X$ and $r \in \mathbb{R}_+$. Let $c, c' \in \partial X$ such that $c \sim c'$. Note that if c belongs to $\mathcal{O}_x(y, r)$, then c' belongs to $\mathcal{O}_x(y, r')$, where $r' = r + \|c - c'\|_\infty$. The result follows from this observation. □

PROPOSITION 4.23. *Let $\omega \in \mathbb{R}_+$ and $\nu = (\nu_x)$ be an ω-conformal density. If (G, ν) satisfies the shadow principle, then the series*

$$\sum_{g \in G} \|\nu_{go}\| e^{-sd(o,go)} \tag{13}$$

converges whenever $s > \omega$. If, in addition, ν_o gives positive measure to the radial limit set $\Lambda_{\mathrm{rad}}(G)$, then the series diverges at $s = \omega$. In particular, its critical exponent is exactly ω.

Proof. Fix $a \in \mathbb{R}_+^*$. According to Corollary 4.18, there is $C \in \mathbb{R}_+^*$ such that for every $\ell \in \mathbb{R}_+$,

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o,go)} \leq C\nu_o(\bar{X} \setminus B(o, \ell)) \leq C.$$

Hence, the series in equation (13) converges whenever $s > \omega$. The second part of the proposition is proved by contraposition. Suppose that the series in equation (13) converges at $s = \omega$. Let $r \in \mathbb{R}_+$. For simplicity, we write $\Lambda = \Lambda_{\mathrm{rad}}(G, o, r)$. Observe that for every $T \in \mathbb{R}_+$,

$$\Lambda \subset \bigcup_{\substack{g \in G \\ d(o,go) \geq T}} \mathcal{O}_o(go, r).$$

It follows from Remark 4.11 that

$$\nu_o(\Lambda) \leq \sum_{\substack{g \in G \\ d(o,go) \geq T}} \nu_o(\mathcal{O}_o(go, r)) \leq e^{2\omega r} \sum_{\substack{g \in G \\ d(o,go) \geq T}} \|\nu_{go}\| e^{-\omega d(o,go)}.$$

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The right-hand side of the inequality is the remainder of the series in which we are interested. Since this series converges at $s = \omega$, we get $\nu_o(\Lambda) = 0$. We observed in Remark 4.21 that

$$\Lambda_{\mathrm{rad}}(G) = \bigcup_{r \geq 0} \Lambda_{\mathrm{rad}}(G, o, r).$$

Hence, $\nu_o(\Lambda_{\mathrm{rad}}(G)) = 0$. □

Remark 4.24. Note that the proof of the second assertion (the series diverges at $s = \omega$, whenever ν_o gives positive measure to the radial limit set) only uses the upper estimate from the shadow lemma. Hence, it holds even if G is virtually cyclic or does not contain a contracting element.

COROLLARY 4.25. *Let $\omega \in \mathbb{R}_+$. Let $\nu = (\nu_x)$ be a G-invariant, ω-conformal density. Then $\omega \geq \omega_G$. Moreover, if ν_o gives positive measure to the radial limit set $\Lambda_{\mathrm{rad}}(G)$, then $\omega = \omega_G$ and the action of G on X is divergent.*

Remark. If the action of G on X is proper and co-compact, one checks that the radial limit set is actually ∂X. Hence, if $\nu = (\nu_x)$ is a G-invariant, ω-conformal density supported on ∂X, then $\omega = \omega_G$.

Proof. By Corollary 4.10, the pair (G, ν) satisfies the shadow principle. Since the density ν is G-invariant, the series in equation (13) which appears in Proposition 4.23 is exactly the Poincaré series of G. The result follows. □

COROLLARY 4.26. *Let $\omega \in \mathbb{R}_+$. Let $\nu = (\nu_x)$ be an ω-conformal density and $\mu = (\mu_x)$ its restriction to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. Assume that (G, ν) satisfies the shadow principle. If μ is almost-fixed by G, then $\omega \geq \omega_G$.*

Proof. According to Lemma 4.4, the map $\chi : G \to \mathbb{R}_+$ sending g to $\ln \|\nu_{go}\|$ is a quasi-morphism. Note that the exponent of the series

$$\sum_{g \in G} \|\nu_{go}\| e^{-sd(o,go)} = \sum_{g \in G} e^{\chi(g)} e^{-sd(o,go)}$$

is exactly ω_χ. We observed earlier that $\omega_\chi \geq \omega_G$. The result follows from Proposition 4.23. □

COROLLARY 4.27. *Let N be a normal subgroup of G such that G/N is amenable. Then $\omega_N = \omega_G$.*

Proof. The proof follows Roblin's argument [42]. Assume first that N is finite. This forces G to be itself amenable. However, since G contains a contracting element, G has to be virtually cyclic (otherwise G would contain a free subgroup). Hence, $\omega_G = 0$ and the result holds. Suppose now that N is infinite. For simplicity, we let $Q = G/N$. Denote by $\nu = (\nu_x)$ an N-invariant, ω_N-conformal density on $\bar{X}$. We are going to 'average' the G-orbit of ν to produce another N-invariant, ω_N-conformal density. The construction goes as follows.

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Choose a Q-invariant mean $M: \ell^\infty(Q) \to \mathbb{R}$. Let $x \in X$ and $f \in C(\bar{X})$. Consider the function

$$\begin{aligned} \psi_{x,f}: \quad G &\to \quad \mathbb{R} \\ u &\mapsto \int f\, d\nu_x^u \end{aligned}$$

According to equation (2), we have $\|\nu_x^u\| \leq e^{\omega_N d(o,x)}$ for every $u \in G$. Consequently, $\psi_{x,f}$ is a bounded function. Since ν is N-invariant, we also observe that $\psi_{x,f}$ induces a map bounded map $Q \to \mathbb{R}$ that we still denote by $\psi_{x,f}$. Using Riesz representation theorem, we define a new measure μ_x by imposing

$$\int f\, d\mu_x = M(\psi_{x,f}) \quad \text{for all } f \in C(\bar{X}).$$

One checks that $\mu = (\mu_x)$ is still ω_N-conformal. Since N is a normal subgroup, it is also N-invariant. In particular, it satisfies the shadow principle (Corollary 4.10). According to Proposition 4.23, the series

$$\sum_{g \in G} \|\mu_{go}\| e^{-sd(o,go)} \tag{14}$$

converges whenever $s > \omega_N$. Consider now the map

$$\begin{aligned} \chi: \quad G &\to \quad \mathbb{R} \\ g &\mapsto M(\ln \psi_{go,\mathbb{1}}) \end{aligned}$$

Note that

$$\psi_{g_1 g_2 o,\mathbb{1}}(u) = \psi_{g_2 o,\mathbb{1}}(u g_1) \psi_{g_1 o,\mathbb{1}}(u) \quad \text{for all } u, g_1, g_2 \in G.$$

Since M is Q-invariant, χ is a homomorphism (factoring through the projection $G \twoheadrightarrow Q$). The exponential is a convex function, and hence the Jensen inequality yields

$$e^{\chi(g)} \leq \|\mu_{go}\| \quad \text{for all } g \in G.$$

Consequently,

$$\sum_{g \in G} \|\mu_{go}\| e^{-sd(o,go)} \geq \sum_{g \in G} e^{\chi(g)} e^{-sd(o,go)}.$$

The critical exponent of the sum on the right-hand side is ω_χ. Combined with the above discussion, it yields $\omega_N \geq \omega_\chi$. We observed previously that for a homomorphism χ, we have $\omega_\chi \geq \omega_G$, and thus $\omega_N \geq \omega_G$. The other inequality follows from the fact that $N \subset G$. □

Remark 4.28. In the proof, we have used the fact that N is normal in a crucial way to ensure that μ is N-invariant and therefore satisfies the shadow principle. It would be nice to generalize this strategy to the case of a *co-amenable* subgroup H, that is, when there is a G-invariant mean on the coset space $H\backslash G$. We know from the Patterson construction that H fixes a point ν in $\mathcal{D}(\omega_H)$. Since G acts on this compact space, it would be tempting to use the fixed point characterization of co-amenability due to Eymard [22, Exposé 1, §2] and conclude that G should fix a point μ in $\mathcal{D}(\omega_H)$. Even though μ may not be H-invariant, that would be enough to ensure that it satisfies the shadow principle. One has to be careful though that the action of G on $\mathcal{D}(\omega_H)$ is not affine. Hence, the fixed point criterion does not apply here. In this context, one can easily prove that the 'average' density μ we have built is G-quasi-invariant. Unfortunately, this information is not sufficient to prove that it satisfies the shadow principle.

COROLLARY 4.29. *Let N be an infinite, normal subgroup of G. Then*

$$\omega(N, X) + \tfrac{1}{2}\omega(G/N, X/N) \geq \omega(G, X).$$

Proof. Let $Q = G/N$. Denote by $\pi : G \twoheadrightarrow Q$ and $\zeta : X \to X/N$ the canonical projections. For simplicity, we write ω_N and ω_G for the growth rates of N and G acting on X, while ω_Q stands for the growth rate of Q acting on X/N. We denote by $\mathcal{H}$ the Hilbert space $\mathcal{H} = \ell^2(Q)$. Given $s, t \in \mathbb{R}_+$, we consider the following maps:

$$\begin{aligned} \phi_s : \ Q &\to \mathbb{R} \\ q &\mapsto \sum_{g \in \pi^{-1}(q)} e^{-sd(o,go)} \end{aligned} \quad \text{and} \quad \begin{aligned} \psi_t : \ Q &\to \mathbb{R} \\ q &\mapsto e^{-td(\zeta(o), q\zeta(o))} \end{aligned}$$

One checks that $\psi_t \in \mathcal{H}$ (respectively $\psi_t \notin \mathcal{H}$) whenever $2t > \omega_Q$ (respectively $2t < \omega_Q$). Similarly, $\phi_s \notin \mathcal{H}$, whenever $s < \omega_N$. We now prove that the converse essentially holds true.

Claim 4.30. If $s > \omega_N$, then $\phi_s \in \mathcal{H}$.

Let $s > \omega_N$. Consider the N-invariant, s-conformal density $\nu^s = (\nu^s_x)$ on $\bar{X}$ defined by

$$\nu^s_x = \frac{1}{\mathcal{P}_N(s)} \sum_{h \in N} e^{-sd(x,ho)} \mathrm{Dirac}(ho).$$

The computation shows that

$$\begin{aligned} \|\phi_s\|^2 = \sum_{\substack{g_1, g_2 \in G \\ \pi(g_1) = \pi(g_2)}} e^{-s[d(o,g_1o) + d(o,g_2o)]} &= \sum_{g \in G} e^{-sd(o,go)} \left(\sum_{h \in N} e^{-sd(go,ho)} \right) \\ &= \mathcal{P}_N(s) \sum_{g \in G} \|\nu^s_{go}\| e^{-sd(o,go)}. \end{aligned}$$

Fix $a \in \mathbb{R}^*_+$. According to Corollary 4.10, ν^s satisfies the shadow principle. By Corollary 4.18, there exists $C \in \mathbb{R}^*_+$ such that for every $\ell \in \mathbb{R}_+$,

$$\sum_{g \in S(\ell,a)} \|\nu^s_{go}\| e^{-sd(o,go)} \leq C \nu^s_o(\bar{X} \setminus B(o, \ell)).$$

Unfolding the definition of ν^s, we get

$$\mathcal{P}_N(s) \sum_{g \in S(\ell,a)} \|\nu^s_{go}\| e^{-sd(o,go)} \leq C \sum_{\substack{h \in N \\ d(o,ho) \geq \ell}} e^{-sd(o,ho)}.$$

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It follows that

$$\|\phi_s\|^2 \leq C \sum_{k\in\mathbb{N}} \sum_{\substack{h\in N \\ d(o,ho)\geq ka}} e^{-sd(o,ho)} \leq C \sum_{h\in N} \sum_{\substack{k\in\mathbb{N} \\ ka\leq d(o,ho)}} e^{-sd(o,ho)}.$$

Up to replacing C by a larger constant, we have

$$\|\phi_s\|^2 \leq C \sum_{h\in N} [1 + d(o, ho)]e^{-sd(o,ho)}. \tag{15}$$

Note that the series

$$-\sum_{h\in N} d(o, ho)e^{-sd(o,ho)}$$

is the derivative of the Poincaré series of N, and hence it converges. Consequently, the right-hand side in equation (15) converges. Thus, $\|\phi_s\|$ is finite, which completes the proof of our claim.

Let $s, t \in \mathbb{R}_+$ with $s > \omega_N$ and $2t > \omega_Q$. The scalar product of ϕ_s and ψ_t can be computed as follows:

$$\begin{aligned}(\phi_s, \psi_t) &= \sum_{q\in Q} e^{-td(\zeta(o),q\zeta(o))} \left(\sum_{g\in\pi^{-1}(q)} e^{-sd(o,go)} \right) \\ &= \sum_{g\in G} e^{-sd(o,go)} e^{-td(\zeta(o),\pi(g)\zeta(o))}.\end{aligned}$$

The projection $\zeta\colon X \twoheadrightarrow X/N$ is 1-Lipschitz. Combined with the Cauchy–Schwartz inequality, it gives

$$\mathcal{P}_G(s + t) \leq (\phi_s, \psi_t) \leq \|\phi_s\|\|\psi_t\| < \infty.$$

Consequently, $s + t \geq \omega_G$. This inequality holds for every $s, t \in \mathbb{R}_+$ with $s > \omega_N$ and $2t > \omega_Q$, whence the result. □

Remark. We keep the notation of the proof. Corollary 4.29 is sharp.

- Indeed, assume that X is a Cayley graph of G. If Q has subexponential growth, then $\omega_Q = 0$ and Q is amenable, so that $\omega_N = \omega_G$ by Corollary 4.27.
- At the other end of the spectrum, assume that $G = \mathbf{F}_r$ is the free group of rank r acting on its Cayley graph X with respect to a free basis. Let $g \in G \setminus \{1\}$. For every $k \in \mathbb{N}$, denote by N_k the normal closure of g^k. Then

  $$\lim_{k\to\infty} \omega_{N_k} = \tfrac{1}{2}\omega_G \quad \text{and} \quad \lim_{k\to\infty} \omega_{G/N_k} = \omega_G.$$

  The first limit is due to Grigorchuk [**24**], see also Champetier [**10**]. The second limit was proved by Shukhov [**45**], see also Coulon [**16**].

5. *Divergent actions*

5.1. *Contracting tails, continued.* We continue here our study of shadows of elements $g \in \mathcal{T}(\alpha, L)$ having a contracting tail. First we recall the proof of the following classical result.

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LEMMA 5.1. *Let $C \in \mathbb{R}_+$. Let $x, y \in X$ and $\gamma: [a, b] \to X$ be a continuous path from x to y. Let $\nu: I \to X$ be a geodesic. Denote by $p = \nu(c)$ and $q = \nu(d)$ respective projections of x and y on ν, and suppose that $c \leq d$. Assume in addition that γ lies in the C-neighborhood of ν. Then ν restricted to $[c, d]$ lies in the $2C$-neighborhood of γ.*

Proof. Fix a point $r = \nu(t_0)$ on ν where $t_0 \in [c, d]$. Consider the set J consisting of all $s \in [a, b]$ such that $\gamma(s)$ admits a projection on ν of the form $\nu(t)$ with $t \leq t_0$. Note that J is non-empty as it contains a. Now set $s_0 = \sup J$. Let $\epsilon > 0$. Since γ is continuous, there is $\eta > 0$ such that for every $s \in [a, b]$, we have $d(\gamma(s), \gamma(s_0)) \leq \epsilon$ provided $|s - s_0| \leq \eta$. By definition of J, there is $s_1 \in (s_0 - \eta, s_0]$ such that $\gamma(s_1)$ admits a projection on ν of the form $\nu(t_1)$ with $t_1 \leq t_0$. However, we claim that there is $s_2 \in [s_0, s_0 + \eta)$ such that $\gamma(s_2)$ admits a projection on ν of the form $\nu(t_2)$ with $t_2 \geq t_0$. Indeed, if $s_0 = b$, we can simply take $s_2 = b$. Otherwise, it follows from the definition of J. By assumption, $d(\nu(t_i), \gamma(s_i)) \leq C$. Hence, the triangle inequality yields

$$d(\nu(t_1), \nu(t_2)) \leq d(\gamma(s_1), \gamma(s_2)) + 2C \leq 2C + 2\epsilon$$

and then

$$d(r, \gamma) \leq \tfrac{1}{2} d(\nu(t_1), \nu(t_2)) + \max\{d(\nu(t_1), \gamma(s_1)), d(\nu(t_2), \gamma(s_2))\} \leq 2C + \epsilon.$$

This inequality holds for every $\epsilon > 0$, whence the result. □

LEMMA 5.2. *Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 16\alpha$. Let $x, y_1, y_2 \in X$ with $d(x, y_1) \leq d(x, y_2)$. Assume that the pairs (x, y_1) and (x, y_2) have an (α, L)-contracting tail. If $\mathcal{O}_x(y_1, r) \cap \mathcal{O}_x(y_2, r)$ is not empty, then $\mathcal{O}_x(y_2, r)$ is contained in $\mathcal{O}_x(y_1, r + 42\alpha)$.*

Proof. By assumption, there is an α-contracting geodesic τ_i ending at y_i and a projection p_i of x on τ_i satisfying $d(p_i, y_i) \geq L$. In addition, we write z_i for the point on τ_i satisfying $d(y_i, z_i) = r + 13\alpha$. We split the proof in several claims.

Claim 5.3. Let $c \in \mathcal{O}_x(y_i, r)$ and $\gamma: I \to X$ a gradient line from x to c. Then z_i lies in the 5α-neighborhood of γ. In particular, $c \in \mathcal{O}(z_i, 5\alpha)$.

According to Lemma 4.14, γ intersects $\mathcal{N}_\alpha(\tau_i)$. Denote by $\gamma(s_i)$ and $\gamma(t_i)$ the entry and exit point of γ in $\mathcal{N}_\alpha(\tau_i)$. We know, again from Lemma 4.14, that $d(\gamma(s_i), p_i) \leq 2\alpha$ and $d(\gamma(t_i), y_i) \leq r + 12\alpha$. In particular, if m_i and n_i stand for projections of $\gamma(s_i)$ and $\gamma(t_i)$ on τ_i, respectively, then m_i, z_i, n_i, and y_i are aligned in this order along τ_i. The path γ restricted to $[s_i, t_i]$ lies in the $5\alpha/2$-neighborhood of τ (Lemma 2.2). It follows that z_i is 5α-close to γ (Lemma 5.1).

Claim 5.4. We have $\langle x, z_2 \rangle_{z_1} \leq 24\alpha$.

According to our assumption, $\mathcal{O}_x(y_1, r) \cap \mathcal{O}_x(y_2, r)$ is not empty. Let $\gamma: I \to X$ be a gradient arc from x to a cocycle c in this intersection. It follows from our previous claim that there is $u_i \in \mathbb{R}_+$ such that $\gamma(u_i)$ is 5α-close to z_i. Hence, the triangle inequality yields

$$\langle x, z_2 \rangle_{z_1} \leq \langle x, \gamma(u_2) \rangle_{\gamma(u_1)} + 10\alpha. \tag{16}$$

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Observe that

$$d(x, z_1) + d(z_1, y_1) - 4\alpha \le d(x, y_1) \le d(x, y_2) \le d(x, z_2) + d(z_2, y_2).$$

Indeed, the first inequality follows from Lemma 2.3 applied to τ_1, the second one is part of our assumptions, while the last one is just the triangle inequality. By construction, $d(y_1, z_1) = d(y_2, z_2)$. Combined with the triangle inequality, it yields

$$d(x, \gamma(u_1)) \le d(x, \gamma(u_2)) + 14\alpha.$$

Therefore, $\langle x, \gamma(u_2)\rangle_{\gamma(u_1)} \le 14\alpha$, which combined with equation (16) completes the proof of our second claim.

Claim 5.5. $\mathcal{O}_x(y_2, r)$ is contained in $\mathcal{O}_x(y_1, r + 42\alpha)$.

Consider $c \in \mathcal{O}_x(y_2, r)$. According to our first claim, $\langle x, c\rangle_{z_2} \le 5\alpha$. Since cocyles are 1-Lipschitz, we get, in combination with the previous claim,

$$\langle x, c\rangle_{z_1} \le \langle x, c\rangle_{z_2} + \langle x, z_2\rangle_{z_1} \le 29\alpha.$$

However, $d(y_1, z_1) = r + 13\alpha$. It follows from equation (8) that $\langle x, c\rangle_{y_1} \le r + 42\alpha$. □

The next statement will be used later to estimate the measures of various saturated sets using a Vitali-type argument.

LEMMA 5.6. *Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 16\alpha$. Let S be a subset of $\mathcal{T}(\alpha, L)$. There is a subset $S^* \subset S$ with the following properties.*

(i) *The collection $(\mathcal{O}_o(go, r))_{g\in S^*}$ is pairwise disjoint.*
(ii) $\bigcup_{g\in S} \mathcal{O}_o(go, r) \subset \bigcup_{g\in S^*} \mathcal{O}_o(go, r + 42\alpha)$.

Proof. Since the action of G on X is proper, we can index the elements $g_0, g_1, g_2 \dots$ of S such that $d(o, g_i o) \le d(o, g_{i+1} o)$ for every $i \in \mathbb{N}$. We build by induction a sequence (i_n) as follows. Set $i_0 = 0$. Let $n \in \mathbb{N}$ for which i_n has been defined. We search for the minimal index $j > i_n$ such that

$$\left(\bigcup_{k=0}^{n} \mathcal{O}_o(g_{i_k} o, r)\right) \cap \mathcal{O}_o(g_j o, r) = \emptyset.$$

If such an index exists, we let $i_{n+1} = j$. Otherwise, we let $i_{n+1} = i_n$, in other words, the sequence (i_n) eventually stabilizes. Finally, we set

$$S^* = \{g_{i_n} \ : \ n \in \mathbb{N}\}.$$

Note that property (i) directly follows from the construction. Let $g \in S$. If g does not belong to S^*, it means that there is $h \in S^*$, with $d(o, ho) \le d(o, go)$ such that $\mathcal{O}_o(go, r) \cap \mathcal{O}_o(ho, r)$ is non-empty. Hence, by Lemma 5.2, the shadow $\mathcal{O}_o(go, r)$ is contained in $\mathcal{O}_o(ho, r + 42\alpha)$. This completes the proof of property (ii). □

LEMMA 5.7. *Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 13\alpha$. Let $x, y \in X$ such that (x, y) has an (α, L)-contracting tail. Let K be a closed ball of radius R centered at x. If $d(x, y) > R + r + 13\alpha$, then for every $c, c' \in \mathcal{O}_x(y, r)$, we have $\|c - c'\|_K \leq 20\alpha$.*

Proof. Denote by τ a contracting tail of (x, y) and let p be a projection of x on τ such that $d(p, y) \geq L$. Fix $x' \in K$ and denote by p' a projection of x' onto τ. Let q be a projection of c on τ, so that $d(y, q) \leq r + 7\alpha$ (Lemma 4.14). We claim that $d(p', q) > 4\alpha$. If $d(p, p') \leq \alpha$, then this is just a consequence of the triangle inequality. Thus, we can assume that $d(p, p') > \alpha$. According to Lemma 2.3, we have $d(x, p') \leq d(x, x') + 2\alpha$. Hence,

$$\begin{aligned} d(p', q) \geq d(x, y) - d(x, p') - d(q, y) &\geq d(x, y) - d(x, x') - d(q, y) - 2\alpha \\ &\geq d(x, y) - (R + r + 9\alpha) > 4\alpha. \end{aligned}$$

It follows from Corollary 3.13 that

$$d(x', q) - 10\alpha \leq c(x', q) \leq d(x', q).$$

This estimate holds for every $x' \in K$. Consider now $x_1, x_2 \in K$. Since c is a cocycle, $c(x_1, x_2) = c(x_1, q) + c(q, x_2)$, thus

$$|c(x_1, x_2) - [d(x_1, q) - d(x_2, q)]| \leq 10\alpha.$$

The same argument holds for c', thus $\|c - c'\|_K \leq 20\alpha$. □

5.2. *The contracting limit set*

5.2.1. *Contracting limit set.* We now introduce a variation of the radial limit set that keeps track of the elements fellow-traveling with some contracting geodesics. Let $\alpha, r, L \in \mathbb{R}_+$ and $x \in X$. The set $\Lambda_{\text{ctg}}(G, x, \alpha, r, L)$ consists of all cocycles $c \in \partial X$ with the following property: for every $T \geq 0$, there exists $g \in G$ such that:

- $d(x, go) \geq T$;
- (x, go) has an (α, L)-contracting tail;
- $c \in \mathcal{O}_x(go, r)$.

We also let

$$\Lambda_{\text{ctg}}(G, x, \alpha, r) = \bigcap_{L \in \mathbb{R}_+} \Lambda_{\text{ctg}}(G, x, \alpha, r, L).$$

Remark. Observe that the set $\Lambda_{\text{ctg}}(G, x, \alpha, r, L)$ is a non-decreasing (respectively non-increasing) function of α and r (respectively L).

Definition 5.8. The *contracting limit set* of G is

$$\Lambda_{\text{ctg}}(G) = \bigcup_{\alpha, r \in \mathbb{R}_+} G\Lambda_{\text{ctg}}(G, o, \alpha, r).$$

It follows from the definition that the contracting limit set is G-invariant and contained in the radial limit set.

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PROPOSITION 5.9. *Let $\alpha \in \mathbb{R}_+^*$. For every $r \in \mathbb{R}_+$ and $L > r + 13\alpha$, we have*

$$G\Lambda_{\mathrm{ctg}}(G, o, \alpha, r, L) \subset \Lambda_{\mathrm{ctg}}(G, o, \alpha, r + 12\alpha, L - \alpha).$$

In particular, $G\Lambda_{\mathrm{ctg}}(G, o, \alpha, r) \subset \Lambda_{\mathrm{ctg}}(G, o, \alpha, r + 12\alpha)$. Moreover, the contracting limit set can also be described as

$$\Lambda_{\mathrm{ctg}}(G) = \bigcup_{\alpha, r \in \mathbb{R}_+} \Lambda_{\mathrm{ctg}}(G, o, \alpha, r).$$

Proof. Let $h \in G$ and c be a cocycle in

$$h\Lambda_{\mathrm{ctg}}(G, o, \alpha, r, L) = \Lambda_{\mathrm{ctg}}(G, ho, \alpha, r, L).$$

For simplicity, we let $b = h^{-1}c$. For every $n \in \mathbb{N}$, we can find an element $g_n \in \mathcal{T}(\alpha, L)$ such that $d(o, g_n o) \geq n$ and $b \in \mathcal{O}_o(g_n o, r)$. Denote by τ_n the contracting tail for the pair $(o, g_n o)$ and p_n a projection of o on τ_n satisfying $d(p_n, g_n o) \geq L$. Up to passing to a subsequence, we can assume that $d(o, g_n o) \geq d(o, h^{-1}o) + L + \alpha$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. We claim that hg_n belongs to $\mathcal{T}(\alpha, L - \alpha)$. Let p'_n be a projection of $h^{-1}o$ on τ_n. It suffices to prove that $d(p'_n, g_n o) \geq L - \alpha$. Indeed, after translating the figure by h, it tells us that $h\tau_n$ is a contracting tail for the pair $(o, hg_n o)$. We distinguish two cases. Assume first that $d(p_n, p'_n) > \alpha$. It follows from Lemma 2.3 that $d(o, p'_n) \leq d(o, h^{-1}o) + 2\alpha$. The triangle inequality yields

$$d(p'_n, g_n o) \geq d(o, g_n o) - d(o, p'_n) \geq d(o, g_n o) - d(o, h^{-1}o) - 2\alpha \geq L - \alpha.$$

Assume now that $d(p_n, p'_n) \leq \alpha$. The triangle inequality yields

$$d(p'_n, g_n o) \geq d(p_n, g_n o) - d(p_n, p'_n) \geq L - \alpha,$$

which completes the proof of our claim.

We now prove that $c \in \mathcal{O}_o(hg_n o, r + 18\alpha)$. Let q_n be a projection of b onto τ_n. According to Lemma 4.14, we have $d(q_n, g_n o) \leq r + 7\alpha$. Combining the above discussion with the triangle inequality, we have

$$d(p'_n, q_n) \geq d(p'_n, g_n o) - d(q_n, g_n o) \geq L - (r + 8\alpha) > 4\alpha.$$

On the one hand, by Corollary 3.13, we have

$$b(h^{-1}o, q_n) \geq d(h^{-1}o, q_n) - 10\alpha,$$

that is, $\langle h^{-1}o, b\rangle_{q_n} \leq 5\alpha$. Consequently,

$$\langle h^{-1}o, b\rangle_{g_n o} \leq \langle h^{-1}o, b\rangle_{q_n} + d(q_n, g_n o) \leq r + 12\alpha.$$

Recall that $b = h^{-1}c$. The above inequality implies that c belongs to the shadow $\mathcal{O}_o(hg_n o, r + 12\alpha)$, which completes the proof of our claim. Note that $d(o, hg_n o)$ diverges to infinity, and hence $c \in \Lambda_{\mathrm{ctg}}(G, o, \alpha, r + 12\alpha, L - \alpha)$. □

The points in the contracting limit set have a specific behavior with respect to the equivalence relation $\sim$ used to define the reduced horoboundary (see §3.2). Proceeding as for the radial limit set (see Lemma 4.22), one checks that the contracting limit set is saturated. The next statement precises this fact.

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PROPOSITION 5.10. *Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 29\alpha$. Let $c, c' \in \partial X$ such that $c \sim c'$. Assume that c belongs to $\Lambda_{\text{ctg}}(G, o, \alpha, r, L)$. Then $\|c - c'\|_\infty \leq 20\alpha$ and c' belongs to $\Lambda_{\text{ctg}}(G, o, \alpha, r + 16\alpha, L)$. In particular, the saturation of $\Lambda_{\text{ctg}}(G, o, \alpha, r)$ is contained in $\Lambda_{\text{ctg}}(G, o, \alpha, r + 16\alpha)$.*

Proof. Since c belongs to $\Lambda_{\text{ctg}}(G, o, \alpha, r, L)$, there exists a sequence of elements $g_n \in \mathcal{T}(\alpha, L)$ such that $d(o, g_n o)$ diverges to infinity and $c \in \mathcal{O}_o(g_n o, r)$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the pair $(o, g_n o)$ has an (α, L)-contracting tail, say τ_n. Let q_n (respectively q'_n) be a projection of c (respectively c') onto τ_n. We break the proof into several steps.

Claim 5.11. For every $n \in \mathbb{N}$, we have $d(q_n, q'_n) \leq 4\alpha$.

Fix $n \in \mathbb{N}$. Let $\gamma : \mathbb{R}_+ \to X$ be a gradient ray for c starting at q_n. Let $t \in \mathbb{R}_+$. According to Lemma 3.9, q_n is a projection of $\gamma(t)$ on τ_n. Assume that, contrary to our claim, $d(q'_n, q_n) > 4\alpha$. It follows from Corollary 3.13 applied with c' that

$$c'(\gamma(t), q'_n) \geq d(q'_n, \gamma(t)) - 10\alpha \geq -10\alpha.$$

On the other hand, since γ is a gradient ray for c, we know that

$$c(\gamma(t), q'_n) = c(\gamma(t), q_n) + c(q_n, q'_n) = -t + c(q_n, q'_n).$$

These two estimates hold for every $t \geq T$, thus contradicting the fact that $\|c - c'\|_\infty < \infty$. It completes the proof of our first claim.

Claim 5.12. For every $n \in \mathbb{N}$, the cocycle c' belongs to $\mathcal{O}_o(g_n o, r + 16\alpha)$. In particular, $c' \in \Lambda_{\text{ctg}}(G, o, \alpha, r + 16\alpha, L)$.

Let $n \in \mathbb{N}$. Since $(o, g_n o)$ has a contracting tail, there is a projection p_n of o on τ_n such that $d(p_n, g_n o) \geq L$. According to Lemma 4.14, $d(q_n, g_n o) \leq r + 7\alpha$. Our previous claim, combined with the triangle inequality, gives

$$d(p_n, q'_n) \geq d(p_n, g_n o) - d(q'_n, q_n) - d(q_n, g_n o) \geq L - (r + 11\alpha) > 4\alpha.$$

Hence, by Corollary 3.13, we have $\langle o, c' \rangle_{q'_n} \leq 5\alpha$, which combined with equation (8) yields

$$\langle o, c' \rangle_{g_n o} \leq \langle o, c' \rangle_{q'_n} + d(q'_n, q_n) + d(q_n, g_n o) \leq r + 16\alpha.$$

This completes the proof of our second claim.

Claim 5.13. $\|c - c'\|_\infty \leq 20\alpha$.

Consider a closed ball K of radius R centered at o. If $n \in \mathbb{N}$ is sufficiently large, then $d(o, g_n o) > R + r + 29\alpha$. According to our previous claim, c and c' both belong to $\mathcal{O}_o(g_n o, r + 16\alpha)$. It follows from Lemma 5.7 that $\|c - c'\|_K \leq 20\alpha$. This estimate does not depend on K, whence the result. □

In a CAT(-1) settings, shadows are known to provide a basis of open neighborhoods for the points in the boundary at infinity. This is no more the case in our context. However, we can still approximate *saturated* subsets of the contracting limit set using shadows. This is the purpose of the next lemma.

COROLLARY 5.14. *Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 13\alpha$. Let B be a saturated subset of $\Lambda_{\mathrm{ctg}}(G, o, \alpha, r, L)$ and V an open subset of $\bar{X}$ containing B. Let $b \in B$. There exists $T \in \mathbb{R}_+$ such that for every $g \in \mathcal{T}(\alpha, L)$ with $d(o, go) \geq T$, if b belongs to $\mathcal{O}_o(go, r)$, then $\mathcal{O}_o(go, r) \subset V$.*

Proof. Assume on the contrary that our statement fails. We can find a sequence of elements $g_n \in \mathcal{T}(\alpha, L)$ such that $d(o, g_n o)$ diverges to infinity, b belongs to $\mathcal{O}_o(g_n o, r)$, and $\mathcal{O}_o(g_n o, r) \setminus V$ is non-empty. For every $n \in \mathbb{N}$, we write c_n for a cocycle in $\mathcal{O}_o(g_n o, r) \setminus V$. Up to passing to a subsequence, we can assume that c_n converges to $c \in \bar{X}$. As V is open, $c \notin V$. We claim that $\|c - b\|_\infty < \infty$. Let K be a closed ball of radius R centered at o. As c_n converges to c, there is $N \in \mathbb{N}$ such that for every $n \geq N$, we have $\|c_n - c\|_K \leq 1$. By construction, b and c_n both belong to $\mathcal{O}_o(g_n o, r)$. According to Lemma 5.7, if n is sufficiently large, $\|b - c_n\|_K \leq 20\alpha$ so that $\|b - c\|_K \leq 20\alpha + 1$. This inequality holds for every compact subset $K \subset X$, which completes the proof of our claim. Since B is saturated, $c \in B$. It contradicts the fact that $B \subset V$. □

5.3. *Measure of the contracting limit set.* From now on, we assume that G is not virtually cyclic and contains a contracting element. According to Corollary 4.10, there are $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ such that any G-invariant, ω_G-conformal density $\nu = (\nu_x)$ satisfies the shadow principle with parameters (ϵ, r_0). The goal of this section is to prove that if the action of G on X is divergent, then any such density gives full measure to the contracting limit set. The proof is an application of the Kochen–Stone theorem, which generalizes the second Borel–Cantelli lemma.

PROPOSITION 5.15. (Kochen–Stone [31]) *Let (Ω, μ) be a probability space. Let (B_n) be a sequence of subsets of Ω such that*

$$\sum_{n \in \mathbb{N}} \mu(B_n) = \infty.$$

Assume that there exists $C \in \mathbb{R}_+^*$ such that for every $N \in \mathbb{N}$,

$$\sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \mu(B_{n_1} \cap B_{n_2}) \leq C \left(\sum_{n=0}^{N} \mu(B_n) \right)^2.$$

Then

$$\mu\left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} B_n \right) \geq \frac{1}{C}.$$

Recall that for every $\alpha, r, \ell, L \in \mathbb{R}_+$, the set $A_\ell(\alpha, r, L)$ is defined by

$$A_\ell(\alpha, r, L) = \bigcup_{g \in S(\ell, r) \cap \mathcal{T}(\alpha, L)} \mathcal{O}_o(go, r).$$

These are the sets with which we will apply the Kochen–Stone theorem. The aim of the next lemmas is to make sure that the hypotheses of Proposition 5.15 are satisfied.

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LEMMA 5.16. *Let $a \in \mathbb{R}_+^*$. There are $\alpha, r_1, C_1 \in \mathbb{R}_+^*$ such that for every $r \geq r_1$ and $L > r + 13\alpha$, the following holds. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. For every integer $N \in \mathbb{N}$,*

$$\sum_{\substack{g \in G \\ d(o,go) \leq Na - L}} e^{-\omega_G d(o,go)} \leq C_1 e^{2\omega_G L} \sum_{n=0}^{N} \nu_o(B_n),$$

where $B_n = A_{na}(\alpha, r, L)$.

Proof. Recall that (ϵ, r_0) are the shadow principle parameters which have been fixed once and for all at the beginning of §5.3. We denote by $\alpha, r_1, C \in \mathbb{R}_+$ the parameters given by Proposition 4.17 applied with ω_G, a, and (ϵ, r_0). Fix $r \geq r_1$ and $L > r + 13\alpha$. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. Recall that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0). It follows from Proposition 4.17 that for every $\ell \in \mathbb{R}_+$,

$$\sum_{g \in S(\ell,a)} e^{-\omega_G d(o,go)} \leq C e^{2\omega L} \nu_o(A_{\ell+L}(\alpha, r, L)).$$

Summing this identity, we get for every $N \in \mathbb{N}$,

$$\sum_{n=0}^{N} \sum_{g \in S(na,a)} e^{-\omega_G d(o,go)} \leq C e^{2\omega L} \sum_{n=0}^{N} \nu_o(A_{na+L}(\alpha, r, L)).$$

Note that $A_{na+L}(\alpha, r, L)$ is covered by $A_{(m-1)a}(\alpha, r, L)$ and $A_{ma}(\alpha, r, L)$, where $m = n + \lceil L/a \rceil$. Hence,

$$\sum_{\substack{g \in G \\ d(o,go) \leq (N+1)a}} e^{-\omega_G d(o,go)} \leq 2C e^{2\omega L} \sum_{n=0}^{N+\lceil L/a \rceil} \nu_o(B_n),$$

whence the result. □

LEMMA 5.17. *Let $a, \alpha, r \in \mathbb{R}_+^*$. There are $b, C_2 \in \mathbb{R}_+$ such that for every $L > r + 13\alpha$, the following holds. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. For every $N \in \mathbb{N}$, we have*

$$\sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \nu_o(B_{n_1} \cap B_{n_2}) \leq C_2 \Bigg(\sum_{\substack{g \in G \\ d(o,go) \leq Na+b}} e^{-\omega_G d(o,go)} \Bigg)^2,$$

where $B_n = A_{na}(\alpha, r, L)$.

Proof. Let $L > r + 13\alpha$. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. Let $N \in \mathbb{N}$. Observe first that

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$$\sum_{n_1=0}^{N}\sum_{n_2=0}^{N} \nu_o(B_{n_1}\cap B_{n_2}) \leq 2\sum_{n_1=0}^{N}\sum_{n_2=n_1}^{N} \nu_o(B_{n_1}\cap B_{n_2})$$
$$\leq 2\sum_{n_1=0}^{N}\sum_{n_2=0}^{N-n_1} \nu_o(B_{n_1}\cap B_{n_1+n_2}). \tag{17}$$

Consider now $n_1, n_2 \in \mathbb{N}$ with $0 \leq n_1 \leq n_1 + n_2 \leq N$. By definition, we have

$$B_{n_1}\cap B_{n_1+n_2} = \bigcup_{(g_1,g_2)\in U} \mathcal{O}_o(g_1o, r)\cap \mathcal{O}_o(g_1g_2o, r),$$

where U is the set of pairs $(g_1, g_2) \in G$ with the following properties:

(U1) $g_1, g_1g_2 \in \mathcal{T}(\alpha, L)$;

(U2) $g_1 \in S(n_1a, r)$ and $g_1g_2 \in S(n_1a + n_2a, r)$.

Let $(g_1, g_2) \in U$ for which $\mathcal{O}_o(g_1o, r)\cap \mathcal{O}_o(g_1g_2o, r)$ is non-empty. According to Lemma 4.15, we have

$$d(o, g_2o) \leq d(g_1o, g_1g_2o) \leq |d(o, g_1g_2o) - d(o, g_1o)| + 4r + 48\alpha.$$

In particular,

$$d(o, g_1g_2o) \geq d(o, g_1o) + d(o, g_2o) - 4r - 48\alpha.$$

Moreover, combined with property (U2), it shows that $g_2 \in S(n_2a, 6r + 48\alpha)$. Using Remark 4.11, we estimate the measure of each shadow:

$$\begin{aligned}\nu_o(\mathcal{O}_o(g_1o, r)\cap \mathcal{O}_o(g_1g_2o, r)) &\leq \nu_o(\mathcal{O}_o(g_1g_2o, r))\\ &\leq e^{2\omega_G r}e^{-\omega_G d(o,g_1g_2o)}\\ &\leq e^{\omega_G(6r+48\alpha)}e^{-\omega_G[d(o,g_1o)+d(o,g_2o)]}.\end{aligned}$$

Consequently,

$$\nu_o(B_{n_1}\cap B_{n_1+n_2}) \leq e^{\omega_G(6r+48\alpha)} \sum_{\substack{g_1\in S(n_1a,r),\\ g_2\in S(n_2a,6r+48\alpha)}} e^{-\omega_G[d(o,g_1o)+d(o,g_2o)]}.$$

Note that for every $d \in \mathbb{R}_+$, an element $g \in G$ belongs to at most $\lceil 2d/a\rceil$ spheres of the form $S(na, d)$ when n runs over $\mathbb{N}$. Summing the previous inequality over n_1 and n_2, and using equation (17), we get

$$\begin{aligned}\sum_{n_1=0}^{N}\sum_{n_2=0}^{N} \nu_o(B_{n_1}\cap B_{n_2}) &\leq C_2 \sum_{\substack{g_1,g_2\in G\\ d(o,g_1o),d(o,g_2o)\leq Na+b}} e^{-\omega_G[d(o,g_1o)+d(o,g_2o)]}\\ &\leq C_2\Bigg(\sum_{\substack{g\in G\\ d(o,go)\leq Na+b}} e^{-\omega_G d(o,go)}\Bigg)^2,\end{aligned}$$

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where

$$b = a + 6r + 48\alpha \quad \text{and} \quad C_2 = 2e^{\omega_G(6r+48\alpha)}\left\lceil \frac{12r + 96\alpha}{a} \right\rceil^2$$

only depend on a, α, and r. □

PROPOSITION 5.18. *Assume that the action of G on X is divergent. There exists $\alpha, r \in \mathbb{R}_+^*$ with the following property. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. For every $L > r + 13\alpha$, we have*

$$\nu_0(\Lambda_{\text{ctg}}(G, o, \alpha, r, L)) > 0.$$

Proof. Fix $a \in \mathbb{R}_+^*$. Let α, r_1, C_1 be the parameters given by Lemma 5.16. Fix $r \geq r_1$. Let b, C_2 be the parameters given by Lemma 5.17 applied with a, α, and r. Choose $L > r + 13\alpha$ and set $a' = \max\{a, b, L\}$. We write C for the constant given by Corollary 4.18 applied with ω_G, a', and (ϵ, r_0).

Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. For simplicity, for every $n \in \mathbb{N}$, we let $B_n = A_{na}(r, \alpha, L)$. Since the action of G is divergent, Lemma 5.16 tells us that

$$\sum_{n\in\mathbb{N}} \nu_o(B_n) = \infty.$$

Recall that (G, ν) satisfies the shadow principle with parameters (ϵ, r_0). By Corollary 4.18,

$$\sum_{g\in S(\ell,a')} e^{-\omega_G d(o,go)} \leq C \quad \text{for all } \ell \in \mathbb{R}_+.$$

Since the Poincaré series of G diverges at $s = \omega_G$, we deduce from Lemmas 5.16 and 5.17 that there exists $C' \in \mathbb{R}_+^*$ such that for every sufficiently large $N \in \mathbb{N}$,

$$\sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \nu_o(B_{n_1} \cap B_{n_2}) \leq C' \left(\sum_{n=0}^{N} \nu_o(B_n) \right)^2.$$

Applying Proposition 5.15, we observe that

$$\nu_0(\Lambda_{\text{ctg}}(G, o, \alpha, r, L)) = \nu_o\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} B_n \right) \geq \frac{1}{C'}.$$ □

COROLLARY 5.19. *Assume that the action of G on X is divergent. There exist $\alpha, r \in \mathbb{R}_+^*$ with the following property. If $\nu = (\nu_x)$ is a G-invariant, ω_G-conformal density, then ν_o gives full measure to*

$$\Lambda_{\text{ctg}}(G, o, \alpha, r) = \bigcap_{L\in\mathbb{R}_+} \Lambda_{\text{ctg}}(G, o, \alpha, r, L).$$

In particular, ν_o gives full measure to the contracting limit set $\Lambda_{\text{ctg}}(G)$ and thus to the radial limit set $\Lambda_{\text{rad}}(G)$.

Remark 5.20. The corollary is reminiscent of the fact that contracting elements are ‘generic’ in G, see [**51**].

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Proof. Let α, r be the parameters given by Proposition 5.18. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density. Let $L > r + 13\alpha$. For simplicity, we let

$$B = G\Lambda_{\rm ctg}(G, o, \alpha, r, L).$$

We claim that ν_o gives full measure to B. Assume on the contrary that the set $A = \bar{X} \setminus B$ has positive measure. We define a new density $\nu^* = (\nu^*_x)$ by

$$\nu^*_x = \frac{1}{\nu_o(A)} \mathbb{1}_A \nu_x.$$

Note that ν^* is ω_G-conformal. By construction, B, and thus A, is G-invariant. Hence, ν^* is also G-invariant. It follows from Proposition 5.18 that the ν^*_o gives positive measure to B, which is a contradiction. According to Proposition 5.9, the set B is contained in $\Lambda_{\rm ctg}(G, o, \alpha, r + 12\alpha, L - \alpha)$. Hence, the latter has full measure as well. These facts hold for every $L > r + 13\alpha$. Thus,

$$\nu_0\bigg(\bigcap_{L \in \mathbb{R}_+} \Lambda_{\rm ctg}(G, o, \alpha, r + 12\alpha, L)\bigg) = 1,$$

that is,

$$\nu_0(\Lambda_{\rm ctg}(G, o, \alpha, r + 12\alpha)) = 1. \qquad \square$$

5.4. *Passing to the reduced horoboundary.* We now study the restriction to the reduced horoboundary of invariant conformal densities. We still assume that G is not virtually cyclic and contains a contracting element. The shadow principle parameters $(\epsilon, r_0) \in \mathbb{R}^*_+ \times \mathbb{R}_+$ are as in the previous section.

PROPOSITION 5.21. *Assume that the action of G on X is divergent. There is $C \in \mathbb{R}^*_+$ with the following property. Let $\nu = (\nu_x)$ and $\nu' = (\nu'_x)$ be two G-invariant, ω_G-conformal densities. Denote by $\mu = (\mu_x)$ and $\mu' = (\mu'_x)$ their respective restrictions to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. Then $\mu_o \leq C\mu'_o$.*

Proof. Let $\alpha, r \in \mathbb{R}^*_+$ be the parameters given by Corollary 5.19. Without loss of generality, we can assume that $r \geq r_0$. Set $r' = r + 16\alpha$. For simplicity, we write Λ and Λ' for the set $\Lambda_{\rm ctg}(G, o, \alpha, r)$ and $\Lambda_{\rm ctg}(G, o, \alpha, r')$, respectively. Let $\nu = (\nu_x)$ and $\nu' = (\nu'_x)$ be as in the statement. Let $B \subset \partial X$ be a saturated Borel subset. We want to compare $\mu_o(B) = \nu_o(B)$ and $\mu'_o(B) = \nu'_o(B)$. According to Corollary 5.19, both ν_o and ν'_o give full measures to Λ. In addition, the saturated set $(B \cap \Lambda)^+$ is contained in Λ', see Proposition 5.10 (note that we do not claim that $(B \cap \Lambda)^+$ is measurable; we will use it only as an auxiliary tool to describe sets and will never compute its measure). Let V be an open subset of $\bar{X}$ containing B. Choose $L > r' + 16\alpha$. Using Corollary 5.14 with $(B \cap \Lambda)^+$, we build a subset $S \subset \mathcal{T}(\alpha, L)$ such that

$$B \cap \Lambda \subset (B \cap \Lambda)^+ \subset \bigcup_{g \in S} \mathcal{O}_o(go, r') \subset V.$$

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According to Lemma 5.6, there is a subset S^* of S such that:

- the collection $(\mathcal{O}_o(go, r'))_{g\in S^*}$ is pairwise disjoint; and
- $B \cap \Lambda$ is covered by $(\mathcal{O}_o(go, r' + 42\alpha))_{g\in S^*}$.

Recall that (G, ν') satisfies the shadow principle. Since Λ has full measure, we get

$$\nu_o(B) \leq \nu_o(B \cap \Lambda) \leq \sum_{g\in S^*} \nu_o(\mathcal{O}_o(go, r' + 42\alpha)) \leq e^{2\omega_G(r'+42\alpha)} \sum_{g\in S^*} e^{-\omega_G d(o,go)}$$
$$\leq C \sum_{g\in S^*} \nu'_o(\mathcal{O}_o(go, r)),$$

where $C = e^{2\omega_G(r'+42\alpha)}/\epsilon$ does not depend on ν and ν'. Hence, $\nu_o(B) \leq C\nu'_o(V)$. This inequality holds for every open subset V containing B, thus $\nu_o(B) \leq C\nu'_o(B)$. □

PROPOSITION 5.22. *Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density and $\mu = (\mu_x)$ its restriction to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. Assume that the action of G on X is divergent. Then:*

(i) *μ_o is supported on the contracting limit set;*
(ii) *μ_o is ergodic;*
(iii) *μ_o is non-atomic, in particular, no equivalence class for $\sim$ has positive measure;*
(iv) *μ is a G-invariant, ω_G-quasi-conformal density;*
(v) *μ is almost-unique, that is, there is $C \in \mathbb{R}_+^*$ such that if $\mu' = (\mu'_x)$ is the restriction to the reduced horocompactification of another G-invariant, ω_G-conformal density, then for every $x \in X$, we have $\mu'_x \leq C\mu_x$.*

Proof. Let $\alpha, r \in \mathbb{R}_+^*$ be the parameters given by Corollary 5.19 and $C \in \mathbb{R}_+$ the one given by Proposition 5.21. Let $\nu = (\nu_x)$ be a G-invariant, ω_G-conformal density and $\mu = (\mu_x)$ its restriction to the reduced horocompactifiction $(\bar{X}, \mathfrak{R})$. It follows from Corollary 5.19 that μ_o gives full measure to the contracting limit set. Let us prove the ergodicity of μ_o. Let B be a G-invariant saturated Borel subset such that $\mu_o(B) > 0$. Consider the density $\nu^* = (\nu^*_x)$ defined by

$$\nu^*_x = \frac{1}{\nu_o(B)} \mathbb{1}_B \nu_x.$$

Since B is G-invariant, ν^* is a G-invariant, ω_G-conformal density. Denote by $\mu^* = (\mu^*_x)$ its restriction to the reduced horocompactifiction. It follows from Proposition 5.21 that $\mu_o \leq C\mu^*_o$. Consequently, $\mu_o(\bar{X} \setminus B) = 0$, that is, $\mu_o(B) = 1$.

We now focus on non-atomicity. Assume on the contrary that $B \in \mathfrak{R}$ is an atom of μ_o. For every $g \in G$, we write $\mathcal{O}^+_o(go, r)$ for the saturation of the shadow $\mathcal{O}_o(go, r)$. Note that $\mathcal{O}^+_o(go, r)$ is measurable (Lemma 3.7). Consequently, the measure of $B \cap \mathcal{O}^+_o(go, r)$ is either zero or equals $\mu_o(B)$. By Corollary 5.19, μ_o only charges the set $\Lambda = \Lambda_{\rm ctg}(G, o, \alpha, r)$, and hence $\Lambda_{\rm rad}(G, o, r)$. Consequently, for every $n \in \mathbb{N}$, there is $g_n \in G$, with $d(o, g_n o) \geq n$ such that

$$\mu_o(\mathcal{O}^+_o(g_n o, r)) \geq \mu_o(B \cap \mathcal{O}^+_o(g_n o, r)) \geq \mu_o(B).$$

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By Proposition 5.10, $O_o^+(g_n o, r) \cap \Lambda$ is contained in $O_o(g_n o, r + 20\alpha)$. Since ν_o gives full measure to Λ, we get

$$0 < \mu_o(B) \leq \nu_o(O_o^+(g_n o, r)) \leq \nu_o(O_o^+(g_n o, r) \cap \Lambda) \leq \nu_o(O_o(g_n o, r + 20\alpha)).$$

Recall that $d(o, g_n o)$ diverges to infinity. Hence, the above inequality contradicts the shadow lemma (Corollary 4.10).

Let us prove item (iv). It follows from the construction that μ is G-invariant, $\|\mu_o\| = 1$, and $\mu_x \ll \mu_y$ for every $x, y \in X$. Hence, we are only left to prove that μ is quasi-conformal. Let $x, y \in X$. We define two auxiliary maps as follows.

$$\begin{array}{rcl} \bar{X} & \to & \mathbb{R} \\ c & \mapsto & \inf\limits_{c' \sim c} c'(x, y) \end{array} \quad \text{and} \quad \begin{array}{rcl} \bar{X} & \to & \mathbb{R} \\ c & \mapsto & \sup\limits_{c' \sim c} c'(x, y) \end{array}.$$

We denote them by $c \mapsto \beta_c^-(x, y)$ and $c \mapsto \beta_c^+(x, y)$, respectively. As $\bar{X}$ is separable, one checks that these maps are $\mathfrak{R}$-measurable. Let B be a saturated Borel subset. Using the conformality of ν, we have

$$\nu_x(B) \leq \int \mathbb{1}_B(c) e^{-\omega_G c(x,y)}\, d\nu_y(c) \leq \int \mathbb{1}_B(c) e^{-\omega_G \beta_c^-(x,y)}\, d\nu_y(c).$$

Since B is saturated and $c \mapsto \beta_c^-(x, y)$ is $\mathfrak{R}$-measurable, we get

$$\mu_x(B) \leq \int \mathbb{1}_B(c) e^{-\omega_G \beta_c^-(x,y)}\, d\mu_y(c).$$

This inequality holds for every $B \in \mathfrak{R}$. Hence,

$$\frac{d\mu_x}{d\mu_y}(c) \leq e^{-\omega_G \beta_c^-(x,y)}, \quad \mu\text{-a.e.}$$

In the same way, we obtain a lower bound for the Radon–Nikodym derivative with $\beta_c^+(x, y)$ in place of $\beta_c^-(x, y)$. By Proposition 5.10, for μ-almost every $c \in \bar{X}$, we have

$$c(x, y) - 20\alpha \leq \beta_c^-(x, y) \leq \beta_c^+(x, y) \leq c(x, y) + 20\alpha.$$

Hence, μ is quasi-conformal. Point (v) now follows from Proposition 5.21 and the quasi-conformality. □

5.5. *More applications*

PROPOSITION 5.23. *Assume that the action of G on X is divergent. For every infinite normal subgroup of G, we have*

$$\omega_N > \tfrac{1}{2}\omega_G.$$

Proof. Let $Q = G/N$ and ω_Q be the growth rate of Q on X/N. According to Corollary 4.29, we have

$$\omega_N + \tfrac{1}{2}\omega_Q \geq \omega_G.$$

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Since the map $X \to X/N$ is 1-Lipschitz, $\omega_Q \leq \omega_G$. Hence,

$$\omega_N \geq \tfrac{1}{2}\omega_G.$$

Suppose now that, in contrast to our claim, $\omega_G = 2\omega_N$. We choose:

- a G-invariant, ω_G-conformal density $\nu = (\nu_x)$; and
- an N-invariant, ω_N-conformal density $\nu' = (\nu'_x)$ such that the action of N on $(\bar{X}, \mathfrak{B}, \nu'_o)$ is ergodic.

We write μ and μ' for their respective restrictions to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. In particular, the action of N on $(\bar{X}, \mathfrak{R}, \mu'_o)$ is ergodic. We claim that μ_0 is absolutely continuous with respect to μ'_0. According to Corollary 4.10, (G, ν') satisfies the shadow lemma for some parameters $(\epsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$. By Corollary 5.19, there exists $\alpha, r \in \mathbb{R}_+^*$ such that ν_o gives full measure to $\Lambda = \Lambda_{\mathrm{ctg}}(G, o, \alpha, r)$. Without loss of generality, we can assume that $r \geq r_0$. For simplicity, we set $r' = r + 16\alpha$ and write Λ' for $\Lambda_{\mathrm{ctg}}(G, o, \alpha, r)$.

Let B be a saturated subset contained in $\Lambda_{\mathrm{ctg}}(G, o, \alpha, r)$. Let V be an open set containing B. Observe that $(B \cap \Lambda)^+$ is contained in Λ' (Proposition 5.10). Fix $L > r' + 16\alpha$. Using Corollary 5.14 with $(B \cap \Lambda)^+$, we build a subset $S \subset \mathcal{T}(\alpha, L)$ such that

$$B \cap \Lambda \subset (B \cap \Lambda)^+ \subset \bigcup_{g \in S} \mathcal{O}_o(go, r') \subset V.$$

According to Lemma 5.6, there is a subset S^* of S such that:

- the collection $(\mathcal{O}_o(go, r'))_{g \in S^*}$ is pairwise disjoint; and
- $B \cap \Lambda$ is covered by $(\mathcal{O}_o(go, r' + 42\alpha))_{g \in S^*}$.

Since ν gives full measure to Λ, we have $\nu_o(B) = \nu_o(B \cap \Lambda)$. Using Remark 4.11 with the density ν, we get

$$\nu_o(B) \leq \sum_{g \in S^*} \nu_o(\mathcal{O}_o(go, r' + 42\alpha)) \leq e^{2\omega_G(r' + 42\alpha)} \sum_{g \in S^*} e^{-\omega_G d(o, go)}.$$

Recall that for every $g \in G$, we have $\|\nu'_{go}\| \geq e^{-\omega_N d(o, go)}$. Since $\omega_G = 2\omega_N$, we obtain

$$\nu_o(B) \leq e^{2\omega_G(r' + 42\alpha)} \sum_{g \in S^*} \|\nu'_{go}\| e^{-\omega_N d(o, go)}.$$

Using now the shadow principle with the density ν', we obtain

$$\nu_o(B) \leq C \sum_{g \in S^*} \nu'_o(\mathcal{O}_o(go, r')) \leq C\nu'_o(V) \quad \text{where } C = \frac{1}{\epsilon} e^{2\omega_G(r' + 42\alpha)}$$

does not depend on B. This inequality holds for every open subset V containing B, and hence $\nu_o(B) \leq C\nu'_o(B)$, that is, $\mu_o(B) \leq C\mu'_o(B)$. This completes the proof of our claim.

Denote by f the Radon–Nikodym derivative $f = d\mu_o/d\mu'_o$. Both μ and μ' are N-invariant. Hence, the set $A = \{c \in \bar{X} \ : \ f(c) > 0\}$ is N-invariant. Note that $\mu'_o(A) > 0$. Indeed otherwise, μ_o would be the zero measure. Since the action of N on $(\bar{X}, \mathfrak{R}, \mu'_o)$ is ergodic, we get $\mu'_o(A) = 1$. Hence, μ_o and μ'_o are in the same class of measures. Since μ_o is G-invariant, μ'_o is G-quasi-invariant. We assumed that μ'_o is ergodic for the action of

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N. It follows from Lemma 4.5 that μ' is almost fixed by G. Thus, $\omega_N \geq \omega_G$ by Corollary 4.26. This contradicts our assumption and completes the proof. □

PROPOSITION 5.24. *Let $H \subset G$ be a subgroup which is not virtually cyclic and contains a contracting element. Let $\nu = (\nu_x)$ be an H-invariant, ω_H-conformal density and $\mu = (\mu_x)$ its restriction to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. Assume that the action of H is divergent. If μ is almost fixed by G, then (G, ν) satisfies the shadow principle.*

Proof. According to Corollary 5.19, there are $\alpha, r_0 \in \mathbb{R}_+^*$ such that ν gives full measure to $\Lambda_{\rm ctg}(H, o, \alpha, r_0)$. Proceeding as in the proof of Corollary 4.10, we show that for every $g \in G$ and $r \in \mathbb{R}_+$,

$$\nu_o(\mathcal{O}_o(go, r)) \geq \|\nu_{go}\| e^{-\omega_H d(o,go)} \nu_o^g(\mathcal{O}_{g^{-1}o}(o, r)). \tag{18}$$

Choose now $r \geq r_0$ and $g \in G$. We denote by $\mathcal{O}^+_{g^{-1}o}(o, r)$ the saturation of the shadow $\mathcal{O}_{g^{-1}o}(o, r)$, which is measurable by Lemma 3.7. According to Proposition 5.10,

$$\mathcal{O}^+_{g^{-1}o}(o, r) \cap \Lambda_{\rm ctg}(H, o, \alpha, r) \subset \mathcal{O}_{g^{-1}o}(o, r + 20\alpha).$$

Recall that ν gives full measure to $\Lambda_{\rm ctg}(H, o, \alpha, r_0)$, and thus to $\Lambda_{\rm ctg}(H, o, \alpha, r)$ as well. Since μ is almost fixed by G, we have

$$\begin{aligned}\nu_o^g(\mathcal{O}_{g^{-1}o}(o, r + 20\alpha)) \geq \mu_o^g(\mathcal{O}^+_{g^{-1}o}(o, r)) &\geq \epsilon \mu_o(\mathcal{O}^+_{g^{-1}o}(o, r)) \\ &\geq \epsilon \nu_o(\mathcal{O}_{g^{-1}o}(o, r)),\end{aligned}$$

where $\epsilon \in \mathbb{R}_+^*$ does not depend on g and r. Combined with equation (18), it shows that for every $r \geq r_0$, for every $g \in G$, we have

$$\nu_o(\mathcal{O}_o(go, r + 20\alpha)) \geq \epsilon \|\nu_{go}\| e^{-\omega_H d(o,go)} \nu_o(\mathcal{O}_{g^{-1}o}(o, r)).$$

According to our assumption, H is not virtually cyclic and contains a contracting element. The conclusion now follows from Proposition 4.9 applied with the group H and the set $\mathcal{D}_0 = \{\nu\}$. □

THEOREM 5.25. *Let H be a commensurated subgroup of G. If the action of H on X is divergent, then the following hold.*

(i) *Any H-invariant, ω_H-conformal density is G-almost invariant when restricted to the reduced horocompactification $(\bar{X}, \mathfrak{R})$.*
(ii) $\omega_H = \omega_G$.
(iii) *The action of G on X is divergent.*

Proof. Let $\nu = (\nu_x)$ be an H-invariant, ω_H-conformal density. We denote by $\mu = (\mu_x)$ its restriction to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. Let $g \in G$. By definition of commensurability, the intersection $H_0 = H^g \cap H$ has finite index in H. In particular, H_0 is divergent and $\omega_{H_0} = \omega_H$. Recall that ν^g is the image of ν under the right action of $g \in G$. It is an H^g-invariant, ω_H-conformal density, and thus an H_0-invariant, ω_{H_0}-conformal

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density. Similarly, ν is an H_0-invariant, ω_{H_0}-conformal density. Since H_0 is divergent, Proposition 5.22 tells us that:

- there is $C \in \mathbb{R}_+$ such that $\mu^g \leq C\mu$;
- the action of $H^g \cap H$ on $(\bar{X}, \mathfrak{R}, \mu_o)$ is ergodic.

Note that C depends *a priori* on H_o and thus on g. Nevertheless, it still proves that μ is G-quasi-invariant. Consequently, μ is C_0-almost fixed by G for some $C_0 \in \mathbb{R}_+^*$ (Lemma 4.5). We deduce from Proposition 5.24 that (G, ν) satisfies the shadow principle. Point (ii) now follows from Corollary 4.26. Recall that $\mathcal{P}_H(s) \leq \mathcal{P}_G(s)$ for every $s \in \mathbb{R}_+$. Since the action of H on X is divergent, $\mathcal{P}_G(s)$ diverges at $s = \omega_H = \omega_G$. Hence the action of G on X is divergent as well, which proves point (iii).

We already know that μ is almost-fixed by G, so that the map $\chi \colon G \to \mathbb{R}$ sending g to $\ln \|\mu_{go}\|$ is a quasi-morphism (Lemma 4.4). We are left to prove that μ is actually G-almost invariant, that is, χ is bounded. Recall that (G, ν) satisfies the shadow principle. It follows from Proposition 4.23 that the critical exponent of the series

$$\sum_{g \in G} e^{\chi(g)} e^{-sd(o,go)} = \sum_{g \in G} \|\nu_{go}\| e^{-sd(o,go)}$$

is exactly ω_H. Hence, $\omega_{-\chi} = \omega_\chi = \omega_H$. Note also that since ν is H-invariant, $\chi(hg) = \chi(g)$ for every $h \in H$ and $g \in G$. Using Proposition 4.3, with the quasi-morphism $-\chi$, we produce an H-invariant, ω_H-conformal density $\nu^* = (\nu_x^*)$ satisfying the following additional property: there is $C_1 \in \mathbb{R}_+^*$ such that for every $g \in G$, for every $x \in X$, we have

$$\frac{1}{C_1} \nu_x \leq e^{\chi(g)} {g^{-1}}_* \nu_{gx} \leq C_1 \nu_x.$$

Denote by μ^* its restriction to the reduced horocompactification $(\bar{X}, \mathfrak{R})$. According to Proposition 5.22(v), there is $C_2 \in \mathbb{R}_+$ such that $\mu \leq C_2 \mu^*$. Recall that μ is C_0-almost fixed by G. Consequently, for every $g \in G$, we have

$$e^{\chi(g)} \mu_o \leq C_0 ({g^{-1}}_* \mu_{go}) \leq C_0 C_2 ({g^{-1}}_* \mu_{go}^*) \leq C_0 C_1 C_2 (e^{-\chi(g)} \mu_o^*).$$

Since μ_o and μ_o^* are probability measures, χ is bounded, whence the result. □

Remark 5.26. As we noticed in the introduction, every finite index and every normal subgroup of G is commensurated. More generally, consider a subgroup H of G and N a normal subgroup of G. If H and N are commensurable (that is, $H \cap N$ has finite index in both H and N), then H is commensurated. However, there are plenty of other examples. Here is a construction suggested by Bader [3]. Consider the free group $\mathbf{F}_2$ and morphism $\phi \colon \mathbf{F}_2 \to M$, where M is a topological group. Let K be an open compact subgroup of M. Then $H = \phi^{-1}(K)$ is commensurated. Now if ϕ has dense image, then H is commensurable with a normal subgroup of $\mathbf{F}_2$ if and only if K is commensurable with a normal subgroup of M. Consider now, for instance, a prime p and a morphism $\phi \colon \mathbf{F}_2 \to \mathrm{PSL}_2(\mathbf{Q}_p)$ with dense image. Then $\phi^{-1}(\mathrm{PSL}_2(\mathbb{Z}_p))$ provides an example of a commensurated subgroup of $\mathbf{F}_2$ that is not commensurable with a normal subgroup of $\mathbf{F}_2$.

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A. *Appendix. Strongly positively recurrent actions*

A.1. *Definition.* Let G be a group acting properly, by isometries on a proper, geodesic, metric space X. Given a compact subset $K \subset X$, we define a subset $G_K \subset G$ as follows: an element $g \in G$ belongs to G_K if there exist $x, y \in K$ and a geodesic γ joining x to gy such that the intersection $\gamma \cap GK$ is contained in $K \cup gK$. Although G_K is not a subgroup of G, its exponential growth rate $\omega(G_K, X)$ is defined in the same way as for the one of G.

Definition A.1. The *entropy at infinity* of the action of G on X is

$$\omega_\infty(G, X) = \inf_K \omega(G_K, X),$$

where K runs over all compact subsets of X. The action of G on X is *strongly positively recurrent* (or *statistically convex co-compact*) if $\omega_\infty(G, X) < \omega(G, X)$.

We refer the reader to [**19**, **44**] for examples of strongly positively recurrent actions in the context of hyperbolic geometry. Arzhantseva, Cashen, and Tao [**1**, §10] also observed that the work of Eskin, Mirzakani, and Rafi [**21**, Theorem 1.7] implies that the action of the mapping class group on the Teichmüller space endowed with the Teichmüller metric is strongly positively recurrent.

A.2. *Divergence*

PROPOSITION A.2. *If the action of G on X is strongly positively recurrent, then it is divergent.*

The statement was proved by Yang [**50**]. We give here an alternative approach in the spirit of Schapira and Tapie [**44**]. The idea is to build a G-invariant, ω_G-conformal density which gives positive measure to the radial limit set. Indeed, according to Proposition 4.23, this will imply that the action of G on X is divergent. As we explained in Remark 4.24, this part of Proposition 4.23 does not require that G contains a contracting element.

First, we give a description of the complement of the radial limit set. To that end, we introduce some notation. Given a compact subset $K \subset X$ and $\epsilon \in \mathbb{R}_+^*$, we denote by $A_{K,\epsilon}$ the set of all cocycles $c \in \partial X$ with the following property: there is a point $x \in K$ such that for every ϵ-quasi-gradient ray $\gamma : \mathbb{R}_+ \to X$ for c starting at x, for every $u \in G$, if the intersection $\gamma \cap uK$ is non-empty, then $d(K, uK) \leq 1$.

LEMMA A.3. *The radial set of G satisfies the following inclusion:*

$$\partial X \setminus \Lambda_{\mathrm{rad}}(G) \subset \bigcap_{K \subset X} G\bigg(\bigcup_{\epsilon > 0} A_{K,\epsilon}\bigg),$$

where K runs over all compact subsets of X.

Proof. The proof is by contraposition. Consider a cocycle $c \in \partial X$ that is not in the set

$$\bigcap_{K \subset X} G\bigg(\bigcup_{\epsilon > 0} A_{K,\epsilon}\bigg).$$

There is a compact subset $K \subset X$ such that for every $g \in G$ and $\epsilon > 0$, the cocycle c does not belong to $gA_{K,\epsilon}$. Fix $\epsilon \in (0, 1)$ and $x_0 \in K$. In addition, we let $g_0 = 1$. We are going to build, by induction, a sequence of points $x_1, x_2 \ldots$ in X, a sequence of elements $g_1, g_2, \ldots$ in G, and a sequence of rays $\gamma_1, \gamma_2, \ldots$, such that for every $i \in \mathbb{N} \setminus \{0\}$, the following hold.

(i) x_i belongs to $g_i K$.

(ii) $c(x_0, x_i) \geq i/2$.

(iii) For every $i \in \mathbb{N} \setminus \{0\}$, the path γ_i is a $2^{-i}\epsilon$-quasi-gradient ray of c starting at x_{i-1} and passing through x_i.

Let $i \in \mathbb{N}$. Assume that $x_i \in X$, $g_i \in G$ have been defined. By assumption, c does not belong to the set

$$g_i A_{K,2^{-(i+1)}\epsilon}.$$

Hence, there exists a $2^{-(i+1)}\epsilon$-quasi-gradient ray $\gamma_{i+1} \colon \mathbb{R}_+ \to X$ for c starting at x_i and an element $u_i \in G$ such that $\gamma_{i+1} \cap g_i u_i K$ is non-empty and $d(g_i K, g_i u_i K) > 1$. We let $g_{i+1} = g_i u_i$ and denote by x_{i+1} a point in $\gamma_{i+1} \cap g_i u_i K$. Since $x_i \in g_i K$ and $x_{i+1} \in g_i u_i K$, we have $d(x_i, x_{i+1}) > 1$. However, γ_{i+1} is a quasi-gradient line. Hence,

$$c(x_i, x_{i+1}) \geq d(x_i, x_{i+1}) - 2^{-(i+1)}\epsilon \geq 1/2.$$

Using the induction hypothesis, we get

$$c(x_0, x_{i+1}) \geq c(x_0, x_i) + c(x_i, x_{i+1}) \geq (i + 1)/2.$$

Consequently, x_{i+1}, g_{i+1}, and γ_{i+1} satisfy the announced properties.

Note that the sequence (x_i) is unbounded. Indeed otherwise, $c(x_0, x_i)$ should be bounded as well. Thus, we can build an infinite path γ by concatenating the restriction of each γ_i between x_{i-1} and x_i. It follows from the construction that γ is an ϵ-quasi-gradient line for c, see Remark 3.3. Moreover, γ intersects $g_i K$ for every $i \in \mathbb{N}$. One proves using the triangle inequality that c belongs to the radial limit set. □

Let $K \subset X$ be a compact subset and $\epsilon \in \mathbb{R}_+^*$. For every compact subset $F \subset X$, we define $U_{K,\epsilon}(F)$ to be the set of cocycles $b \in \bar{X}$ for which there is a cocycle $c \in A_{K,\epsilon}$ satisfying $\|b - c\|_F < \epsilon$. Observe that $U_{K,\epsilon}(F)$ is an open subset of $\bar{X}$ containing $A_{K,\epsilon}$.

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LEMMA A.4. *Let $K \subset X$ be a compact set and $\epsilon \in \mathbb{R}_+^*$. Fix a base point $o \in K$. There exist $r \in \mathbb{R}_+$ and a finite subset $S \subset G$ such that for every $T \geq \epsilon$, if F stands for the closed ball of radius $T + 2r$ centered at o, then*

$$U_{K,\epsilon}(F) \cap Go \subset S\Bigg(\bigcup_{\substack{k \in G_K \\ d(o,ko) \geq T}} O_o(ko, r) \Bigg).$$

Proof. Since the action of G on X is proper, the set

$$S = \{u \in G \ : \ d(K, uK) \leq 1\}$$

if finite. We fix $r > 2 \operatorname{diam} K + 1$. Let $T \geq \epsilon$ and F be the closed ball of radius $R = T + 2r$ centered at o. Let $g \in G$ such that go belongs to $U_{K,\epsilon}(F)$. We write $b = \iota(go)$ for the corresponding cocycle. By definition, there is $c \in A_{K,\epsilon}$ such that $\|b - c\|_F < \epsilon$. Observe first that $d(o, go) > R - \epsilon$. Indeed, the map $x \mapsto b(x, go)$ admits a global minimum at go, while there exists a c-gradient line starting at go. We cannot have at the same time $d(o, go) \leq R - \epsilon$ and $\|b - c\|_F < \epsilon$. In particular, $g \notin S$.

Since $c \in A_{K,\epsilon}$, there exists $x \in K$, such that for every ϵ-quasi-gradient ray $\gamma\colon \mathbb{R}_+ \to X$ for c, starting at x, if γ intersects uK for some $u \in G$, then $u \in S$. Consider now a geodesic $\alpha\colon [0\,,\ell] \to X$ from x to go. We denote by $s \in [0\,,\ell]$ the largest time such that the point $y = \alpha(s)$ belongs to SK. We now denote by $t \in [s\,,\ell]$ the smallest time such that the point $z = \alpha(t)$ lies in hK for some $h \in G \setminus S$ (such a time t exists since $\alpha(\ell)$ belongs to gK). It follows from the construction that h can be written $h = uk$ with $u \in S$ and $k \in G_K$. Moreover, $y \in uK$. Observe that $\langle y, go\rangle_z = 0$, while $d(y, uo) \leq r/2$ and $d(z, uko) \leq r/2$. The triangle inequality yields $\langle uo, go\rangle_{uko} \leq r$, that is, go belongs to $uO_o(ko, r)$.

We are left to prove that $d(o, ko) \geq T$. By construction, $d(o, uo) \leq r$. Thanks to the triangle inequality, it suffices to show that $d(o, z) \geq R$. Assume on the contrary that $d(o, z) < R$. In particular, both x and z belong to F. Since b and c differ by at most ϵ on F, we get that $c(x, z) \geq d(x, z) - \epsilon$. Hence, any geodesic from x to z is an ϵ-quasi-gradient arc for c. If we concatenate this path with a gradient ray for c starting at z, we obtain an ϵ-quasi-gradient ray for c starting at x and intersecting hK with $d(K, hK) > 1$. This contradicts the fact that c belongs to $A_{K,\epsilon}$, and completes the proof. □

PROPOSITION A.5. *If the action of G on X is strongly positively recurrent, then there is a G-invariant, ω_G-conformal density which gives full measure to the radial limit set $\Lambda_{\mathrm{rad}}(G)$.*

Proof. By definition, there is a compact subset $K \subset X$ such that $\omega_{G_K} < \omega_G$. We fix once and for all a base point $o \in K$. The argument relies on Patterson's construction recalled in the proof of Proposition 4.3 with $H = G$ and χ the trivial morphism. In particular, $\mathcal{Q}(s)$ stands for the weighted Poincaré series defined in equation (5). For every $s > \omega_G$, we consider the density $\nu^s = (\nu_x^s)$ defined as in equation (6). As we explained, there is a sequence (s_n) converging to ω_G from above such that ν^{s_n} converges to a G-invariant, ω_G-conformal density ν supported on ∂X.

Let $\eta > 0$ such that $\omega_{G_K} < \omega_G - \eta$. The weight θ used to construct ν is slowly increasing. More precisely, according to property (P1), there exists t_0 such that for every

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$t \geq t_0$ and $u \in \mathbb{R}_+$, we have $\theta(t+u) \leq e^{\eta u}\theta(t)$. Let $\epsilon > 0$. Let $r \in \mathbb{R}_+$ and $S \subset G$ be the data provided by Lemma A.4 applied with K and ϵ. For every $T \in \mathbb{R}_+$, we write F_T for the closed ball of radius $R = T + 2r$ centered at o. Let $s > \omega_G$ and $T \geq \max\{t_0, \epsilon\}$. In view of Lemma A.4, we have

$$\nu_o^s(U_{K,\epsilon}(F_T)) \leq |S| \sum_{\substack{k \in G_K \\ d(o,ko) \geq T}} \nu_o^s(\mathcal{O}_o(ko, r)).$$

Let us estimate the measures of the shadows in the sum. Let $k \in G_K$, such that $d(o, ko) \geq T$. Any element $g \in G$ such that $go \in \mathcal{O}_o(ko, r)$ can be written $g = ku$ with $u \in G$ and

$$d(o, ko) + d(o, uo) - 2r \leq d(o, go) \leq d(o, ko) + d(o, uo).$$

Unfolding the definition of ν^s, we get

$$\nu_o^s(\mathcal{O}_o(ko, r)) \leq \frac{e^{2sr} e^{-sd(o,ko)}}{\mathcal{Q}(s)} \sum_{u \in G} \theta(d(o, go)) e^{-sd(o,uo)}. \tag{A.1}$$

Observe that if $d(o, uo) \geq t_0$, then it follows from our choice of t_0 that

$$\theta(d(o, go)) \leq \theta(d(o, ko) + d(o, uo)) \leq e^{\eta d(o,ko)}\theta(d(o, uo)).$$

Otherwise, since $d(o, ko) \geq T \geq t_0$, we have

$$\theta(d(o, go)) \leq \theta(t_0 + d(o, uo) + d(o, ko) - t_0) \leq e^{\eta d(o,ko)}\theta(t_0).$$

We break the sum in equation (A.1) according to the length of u and get

$$\nu_o^s(\mathcal{O}_o(ko, r)) \leq \frac{e^{2sr} e^{-(s-\eta)d(o,ko)}}{\mathcal{Q}(s)} [\theta(t_0)\Sigma_1(s) + \Sigma_2(s)],$$

where

$$\Sigma_1(s) = \sum_{\substack{u \in G \\ d(o,uo) \leq t_0}} e^{-sd(o,uo)} \quad \text{and}$$

$$\Sigma_2(s) = \sum_{\substack{u \in G \\ d(o,uo) > t_0}} \theta(d(o, uo)) e^{-sd(o,uo)}.$$

Note that $\Sigma_1(s)$ is a finite sum that does not depend on k, while $\Sigma_2(s)$ is the remainder of the series $\mathcal{Q}(s)$. Hence,

$$\nu_o^s(\mathcal{O}_o(ko, r)) \leq e^{2sr}\left[\frac{\theta(t_0)}{\mathcal{Q}(s)}\Sigma_1(s) + 1\right] e^{-(s-\eta)d(o,ko)}.$$

Summing over all long elements $k \in G_K$, we get

$$\nu_o^s(U_{K,\epsilon}(F_T)) \leq |S| e^{2sr}\left[\frac{\theta(t_0)}{\mathcal{Q}(s)}\Sigma_1(s) + 1\right] \sum_{\substack{k \in G_K \\ d(o,ko) \geq T}} e^{-(s-\eta)d(o,ko)}.$$

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Note that $\Sigma_1(s)$ is bounded, while $Q(s)$ diverges to infinity. Since $U_{K,\epsilon}(F_T)$ is an open subset of $\bar{X}$, we can pass to the limit and get

$$\nu_o(U_{K,\epsilon}(F_T)) \leq |S|e^{2\omega_G r} \sum_{\substack{k \in G_K \\ d(o,ko) \geq T}} e^{-(\omega_G - \eta)d(o,ko)}.$$

The sum corresponds to the remainder of the Poincaré series of G_K at $s = \omega_G - \eta$. However, $\omega_G - \eta > \omega_{G_K}$. Hence, this series converges, and its remainder tends to zero when T approaches infinity. Consequently, for every $\epsilon > 0$,

$$\nu_o\bigg(\bigcap_{T \geq 0} U_{K,\epsilon}(F_T)\bigg) = 0.$$

By construction, the set $A_{K,\epsilon}$ is contained in $U_{K,\epsilon}(F_T)$ for every $T \in \mathbb{R}_+$. It follows from Lemma A.3 that

$$\partial X \setminus \Lambda_{\text{rad}}(G) \subset G\bigg(\bigcup_{\epsilon > 0} \bigcap_{T \geq 0} U_{K,\epsilon}(F_T)\bigg).$$

Since G is countable, we conclude that $\nu_o(\partial X \setminus \Lambda_{\text{rad}}(G)) = 0$. Recall that ν_o is supported on ∂X, thus $\nu_o(\Lambda_{\text{rad}}(G)) = 1$. □

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