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GORENSTEIN QUOTIENTS BY PRINCIPAL IDEALS OF FREE KOSZUL HOMOLOGY

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Abstract. Let A be a noetherian local ring, x a non-unit element of A, $B = A/(x)$. Let E be the Koszul complex associated to an arbitrary set of generators of the ideal (x) of A. Assume that $H_1(E)$ is a free B-module. Then A is Gorenstein if and only if B is also.

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Introduction. Let A be a noetherian local (commutative with unit) ring, I an ideal of A, and $B = A/I$. Let E be the Koszul complex associated with a set of generators of the ideal I. Assume that $H_1(E)$ is a free B-module, and that $H_2(E)/H_1(E)^2 = 0$ (both properties are independent of the choice of the set of generators of I). Then André proves in [**1**] that A is a complete intersection if and only if B is a complete intersection.

In this paper we consider the analogous question for the Gorenstein property, and we answer it for principal ideals as follows.

THEOREM. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, $x \in \mathfrak{m}$, $B = A/(x)$. Let E be the Koszul complex associated with a set of generators of the ideal (x) of A. Assume that $H_1(E)$ is a free B-module. Then A is Gorenstein if and only if B is.*

In fact, in the notation of [**3**], we prove that under the above hypotheses the homomorphism $A \to B$ is quasi-Gorenstein and has Gorenstein dimension 0 or 1, according as x is a zero-divisor or not. In particular, denoting the Gorenstein dimension by G-dim, we obtain, using [**3**],

$$\text{G-dim}_A(M) = \begin{cases} \text{G-dim}_B(M) & \text{if } \; x \text{ is a zero-divisor} \\ \text{G-dim}_B(M) + 1 & \text{if } \; x \text{ is a non-zero-divisor} \end{cases}$$

for any B-module of finite type M. In the case when x is a non-zero-divisor and $\text{G-dim}_B(M) < \infty$, this equality was obtained in [**2**, (4.32)].

Proof. If x is a non-zero-divisor, then the result is well known: by [**7**] we have, for any B-module M,

$$\text{Ext}^p_A(M, A) = \text{Ext}^{p-1}_B(M, B). \qquad (*)$$

From this isomorphism with $M = k$ we deduce that A is Gorenstein if and only if B is (moreover, from the same isomorphism with $M = B$, we deduce from [**2**, (3.14), (4.13(a), (i)$\Longleftrightarrow$(iv))] that the Gorenstein dimension of the A-module B is 1, and from [**3**, (7.5)], using again $(*)$ with $M = k$, we deduce that $A \to B$ is quasi-Gorenstein).

So assume that x is a zero-divisor. The property of $H_1(E)$ being free does not depend on the choice of the set of generators of the ideal (x), so we can assume that E is the Koszul complex associated with the element x. Then $H_1(E) = (0 : x) \neq 0$. Now the theorem follows from the two following propositions.

Proposition 1. *Let* $(A, \mathfrak{m}, k)$ *be a noetherian local ring,* $x \in \mathfrak{m}, B = A/(x)$. *Assume that* x *is a zero-divisor. Then* $(0 : x)$ *is a free* B*-module if and only if there exists* $a \in A$ *such that* $(0 : x) = (a)$ *and* $(0 : a) = (x)$.

Proof. Assume $(0 : x)$ is B-free. First we use an argument from [**4**] to show that $(0 : x)$ is, as a B-module, free of rank 1. We study separately the cases when the Krull dimension of A is 0 or 1.

If $\dim(A) = 0$, A is artinian and so the lengths of the A-modules in the exact sequence

$$0 \to (A/(x))^n = (0 : x) \to A \to (x) \to 0$$

are finite, where $n = \mathrm{rank}_B(0 : x)$. We have

$$L(A) = L((A/(x))^n) + L((x)) = n(L(A) - L(x))) + L((x))$$

and, since $L(A) > L((x))$ (x is not a unit), we obtain $n = 1$.

If $\dim(A) = 1$ and x is contained in some minimal prime ideal of A, localizing at that prime ideal reduces the problem to the case when $\dim(A) = 0$. If x is not contained in any minimal prime ideal of A, then $A/(x)$ is artinian, and so $A/(x)$ and $(0 : x) = (A/(x))^n$ are $A/(x)$-modules of finite length. So by [**5**, Definition A.2] we can define $e_A(x, A) = L(A/(x)) - L((0:x)) = L(A/(x)) - L((A/(x))^n) = L(A/(x)) - nL((A/(x)))$, and from [**5**, Lemma A.2.7], under our hypothesis we deduce that $e_A(x, A) \geq 0$. Therefore $n = 1$.

In the general case ($\dim(A)$ arbitrary), by the Krull principal ideal theorem [**6**], x is contained in a prime ideal of height at most 1. Localizing at that prime ideal we are in the case $\dim(A) \leq 1$ already studied.

So $(0 : x)$ is a free B-module of rank 1, and so a principal ideal of A, say $(0 : x) = (a)$, $a \in A$. It is clear that $(x) \subset (0 : (0 : x)) = (0 : a)$. Therefore we have a commutative diagram of exact rows

$$\begin{array}{ccccccc} 0 \to (x) & \to & A & \to & B = (0 : x) = (a) & \to & 0 \\ \downarrow & & \| & & \| & & \\ 0 \to (0 : a) & \to & A & \xrightarrow{\cdot a} & (a) & \to & 0 \end{array}$$

where $B = (0 : x) = (a)$ is the isomorphism taking 1 into a. We deduce that $(x) = (0 : a)$.

The converse follows from a similar diagram

$$\begin{array}{ccccccc} 0 \to (x) & \to & A & \longrightarrow & B & \to & 0 \\ \| & & \| & & \downarrow \lambda & & \\ 0 \to (0 : a) & \to & A & \xrightarrow{\cdot a} & (a) & \to & 0 \end{array}$$

where λ is the homomorphism taking 1 into a.

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Proposition 2. *Let* $(A, \mathfrak{m}, k)$ *be a noetherian local ring,* $x, a \in \mathfrak{m}$ *such that* $(0 : x) = (a)$ *and* $(0 : a) = (x)$, *and let* $B = A/(x)$. *Then we have*

(i) $\text{G-dim}_A(B) = 0$;

(ii) *the homomorphism* $A \to B$ *is quasi-Gorenstein;*

(iii) *A is Gorenstein if and only if B is.*

Proof. First we show that $\text{Ext}_A^q(B, A) = 0$ for $q > 0$. From the cohomology long exact sequences associated with the exact sequences of A-modules

$$0 \to B = (0 : x) \to A \xrightarrow{\cdot x} (x) \to 0 \tag{I}$$

$$0 \to (x) \to A \to B \to 0 \tag{II}$$

we see that it suffices to show that $\text{Ext}_A^q(B, A) = 0$ for $q = 1, 2$.

The homomorphism ϕ in the cohomology long exact sequence associated with (II)

$$0 \to \text{Hom}_A(B, A) \to \text{Hom}_A(A, A) \xrightarrow{\phi} \text{Hom}_A((x), A) \to \text{Ext}_A^1(B, A) \to 0$$

can be identified with the homomorphism $A \xrightarrow{\cdot x} (0 : (0 : x)) = (x)$, and so it is surjective; thus $\text{Ext}_A^1(B, A) = 0$.

From the (continuation of) the same cohomology exact sequence we obtain $\text{Ext}_A^2(B, A) = \text{Ext}_A^1((x), A)$. Similarly, from the cohomology long exact sequence associated with (I)

$$0 \to \text{Hom}_A((x), A) \to \text{Hom}_A(A, A) \xrightarrow{\psi} \text{Hom}_A((0 : x), A) = (a) \to \text{Ext}_A^1((x), A) \to 0.$$

we obtain that $\text{Ext}_A^1((x), A) = 0$. Therefore $\text{Ext}_A^q(B, A) = 0$ for $q > 0$.

Now, in the change of rings spectral sequence

$$E_2^{p,q} = \text{Ext}_B^p(k, \text{Ext}_A^q(B, A)) \Rightarrow \text{Ext}_A^{p+q}(k, A),$$

we have $\text{Ext}_A^q(B, A) = 0$ for $q > 0$, and so

$$\text{Ext}_A^p(k, A) = \text{Ext}_B^p(k, \text{Hom}_A(B, A)) = \text{Ext}_B^p(k, B),$$

since we have A-module isomorphisms $\text{Hom}_A(B, A) = (0 : x) = B$.

This shows that A is Gorenstein if and only if B is. From the exact sequence $A \xrightarrow{\cdot x} A \to B \to 0$, we see that we can take an Auslander-Bridger dual $D(B)$ isomorphic to B, so by [**2**, (3.8) (a)$\Longleftrightarrow$(b)] we deduce that $\text{G-dim}_A(B) = 0$. Finally, from the isomorphism $\text{Ext}_A^p(k, A) = \text{Ext}_B^p(k, B)$ and [**3**, (7.5)], we deduce that $A \to B$ is quasi-Gorenstein.

Corollary. *Let* $(A, \mathfrak{m}, k)$ *be a noetherian local ring, x an element of* $\mathfrak{m}$ *which is not a zero-divisor or such that* $(0 : x) = (a)$ *and* $(0 : a) = (x)$ *for some* $a \in A$. *Let* $B = A/(x)$ *and let M be a finite B-module. Then*

$$\text{G-dim}_A(M) = \begin{cases} \text{G-dim}_B(M) & \text{if } x \text{ is a zero-divisor,} \\ \text{G-dim}_B(M) + 1 & \text{if } x \text{ is a non- zero-divisor.} \end{cases}$$

Proof. We have shown that $A \to B$ is quasi-Gorenstein, so that the result follows from [**3**, (7.11)]

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