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# RECOVERY OF THE TEMPERATURE DISTRIBUTION IN A MULTILAYER FRACTIONAL DIFFUSION EQUATION

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**Abstract**

We study the inverse boundary value problem for fractional diffusion in a multilayer composite medium. Given data in the right boundary of the second layer, the problem is to recover the temperature distribution in the first layer, which is inaccessible for measurement. The problem is ill-posed and we propose a Fourier spectral approach to achieve Hölder approximations. The convergence analysis is performed in both the $L^2$- and $L^\infty$-settings.

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## 1. Introduction

The field of fractional diffusion equations (FDEs) presents challenges and promising applications in the real world (see, for example, [2, 3, 5]). Inverse problems for fractional diffusion equations are now being studied, but most of the work concentrates on single-layer domains. For example, Zheng and Wei [9] studied the inverse problem of recovering the temperature distribution in the domain $0 \le x < 1$ for the time-fractional diffusion equation from the boundary data at $x = 1$. Xiong *et al.* [7] applied the modified kernel method to regularise the fractional sideways diffusion equation where the spatial domain is the interval $[0, 1]$ and Tuan *et al.* [6] extended this work to the inhomogeneous case. In contrast to the previous work, we consider an inverse boundary value problem for an FDE in a multilayer domain. To the best of our knowledge, this is the first investigation of such a problem. More precisely, let $D_t^\gamma$ denote the Caputo fractional derivative of order $\gamma$ $(0 < \gamma < 1)$ defined by

$$\partial_t^\gamma u := \partial_t^\gamma u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u_\tau(x, \tau)}{(t-\tau)^\gamma}\, d\tau.$$

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We consider a composite body consisting of two layers, $\mathbb{D}_1 := \{x \mid 0 \le x \le l_1\}$ and $\mathbb{D}_2 := \{x \mid l_1 \le x \le l_2\}$. These two layers are in perfect thermal contact at the intersection point $x = l_1$. Let $k_1, k_2 > 0$ be the thermal conductivities and $\alpha_1, \alpha_2 > 0$ the thermal diffusivities of the first and second layers, respectively. The temperature distributions in these layers, say $u_1(x,t)$ and $u_2(x,t)$, satisfy the following equations:

- on the first layer $\mathbb{D}_1$,

$$\begin{cases} \partial_t^\gamma u_1(x,t) = \alpha_1 \partial_x^2 u_1(x,t), & x \in \mathbb{D}_1, t > 0, \\ u_1(l_1,t) = u_2(l_1,t), & t > 0, \\ k_1 \partial_x u_1(l_1,t) = k_2 \partial_x u_2(l_1,t), & t > 0; \end{cases} \tag{1.1}$$

- on the second layer $\mathbb{D}_2$,

$$\begin{cases} \partial_t^\gamma u_2(x,t) = \alpha_2 \partial_x^2 u_2(x,t), & x \in \mathbb{D}_2, t > 0, \\ u_2(l_2,t) = g(t), & t > 0, \\ \partial_x u_2(l_2,t) = 0, & t > 0; \end{cases} \tag{1.2}$$

- subject to the homogeneous initial conditions

$$u_1(x,0) = u_2(x,0) = 0, \quad x \in (0, l_2). \tag{1.3}$$

The problem (1.1)–(1.3) can be considered as a Cauchy boundary value problem where the data imposed at the right boundary of the second layer is accessible for measurement. The objective is to reconstruct the whole structure in the inaccessible first layer. The case $\gamma = 1$ has been studied in [8], where the problem (1.1)–(1.3) is shown to be ill-posed in the sense of Hadamard. Thus, it is of interest to know whether the same property remains in the fractional case $0 < \gamma < 1$ and, if so, whether this ill-posedness can be diminished with regularisation techniques.

In the current paper, we aim to answer these questions. We show that the fractional problem is ill-posed in the sense of Hadamard, consistent with the result for standard diffusion (see [8]). We propose a simple but effective Fourier truncation method to overcome the ill-posedness and derive convergence estimates of Hölder type as the noise level tends to zero, in both $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$. Numerical simulations to illustrate the theoretical findings will be provided in forthcoming work.

Throughout the paper, we assume that $k_1 = k_2$ for simplicity in the presentation. As usual, the measurement $g_\delta$ of $g$ may contain an error satisfying

$$\|g_\delta - g\|_{L^2(\mathbb{R})} < \delta, \tag{1.4}$$

where $\delta \in (0,1)$ is a bound for the measurement error. Denote by $\mathbf{H}^p(\mathbb{R})$ the standard Sobolev space. We require the *a priori* information

$$\|u_1(0,\cdot)\|_{\mathbf{H}^p(\mathbb{R})} := \left( \int_{\mathbb{R}} (1+\xi^2)^p |\widehat{u_1}(0,\xi)|^2 \, d\xi \right)^{1/2} \le \mathbf{E} \quad \text{with } p \ge 0. \tag{1.5}$$

## 2. The ill-posedness

We extend all the functions above to the whole line $-\infty < t < \infty$ by making them zero outside the original domains, if necessary. Let

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t}\, dt$$

denote the Fourier transform of a function $f \in L^2(\mathbb{R})$. Applying the Fourier transform with respect to $t$ to both sides of (1.2), we obtain the solution in the second layer,

$$\widehat{u_2}(x,\xi) = \cosh\Big(\sqrt{\frac{(i\xi)^\gamma}{\alpha_2}}(l_2 - x)\Big)\widehat{g}(\xi), \tag{2.1}$$

for $l_1 \le x \le l_2$, where

$$\sqrt{\frac{(i\xi)^\gamma}{\alpha_j}} = \Big(\frac{|\xi|^\gamma}{2^\gamma \alpha_j}\Big)^{1/2}(1 + i\mathrm{sign}(\xi))^\gamma = \frac{|\xi|^{\gamma/2}}{\sqrt{\alpha_j}}\Big(\cos\Big(\frac{\pi}{4}\gamma\Big) + i\sin\Big(\frac{\pi}{4}\gamma\Big)\mathrm{sign}(\xi)\Big)$$

for $j \in \{1, 2\}$. Thanks to the representation (2.1), we can solve the problem in the first layer to obtain

$$\widehat{u_1}(x,\xi) = \widehat{\Theta}(x,\xi)\widehat{g}(\xi), \tag{2.2}$$

where

$$\widehat{\Theta}(x,\xi) = \cosh\Big(\sqrt{\frac{(i\xi)^\gamma}{\alpha_1}}(l_1 - x)\Big)\cosh\Big(\sqrt{\frac{(i\xi)^\gamma}{\alpha_2}}(l_2 - l_1)\Big) + \sqrt{\frac{\alpha_1}{\alpha_2}}\sinh\Big(\sqrt{\frac{(i\xi)^\gamma}{\alpha_1}}(l_1 - x)\Big)\sinh\Big(\sqrt{\frac{(i\xi)^\gamma}{\alpha_2}}(l_2 - l_1)\Big).$$

The solution $u_1(x,t)$ can be recovered by taking the inverse Fourier transform of (2.2). The following lemmas give some useful estimates.

LEMMA 2.1. *For arbitrary $z \in \mathbb{C}$,*

$$\sinh|\Re(z)| \le |\sinh z| \le \cosh \Re(z) \le e^{|\Re(z)|},$$
$$\sinh|\Re(z)| \le |\cosh z| \le \cosh \Re(z) \le e^{|\Re(z)|}.$$

PROOF. The inequalities follow from the definitions by elementary calculations. □

LEMMA 2.2. *Set*

$$\ell(x) := \Big(\frac{l_1 - x}{\sqrt{\alpha_1}} + \frac{l_2 - l_1}{\sqrt{\alpha_2}}\Big)\cos\Big(\frac{\pi}{4}\gamma\Big), \quad \ell_0 = \ell(0).$$

*Then*

$$|\widehat{\Theta}(x,\xi)| \le \Big(1 + \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{|\xi|^{\gamma/2}\ell(x)} \quad \text{for } 0 \le x \le l_1.$$

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Proof. Observe that

$$|\widehat{\Theta}(x,\xi)| \le |\cosh(a_x(\xi)+i\bar{a}_x(\xi))\cosh(b(\xi)+i\bar{b}(\xi))| + \left|\sqrt{\frac{\alpha_1}{\alpha_2}}\sinh(a_x(\xi)+i\bar{a}_x(\xi))\sinh(b(\xi)+i\bar{b}(\xi))\right|,$$

where

$$a_x(\xi)=\sqrt{\frac{|\xi|^\gamma}{\alpha_1}}\cos\Big(\frac{\pi}{4}\gamma\Big)(l_1-x),\quad \bar{a}_x(\xi)=\sqrt{\frac{|\xi|^\gamma}{\alpha_1}}\sin\Big(\frac{\pi}{4}\gamma\Big)\mathrm{sign}(\xi)(l_1-x),$$
$$b(\xi)=\sqrt{\frac{|\xi|^\gamma}{\alpha_2}}\cos\Big(\frac{\pi}{4}\gamma\Big)(l_2-l_1),\quad \bar{b}(\xi)=\sqrt{\frac{|\xi|^\gamma}{\alpha_2}}\sin\Big(\frac{\pi}{4}\gamma\Big)\mathrm{sign}(\xi)(l_2-l_1).$$

By Lemma 2.1,

$$\begin{aligned}|\widehat{\Theta}(x,\xi)| &\le |\cosh(a_x(\xi)+i\bar{a}_x(\xi))||\cosh(b(\xi)+i\bar{b}(\xi))| \\ &\quad + \sqrt{\frac{\alpha_1}{\alpha_2}}|\sinh(a_x(\xi)+i\bar{a}_x(\xi))||\sinh(b(\xi)+i\bar{b}(\xi))| \\ &\le \Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{a_x(\xi)+b(\xi)} = \Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{|\xi|^{\gamma/2}\ell(x)},\end{aligned}$$

which completes the proof. □

Lemma 2.3. *For $0\le x<l_1$, put*

$$\Lambda(x)=\max\bigg\{\Big(\frac{\sqrt{\alpha_1}\ln 2}{(l_1-x)\cos(\pi\gamma/4)}\Big)^{2/\gamma},\Big(\frac{\sqrt{\alpha_2}\ln 2}{(l_2-l_1)\cos(\pi\gamma/4)}\Big)^{2/\gamma}\bigg\},\quad \Lambda_0=\Lambda(0).$$

*Then, for $|\xi|\ge\Lambda(x)$,*

$$|\widehat{\Theta}(x,\xi)| \ge \frac{1}{16}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{|\xi|^{\gamma/2}\ell(x)}.$$

Proof. Observe that

$$\widehat{\Theta}(x,\xi) = \underbrace{\cosh(a_x(\xi)+i\bar{a}_x(\xi))\cosh(b(\xi)+i\bar{b}(\xi))}_{B_1(x,\xi)} + \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\underbrace{\sinh(a_x(\xi)+i\bar{a}_x(\xi))\sinh(b(\xi)+i\bar{b}(\xi))}_{B_2(x,\xi)}.$$

By direct computation,

$$2B_1(x,\xi)=\cosh((a_x(\xi)+b(\xi))+i(\bar{a}_x(\xi)+\bar{b}(\xi))) + \cosh((a_x(\xi)-b(\xi))+i(\bar{a}_x(\xi)-\bar{b}(\xi)))$$

and

$$2B_2(x,\xi)=\cosh((a_x(\xi)+b(\xi))+i(\bar{a}_x(\xi)+\bar{b}(\xi))) - \cosh((-a_x(\xi)+b(\xi))+i(-\bar{a}_x(\xi)+\bar{b}(\xi))).$$

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By Lemma 2.1,

$$\begin{aligned}2|\widehat{\Theta}(x,\xi)| &\geq \Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)|\cosh((a_x(\xi)+b(\xi))+i(\bar{a}_x(\xi)+\bar{b}(\xi)))| \\ &\quad -|\cosh((a_x(\xi)-b(\xi))+i(\bar{a}_x(\xi)-\bar{b})(\xi))| \\ &\quad -\sqrt{\frac{\alpha_1}{\alpha_2}}|\cosh((-a_x(\xi)+b(\xi))+i(-\bar{a}_x(\xi)+\bar{b}(\xi)))| \\ &\geq \frac{1}{2}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)(e^{a_x(\xi)+b(\xi)}-e^{-a_x(\xi)+b(\xi)}-e^{a_x(\xi)-b(\xi)}-e^{-a_x(\xi)-b(\xi)}).\end{aligned}$$

Since $|\xi| \geq \Lambda(x)$,

$$e^{-a_x(\xi)+b(\xi)} \leq \tfrac{1}{4}e^{a_x(\xi)+b(\xi)}, \quad e^{a_x(\xi)-b(\xi)} \leq \tfrac{1}{4}e^{a_x(\xi)+b(\xi)}.$$

Thus, we arrive at the final estimate

$$|\widehat{\Theta}(x,\xi)| \geq \frac{1}{16}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{a_x(\xi)+b(\xi)} = \frac{1}{16}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{|\xi|^{\gamma/2}\ell(x)}.$$

The lemma is proved. □

The following theorem answers the first question raised in the paper.

THEOREM 2.4. *The problem (1.1)–(1.3) is ill-posed in the sense of Hadamard.*

PROOF. We give an example to demonstrate that the problem (1.1)–(1.3) is ill-posed. For any $n \in \mathbb{N}$ with $n \geq \Lambda(x)$, where $\Lambda$ is the same function as in Lemma 2.3, define $\Omega_n := \{\xi \in \mathbb{R}; n \leq \xi \leq n+1\}$. Let $g_n \in L^2(\mathbb{R})$ be the measured data such that

$$\widehat{g_n}(\xi) = \begin{cases}\widehat{g}(\xi)+1/n & \text{if } \xi \in \Omega_n, \\ \widehat{g}(\xi) & \text{if } \xi \in \mathbb{R}\backslash\Omega_n.\end{cases}$$

By Parseval's identity, $\|g_n - g\|_{L^2(\mathbb{R})} = (\int_{\Omega_n} n^{-2}\,d\xi)^{1/2} = n^{-1} \to 0$ as $n \to \infty$. Let $u_1$ and $u_{1n}$ be solutions of (1.1)–(1.3) corresponding to the data $g$ and $g_n$, respectively, so that

$$\widehat{u_1}(x,\xi) = \widehat{\Theta}(x,\xi)\widehat{g}(\xi) \quad \text{and} \quad \widehat{u_{1n}}(x,\xi) = \widehat{\Theta}(x,\xi)\widehat{g_n}(\xi).$$

By Parseval's identity and Lemma 2.3,

$$\lim_{n\to\infty}\|(u_{1n}-u_1)(x,\cdot)\|_{L^2(\mathbb{R})} = \lim_{n\to\infty}\frac{1}{n}\Big(\int_{\Omega_n}|\widehat{\Theta}(x,\xi)|^2\,d\xi\Big)^{1/2} \geq \frac{1}{16}\lim_{n\to\infty}\frac{e^{n^{\gamma/2}\ell(x)}}{n} = +\infty.$$

This proves the ill-posedness of the problem. □

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## 3. The Fourier spectral method

Since the problem (1.1)–(1.3) is ill-posed, any tiny noise in the data may produce a solution which is far away from its exact value. The Fourier spectral method is a very simple, but effective, method to deal with such problems. The idea is to cut off the solution at high frequencies because in this ill-posed problem the information at high frequencies is not a true reflection of the solution. Let $\mathbb{E}_\beta = [-\beta_\delta, \beta_\delta]$ denote the regularisation domain, where $\beta_\delta := \beta(\delta)$ is the regularisation parameter that will be chosen later. In the spirit of Fourier's truncation method (see, for example, [1, 4]), we consider the regularised solution

$$\widehat{\mathbf{u}}^\delta_{1\beta}(x,\xi) = \widehat{\Theta}(x,\xi)\widehat{g}_\delta(\xi)\mathcal{I}_{\mathbb{E}_\beta}(\xi), \tag{3.1}$$

where $\mathcal{I}_{\mathbb{E}_\beta}$ denotes the characteristic function of the interval $\mathbb{E}_\beta$. In other words, by using the inverse Fourier transform, the regularised solution can be represented as

$$\mathbf{u}^\delta_{1\beta}(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{E}_\beta}\widehat{\Theta}(x,\xi)\widehat{g}_\delta(\xi)e^{it\xi}\,d\xi. \tag{3.2}$$

Put

$$\mathbf{D}_p(x) = \|\mathbf{u}^\delta_{1\beta}(x,\cdot) - u_1(x,\cdot)\|_{L^p(\mathbb{R})},$$

the $L^p$-distance between the exact and regularised solutions. The rest of this section is devoted to estimating the distances $\mathbf{D}_2(x)$ and $\mathbf{D}_\infty(x)$.

THEOREM 3.1 (The $L^2$-distance). *Let $u_1$ be the solution of problem (1.1)–(1.3) and $\mathbf{u}^\delta_{1\beta}$ be the regularised solution given by (3.2). Assume that the measured data $g_\delta$ fulfils (1.4). If the exact solution $u_1$ satisfies (1.5) and the regularisation parameter $\beta_\delta$ is given by*

$$\beta_\delta = \Big(\frac{1}{\ell_0}\Big)^{2/\gamma}\Big(\ln\frac{\overline{\mathbf{E}}}{\delta} - \frac{2p}{\gamma}\ln\Big(\ln\frac{\overline{\mathbf{E}}}{\delta}\Big)\Big)^{2/\gamma}, \tag{3.3}$$

*where $\overline{\mathbf{E}} = \mathbf{E} + \delta(e^{e^{8p/\gamma}} + e^{2\ell_0(\Lambda_0)^{\gamma/2}})$, then, for every $x \in [0, l_1)$, we obtain the Hölder convergence estimate*

$$\mathbf{D}_2(x) \le \Big(1 + \sqrt{\frac{\alpha_1}{\alpha_2}} + 16(2\ell_0)^{2p/\gamma}\Big)\overline{\mathbf{E}}^{\ell(x)/\ell_0}\delta^{1-\ell(x)/\ell_0}\Big(\ln\frac{\overline{\mathbf{E}}}{\delta}\Big)^{-2p\ell(x)/\gamma\ell_0}. \tag{3.4}$$

PROOF. From Parseval's identity and the triangle inequality,

$$\mathbf{D}_2(x) \le \underbrace{\|(\widehat{\mathbf{u}}_{1\beta} - \widehat{\mathbf{u}}^\delta_{1\beta})(x,\cdot)\|_{L^2(\mathbb{R})}}_{\mathcal{S}_1(x)} + \underbrace{\|(\widehat{\mathbf{u}}_{1\beta} - \widehat{u}_1)(x,\cdot)\|_{L^2(\mathbb{R})}}_{\mathcal{S}_2(x)}, \tag{3.5}$$

where $\mathbf{u}_{1\beta}$ is the regularised solution (3.1) with respect to the exact data $g$. Let us first evaluate $\mathcal{S}_1(x)$. By Lemma 2.2, (1.4) and (3.3),

$$\begin{aligned}\mathcal{S}_1(x) &= \Big(\int_{\mathbb{E}_\beta}|\widehat{\Theta}(x,\xi)|^2|\widehat{g}_\delta(\xi) - \widehat{g}(\xi)|^2\,d\xi\Big)^{1/2}\\ &\le \Big(1 + \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{(\beta_\delta)^{\gamma/2}\ell(x)}\Big(\int_{\mathbb{E}_\beta}|\widehat{g}_\delta(\xi) - \widehat{g}(\xi)|^2\,d\xi\Big)^{1/2} \le \Big(1 + \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)e^{(\beta_\delta)^{\gamma/2}\ell(x)}\delta.\end{aligned} \tag{3.6}$$

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We are now in a position to estimate $\mathcal{S}_2(x)$. By Lemma 2.3 and (1.5),

$$\mathcal{S}_2(x) \le \bigg(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \frac{1}{|\xi|^{2p}}\bigg|\frac{\widehat{\Theta}(x,\xi)}{\widehat{\Theta}(0,\xi)}\bigg|^2 (1+\xi^2)^p|\widehat{u_1}(0,\xi)|^2\,d\xi\bigg)^{1/2}$$
$$\le 16\bigg(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \frac{e^{2|\xi|^{\gamma/2}(\ell(x)-\ell_0)}}{|\xi|^{2p}}(1+\xi^2)^p|\widehat{u_1}(0,\xi)|^2\,d\xi\bigg)^{1/2}$$
$$\le 16\,\frac{e^{(\beta_\delta)^{\gamma/2}(\ell(x)-\ell_0)}}{(\beta_\delta)^p}\mathbf{E} \le \frac{16}{(\beta_\delta)^p}\overline{\mathbf{E}}^{\ell(x)/\ell_0}\delta^{1-\ell(x)/\ell_0}\Big(\ln\frac{\overline{\mathbf{E}}}{\delta}\Big)^{2p(\ell_0-\ell(x))/\gamma\ell_0}.$$

Since $\overline{\mathbf{E}} > \delta e^{e^{8p/\gamma}}$, it remains to prove that

$$\ln\frac{\overline{\mathbf{E}}}{\delta} \ge \frac{4p}{\gamma}\ln\Big(\ln\frac{\overline{\mathbf{E}}}{\delta}\Big), \quad \text{that is,} \quad \beta_\delta \ge \Big(\frac{1}{2\ell_0}\Big(\ln\frac{\overline{\mathbf{E}}}{\delta}\Big)\Big)^{2/\gamma}. \tag{3.7}$$

Put $\alpha = 8p/\gamma$, $y = \ln\overline{\mathbf{E}}/\delta$ and $h(y) = y - (\alpha/2)\ln(y)$. Since $\alpha > 0$, we claim that $h(y) \ge 0$ for $y \in (e^\alpha, +\infty)$. This follows since $h'(y) = 1 - \alpha/2y > 1 - \alpha/2e^\alpha > 0$ for all $y \in (e^\alpha, +\infty)$. Then $h(y) > h(e^\alpha) = e^\alpha - \frac{1}{2}\alpha^2 \ge 0$. Thus, the estimate (3.7) holds. Having disposed of this preliminary step, we can estimate $\mathcal{S}_2$ by

$$\mathcal{S}_2(x) \le 16(2\ell_0)^{2p/\gamma}\overline{\mathbf{E}}^{\ell(x)/\ell_0}\delta^{1-\ell(x)/\ell_0}\Big(\ln\frac{\overline{\mathbf{E}}}{\delta}\Big)^{-2p\ell(x)/\gamma\ell_0}. \tag{3.8}$$

Substituting (3.6) and (3.8) into (3.5) and noting that the parameter $\beta_\delta$ is chosen as in (3.3), we obtain (3.4), which is the desired conclusion. The theorem is proved. □

Theorem 3.2 (The $L^\infty$-distance). *Let $u_1$ be the solution of problem (1.1) and $\mathbf{u}^\delta_{1\beta}$ be as in Theorem 3.1. Let the measured data $g_\delta$ satisfy (1.4). Suppose that the exact solution $u_1$ satisfies the a priori bound (1.5) for $p > \frac{1}{2}$. Denote*

$$\widetilde{\ell}(x) = \ell(x) + \frac{1}{\gamma} - \frac{1}{2} \quad \text{and} \quad \widetilde{\mathbf{E}} = \mathbf{E} + \delta(e^{e^{4(2p-1)/\gamma}} + e^{2\widetilde{\ell}_0(\Lambda(0))^{\gamma/2}}).$$

*If the regularisation parameter $\beta_\delta$ is selected by*

$$\beta_\delta = \Big(\frac{1}{\widetilde{\ell}_0}\Big)^{2/\gamma}\Big(\ln\frac{\widetilde{\mathbf{E}}}{\delta} - \frac{2p-1}{\gamma}\ln\Big(\ln\frac{\widetilde{\mathbf{E}}}{\delta}\Big)\Big)^{2/\gamma}, \tag{3.9}$$

*then, for every $x \in [0, l_1)$,*

$$\mathbf{D}_\infty(x) \le \bigg[\sqrt{\frac{2}{\gamma\ell(l_1)}}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big) + 16\sqrt{\frac{2}{2p-1}}(2\widetilde{\ell}_0)^{(2p-1)/\gamma}\bigg]$$
$$\times\,\widetilde{\mathbf{E}}^{\widetilde{\ell}(x)/\widetilde{\ell}_0}\delta^{1-\widetilde{\ell}(x)/\widetilde{\ell}_0}\Big(\ln\frac{\widetilde{\mathbf{E}}}{\delta}\Big)^{(1-2p)\widetilde{\ell}(x)/\gamma\widetilde{\ell}_0}. \tag{3.10}$$

PROOF. First, $\mathbf{D}_\infty(x) \le \mathcal{T}_1(x) + \mathcal{T}_2(x)$ because

$$\left|\int_{\mathbb{R}} \widehat{\Theta}(x,\xi)(\widehat{g}_\delta \mathcal{I}_{\beta_\delta} - \widehat{g})e^{it\xi}\, d\xi\right| \le \underbrace{\int_{\mathbb{E}_\beta} |\widehat{\Theta}(x,\xi)(\widehat{g}_\delta - \widehat{g})|\, d\xi}_{\mathcal{T}_1(x)} + \underbrace{\int_{\mathbb{R}\setminus\mathbb{E}_\beta} |\widehat{\Theta}(x,\xi)\widehat{g}|\, d\xi}_{\mathcal{T}_2(x)}.$$

The proof now naturally falls into two steps. For the first step, it follows immediately from Hölder's inequality and the choice of regularisation parameter (3.9) that

$$\begin{aligned}
\mathcal{T}_1(x) &\le \sqrt{2}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)\Big(\int_0^{\beta_\delta} e^{2\xi^{\gamma/2}\ell(x)}\, d\xi\Big)^{1/2}\delta \\
&= \sqrt{2}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)\Big(\int_0^{\beta_\delta} \frac{\xi^{1-\gamma/2}}{\gamma\ell(x)}\, d(e^{2\xi^{\gamma/2}\ell(x)})\Big)^{1/2}\delta \\
&\le \sqrt{2}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)\frac{(\beta_\delta)^{(2-\gamma)/4}}{\sqrt{\gamma\ell(l_1)}} e^{(\beta_\delta)^{\gamma/2}\ell(x)}\delta \\
&\le \sqrt{\frac{2}{\gamma\ell(l_1)}}\Big(1+\sqrt{\frac{\alpha_1}{\alpha_2}}\Big)\widetilde{\mathbf{E}}^{\widetilde{\ell}(x)/\widetilde{\ell}_0}\delta^{1-\widetilde{\ell}(x)/\widetilde{\ell}_0}\Big(\ln\frac{\widetilde{\mathbf{E}}}{\delta}\Big)^{(1-2p)\widetilde{\ell}(x)/\gamma\widetilde{\ell}_0}. \qquad (3.11)
\end{aligned}$$

To reach the conclusion, it is necessary to estimate $\mathcal{T}_2$. Again, in view of Hölder's inequality and the parameter choice (3.9),

$$\begin{aligned}
\mathcal{T}_2(x) &= \int_{\mathbb{R}\setminus\mathbb{E}_\beta} \frac{1}{(1+\xi^2)^{p/2}}\left|\frac{\widehat{\Theta}(x,\xi)}{\widehat{\Theta}(0,\xi)}\right|(1+\xi^2)^{p/2}|\widehat{u}_1(0,\xi)|\, d\xi \\
&\le 16\Big(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \frac{e^{2|\xi|^{\gamma/2}(\ell(x)-\ell_0)}}{|\xi|^{2p}}\, d\xi\Big)^{1/2}\|u_1(0,\cdot)\|_{\mathbf{H}^p(\mathbb{R})} \\
&\le 16\sqrt{2}e^{(\beta_\delta)^{\gamma/2}(\ell(x)-\ell_0)}\Big(\int_{\beta_\delta}^{\infty} \frac{1}{\xi^{2p}}\, d\xi\Big)^{1/2}\mathbf{E} \\
&\le \frac{16\sqrt{2}}{\sqrt{2p-1}}\,\frac{\widetilde{\mathbf{E}}^{\widetilde{\ell}(x)/\widetilde{\ell}_0}\delta^{1-\widetilde{\ell}(x)/\widetilde{\ell}_0}(\ln\widetilde{\mathbf{E}}/\delta)^{(1-2p)(\widetilde{\ell}(x)-\widetilde{\ell}_0)/\gamma\widetilde{\ell}_0}}{(\beta_\delta)^{(2p-1)/2}} \\
&\le \frac{16\sqrt{2}}{\sqrt{2p-1}}(2\widetilde{\ell}_0)^{(2p-1)/\gamma}\widetilde{\mathbf{E}}^{\widetilde{\ell}(x)/\widetilde{\ell}_0}\delta^{1-\widetilde{\ell}(x)/\widetilde{\ell}_0}\Big(\ln\frac{\widetilde{\mathbf{E}}}{\delta}\Big)^{(1-2p)\widetilde{\ell}(x)/\gamma\widetilde{\ell}_0}, \qquad (3.12)
\end{aligned}$$

where the last inequality is obtained by using an argument similar to that for (3.7). By substituting (3.11) and (3.12) into the estimate $\mathbf{D}_\infty(x) \le \mathcal{T}_1(x) + \mathcal{T}_2(x)$, we arrive at the final conclusion (3.10). □

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