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MAPS WITH LOCALLY FLAT SINGULAR SETS

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1. Introduction. A map $f: M \to N$ is *topologically equivalent* to $g: X \to Y$ if there exist homeomorphisms $\alpha: M \to X$ and $\beta: N \to Y$ such that $\beta f \alpha^{-1} = g$. At $x \in M$, f is *locally topologically equivalent* to g if, for every neighbourhood $W \subset M$ of x, there exist neighbourhoods $U \subset W$ of x and V of $f(x)$ such that $f|U: U \to V$ is topologically equivalent to g.

1.1. *Definition.* Given a map $f: M \to N$ and $x \in M$, let F be the component of $f^{-1}(f(x))$ containing x. The *singular set* A_f is defined as follows: $x \in M - A_f$ if and only if there are neighbourhoods U of F and V of $f(x)$ such that $f|U: U \to V$ is topologically equivalent to the product projection map of $V \times F$ onto V.

Given maps $\psi: P \to Q$ and $\omega: R \to S$, define $\psi \times \omega: P \times R \to Q \times S$ by $\psi \times \omega(p, r) = (\psi(p), \omega(r))$. Define the *open cone* $c(P)$ to be the identification space obtained from $P \times [0, 1)$ by identifying $P \times \{0\}$ to a point p^*. The cone of the empty set will be a point. Let ι be the identity map on $[0, 1)$, and let the *cone map* $c(\psi): c(P) \to c(Q)$ be the map induced by $\psi \times \iota$. Let ι_k be the identity map on E^k.

A symbol such as N^p will denote a manifold of dimension p. A submanifold K^q of N^p is said to be *locally flat* if for each $x \in K^q$ there exist a neighbourhood U of x in N^p and a homeomorphism $\alpha: (U, U \cap K^q) \to (E^p, E^q)$. Let G be the ring of integers Z or a field of characteristic p, p prime or zero. For the definition of a *cohomology n-manifold* (denoted by n-cm) over G see [**3**, p. 9, Definition 3.3]. An n-cm is *sphere-like* if it has the cohomology groups of an n-sphere.

If $A \subset M$, then $M - A$ is said to be *locally simply connected* at $x \in A$ if for each open neighbourhood W of x there exists an open neighbourhood $U \subset W$ of x such that continuous images of S^1 in $U - A$ are null-homotopic in $W - A$.

A map $f: M \to N$ is *proper* if for each compact set $K \subset N$, $f^{-1}(K)$ is compact.

1.2. Theorem. *Let M be an n-*cm *over G and $f: M \to N^p$ a proper map such that*

(1) *$f|A_f$ is a homeomorphism, $f^{-1}(f(A_f)) = A_f$, and*

(2) *$f(A_f)$ is a locally flat q-manifold.*

Then $q < p$, and at $x \in A_f$ the map f is locally topologically equivalent to $c(\psi) \times \iota_q$, where $\psi: K \to S^{p-q-1}$ is a bundle map, $K = \phi$ when $n < p$, and K

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is a sphere-like $(n - q - 1)$-cm *when* $n \geqq p$. *In addition, if* $n \geqq p$ *and* $f^{-1}(y)$ *is a manifold for each* y *in a neighbourhood of* $f(x)$, *then*

(a) *when* $q = p - 2$, $\psi: S^1 \to S^1$ *is a d-to-*1 *covering map;*

(b) *if* $M^n - A_f$ *is locally simply connected at* x *and* $G = Z$, *then* K *is a homotopy sphere;*

(c) *if* $q \leqq p - 3$ *and the hypothesis of* (b) *holds, then* $q = p - 3$, $p - 5$, *or* $p - 9$, *and* $\psi: S^{2p-2q-3} \to S^{p-q-1}$ *has fibre* S^1, T^3 (*homotopy* 3-*sphere*) *or* S^7.

1.3. *Remark.* Note that f in Theorem 1.2 is actually a singular fibering when $n \geqq p$ [**5**, p. 71]. The additional hypothesis in (b) is satisfied if M^n is a manifold and f is C^n [**5**, p. 72, Theorem 1.3], or if $f^{-1}(y)$ is a manifold and A_f is locally flat. When $q = p - 3$ in (c), the map ψ can be taken to be the Hopf map [**10**, p. 64, Lemma 2.7]. Antonelli [**1**; **2**] has classified singular fiberings of spheres when A_f and $f(A_f)$ are locally flat manifolds.

A compact set is said to be *G-acyclic* if it has the G-cohomology groups of a point.

1.4. Theorem. *Let* $f: M^n \to N^p$, $n > p$, *be a proper* C^n *map such that every component of* $f^{-1}(y) \cap A_f$ *is* Z_2-*acyclic for each* $y \in N^p$. *Then there exists a closed set* $Y \subset f(A_f)$ *with* $\dim Y < \max(0, \dim f(A_f))$ *so that if* $y \in N^p - Y$ *and* F *is a component of* $f^{-1}(y)$, *then there are neighbourhoods* U *of* F *and* V *of* y *such that* $f|U: U \to V$ *is topologically equivalent to* $\theta\lambda$, *where*

(a) $\lambda: U \to X$ *is a monotone map onto the n*-cm (*over* Z_2) X *with* $A_\lambda \subset A_f$, *and*

(b) $\theta: X \to E^p$ *satisfies the hypothesis of Theorem* 1.2.

1.5. *Remark.* Theorem 1.4 was proved in [**6**] for singular fiberings. Note that if $F \subset M^n - A_f$, then λ can be taken to be the identity homeomorphism and $A_\theta = \emptyset$.

2. Proofs of Theorems 1.2 and 1.4.

2.1. Lemma. *If* $\theta: X \to c(K) \times E^q$ *is a proper singular fibering with* K *a compact manifold and* $\theta(A_\theta) = \{k^*\} \times E^q$, *then there exists a bundle with total space* L, *base space* K, *and map* $\psi: L \to K$ *such that* θ *is topologically equivalent to*

$$c(\psi) \times \iota_q: c(L) \times E^q \to c(K) \times E^q.$$

Proof. Consider the fiber bundle ξ with map

$$i^{-1} \circ \theta|X - A_\theta: A - A_\theta \to K \times (0, 1) \times E^q,$$

where

$$i: K \times (0, 1) \times E^q \to c(K) \times E^q - \theta(A_\theta)$$

is the inclusion map. It follows from [**9**, p. 53, Theorem 11.4] that ξ is equivalent to a bundle of the form $\xi' \times (0, 1) \times E^q$, where ξ' is a bundle with total space L, base space K, and map $\psi: L \to K$. Thus by definition of bundle equivalence, there exists a homeomorphism h such that the following diagram commutes:

$$\begin{array}{ccc} X - A_\theta & \xrightarrow{h} & L \times (0, 1) \times E^q \\ \Big\downarrow \theta|X - A_\theta & & \Big\downarrow \psi \times \iota \times \iota_q \\ c(K) \times E^q - \theta(A_\theta) & \xrightarrow{i^{-1}} & K \times (0, 1) \times E^q \end{array}$$

The map $\psi \times \iota \times \iota_q$ has an extension to $c(\psi) \times \iota_q$: $c(L) \times E^q \to c(K) \times E^q$. Extend the map h to a homeomorphism $\bar{h}$ of X onto $c(L) \times E^q$ by defining $\bar{h}(x) = (c(\psi) \times \iota_q)^{-1}(\theta(x))$ for $x \in A_\theta$, $\bar{h} = h$ otherwise. Since $\theta(A_\theta)$ and $c(\psi) \times \iota_q|l^* \times E^q$ are both one-to-one, $\bar{h}$ is well-defined and one-to-one. If $y \in l^* \times E^q$, then $y = (l^*, t)$ for some $t \in E^q$. Then $x = \theta^{-1}((c(\psi) \times \iota_q)(l^*, t))$ is mapped by $\bar{h}$ into y, and so $\bar{h}$ is onto. The continuity of $\bar{h}$ and $\bar{h}^{-1}$ follows from the condition that θ and $c(\psi) \times \iota_q$ are proper.

2.2. *Proof of Theorem* 1.2. Suppose that $q = p$. Then there is an open set $V \subset f(A_f)$, and if U is a component of $f^{-1}(V)$, then $f|U$ is a homeomorphism. Since U is open in M, $U \subset M - A_f$, which is a contradiction; hence $q < p$. If $n < p$, then $M \subset A_f$, and so conclusions of the theorem are satisfied with $K = \phi$.

Assume that $n \geqq p$. Lemma 2.1 implies that since $f(A_f)$ is locally flat, if $x \in A_f$ there exist a neighbourhood V of $f(x)$, a homeomorphism α sending $(V, V \cap f(A_f))$ onto $(c(S^{p-q-1}) \times E^q, s^* \times E^q)$, and a component U of $f^{-1}(V)$ containing x such that $f|U$ is topologically equivalent to $c(\psi) \times \iota_q$, where ψ: $K \to S^{p-q-1}$ is a bundle map. Since $c(K) \times E^q$ is an n-cm over G, $c(K)$ is an $(n - q)$-cm over G [**3**, p. 15, Theorem 4.10]. From the cohomology sequence with compact supports of the pair $(c(K), k^*)$ and the Künneth formula, it follows that K is a sphere-like $(n - q - 1)$-cm over G.

Assume that $n \geqq p$ and $f^{-1}(y)$ is a manifold for each y in a neighbourhood of $f(x)$. If $q = p - 2$ and $n \geqq p$, then the bundle map ψ: $K \to S^1$ can be factored into a monotone bundle map g onto S^1 followed by a d-to-1 covering map. By the homotopy sequence for a fibering [**8**, p. 377, Theorem 10] and [**6**, p. 45, Theorem 6.1], $H_1(K; Z)$ has a summand Z. Thus $\bar{H}^1(K; Z)$ has a summand Z, and hence K is S^1 and ψ is a d-to-1 covering map.

Let $M - A_f$ be locally simply connected at x. Let $W \subset U$ be a neighbourhood of x such that i_*: $\pi_1(W - A_f) \to \pi_1(U - A_f)$ induced by inclusion is the zero map. There exists a neighbourhood $V' \subset V$ so that the component U' of $f^{-1}(V')$ containing x is contained in W and $U' - A_f$ is a deformation retract of $U - A_f$. The inclusion map j: $U' - A_f \to U - A_f$ induces an isomorphism on fundamental groups, but since j can be factored through $W - A_f$, $\pi_1(U - A_f) = 0$ and K is a homotopy sphere. The conclusions desired in (c) follow from [**4**] (see [**10**, p. 64, Lemma 2.7]).

A map f is *quasi-monotone* if for each region V in the range and component U of $f^{-1}(V)$, $f(U) = V$.

2.3. *Remark.* Let M be an orientable n-cm over G, $n \geqq p$, N^p connected, and let $f: M \to N^p$ be a proper map. If each component of $f^{-1}(y) \cap A_f$ is G-acyclic, $y \in N^p$, then

(1) if $\dim f(A_f) = p$, $n = p$;

(2) if $\dim f(A_f) < p - 1$, then

(a) f is quasi-monotone, and

(b) there exists a positive integer k such that if $y \in N^p - f(A_f)$, then $f^{-1}(y)$ has exactly k components, while if $y \in f(A_f)$, $f^{-1}(y)$ has at most k components.

Proof. We may as well assume that M is connected. Let hg be the monotone light factorization of f [**13**, p. 141, Theorem 4.1]. Since M is orientable, $H_c^n(U; G) = G$ for any connected open subset U of M [**3**, p. 11, Theorem 4.3]. Thus $H_c^n(g(U); G) = G$ [**8**, p. 346, Theorem 18]. Since light maps cannot lower dimension [**7**, p. 91, Theorem VI7] we have $\dim g(U) \leqq \dim f(U)$. If V is an open euclidean subset of N^p contained in $f(A_f)$, then let U be a component of $f^{-1}(V)$. Then $\dim g(U) \geqq n$, but $\dim f(U) = p$, which implies that $n = p$ when $\dim f(A_f) = p$.

Suppose that $\dim f(A_f) < p - 1$. If V is a region in N^p and U a component of $f^{-1}(V)$, then $f| U - f^{-1}(f(A_f))$ is a proper open map into the connected set $V - f(A_f)$. If $f(U) \neq V$, then $f(U) \subset V \cap f(A_f)$, since any point in $V \cap f(A_f)$ is a limit point of $V - f(A_f)$. Thus $\dim g(U) \leqq p - 2$, which is a contradiction to $n \geqq p$. Hence $f(U) = V$ and f is quasi-monotone. The proof of (2) (b) is similar to the second paragraph of the proof for [**10**, p. 64, Lemma 2.5].

2.4. Lemma. *Suppose that $f: M \to N^p$ is a proper map, M an orientable n-*cm *over G, $n > p$, $f(A_f)$ a locally flat q-manifold, and $f^{-1}(y)$ is G-acyclic for $y \in f(A_f)$. Then $f = \theta\lambda$, where*

(a) *$\lambda: M \to X$ is a monotone map onto the orientable n-*cm *X with $A_\lambda \subset A_f$, and*

(b) *$\theta: X \to N^p$ satisfies the hypothesis of Theorem* 1.2.

Proof. Let λ be the map corresponding to a decomposition of M with non-degenerate elements consisting of inverse images of points in $f(A_f)$. Since λ is acyclic, $\lambda(M) = X$ is an orientable n-cm over G [**14**, p. 21, Theorem 2], and $A_\lambda \subset A_f$. Let θ correspond to the decomposition of M whose non-degenerate elements are inverse images of points in $N^p - f(A_f)$. Then $f = \theta\lambda$ and $\theta|A_\theta: A_\theta \to f(A_f)$ is a homeomorphism. In addition, $\theta^{-1}(\theta(A_\theta)) = A_\theta$.

2.5. *Proof of Theorem* 1.4. Follows immediately from Lemma 2.4 and [**12**, Theorem 1.2].

2.6. *Definition.* Let $f: M^n \to N^p$. The *branch set* $B_f \subset M^n$ is defined by: $x \in M^n - B_f$ if and only if f at x is locally topologically equivalent to the natural product projection map of E^n onto E^p.

2.7. COROLLARY. *Let $f: M^{n+1} \to N^n$ satisfy the hypothesis of Lemma 2.4 with $G = Z$. Then*

(a) *X is an $(n + 1)$-manifold, and*

(b) *at $x \in B_\theta$, θ is locally topologically equivalent to $c(\psi) \times \iota_q$, where $q = n - 3$, θ is open, and $\psi: S^3 \to S^2$ is the Hopf map, or $q = n - 1$ and $\psi: S^1 \to S^0$ is a constant map.*

Proof. By Lemma 2.4 we know that X is an $(n + 1)$-cm over Z and that θ satisfies the hypothesis of Theorem 1.2. Thus at $x \in A_\theta$, θ is locally topologically equivalent to $c(\psi) \times \iota_q$, where $\psi: K \to S^{n-q-1}$ is a bundle map and K is an $(n - q)$-manifold with the Z cohomology groups of a sphere. If $q = n - 1$, then clearly $\psi: S^1 \to S^0$ is a constant map and X is an $(n + 1)$-manifold; thus suppose that $q \leqq n - 2$. The bundle map ψ can be factored into a monotone bundle map followed by a finite covering map. Since the intermediate space is always homeomorphic to S^{n-q-1}, we will consider only the situation in which ψ is itself monotone; thus the fibre is S^1. If $n - q - 1 = 1$, then K is either $S^1 \times S^1$ or the Klein bottle, neither of which is a cohomology sphere. If $n - q - 1 \geqq 2$, we can reduce the structure group of the bundle to S^1 (see third paragraph of proof for [**10**, p. 64, Lemma 2.7]). Now by [**9**, p. 99, Theorem 18.5], K is $S^{n-1} \times S^1$ for $n - q - 1 > 2$, while K is a lens space for $n - q - 1 = 2$ [**9**, p. 135, 26.2]; thus the Hopf map is the only possibility for ψ and X is an $(n + 1)$-manifold. It follows from [**11**, Proposition 2.1] that $A_\theta = B_\theta$.

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