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ADDITIVE GROUPS OF RINGS WHOSE SUBRINGS ARE IDEALS

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An Abelian group G is called an SI-group if for every ring R with additive group $R^+ = G$, every subring S of R is an ideal in R. A complete description is given of the torsion SI-groups, and the completely decomposable torsion free SI-groups. Results are obtained in other cases as well.

All groups considered in this paper are Abelian, with addition the group operation. Rings are assumed to be associative, but need not possess a unity. Most of the results which will be obtained remain true for non-associative rings. R^+ will denote the additive group of a ring R. The type function will be denoted by t. Both the ring and the additive group of integers and rational numbers will be denoted by $\mathbb{Z}$ and $\mathbb{Q}$ respectively. Terminology and notation will mostly follow [**3, 4**]. One departure is the product of types in [**4**], which will be called the sum of types.

The object of this paper is to describe groups G satisfying the following property: If R is a ring with additive group G, then every subring of R is an ideal in R.

DEFINITION: A ring R is an SI-ring if all of its subrings are ideals. A group G is an SI-group if every ring R with additive group $R^+ = G$ is an SI-ring.

LEMMA 1. *Let R be a commutative ring with $R^+ = G$, and let M be an R-module, satisfying $RM \neq 0$, with $M^+ = H$. Then $G \oplus H$ is not an SI-group.*

PROOF: For $r_1, r_2 \in R$, and $m_1, m_2 \in M$ define $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. This multiplication induces a ring structure S on $G \oplus H$. However the subring $T = \{(r, 0) \mid r \in R\}$ is not an ideal in R, because choosing $r \in R$, and $m \in M$ satisfying $rm \neq 0$, yields that $(r, 0)(0, m) = (0, rm) \notin T$. ∎

COROLLARY 2. *For every Abelian group $H \neq 0$, the group $\mathbb{Z} \oplus H$ is not an SI-group.*

LEMMA 3. *A direct summand of an SI-group is an SI-group.*

PROOF: Let $G = H \oplus K$, and suppose that there exists a non SI-ring S with $S^+ = H$. Let R be the ring with $R^+ = G$, and multiplication defined by $(h_1, k_1)(h_2, k_2) =$

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$(h_1h_2, 0)$, where h_1h_2 is the product in S. Let T be a subring of S which is not an ideal in S, then $\{(t,0) \mid t \in T\}$ is a subring of R which is not an ideal in R.

It is well known, [**3**, Theorem 21.2], that a divisible subgroup of a group G is a direct summand of G. This coupled with Lemma 3 yields:

COROLLARY 4. *If $\mathbb{Q}$ is a subgroup of G, then G is not an SI-group.*

An immediate consequence of Corollary 4 is:

COROLLARY 5. *A torsion free SI-group is reduced.*

A group G is nil if the only ring R satisfying $R^+ = G$ is the ring with trivial multiplication, $R^2 = 0$.

LEMMA 6. *Let p be a prime. A p-group G is an SI-group if and only if either $G = Z(p^n)$, n a positive integer, or $G = D \oplus Z(p^n)$, with D a divisible p-group, and $n = 0$ or 1.*

PROOF: Suppose that G is an SI-group, and let D be the maximal divisible subgroup of G. Then $G = D \oplus H$. If $H \neq Z(p^n)$, n a non-negative integer, then there exists positive integers $n \leqslant m$ such that $Z(p^m) \oplus Z(p^n)$ is a direct summand of G, [**3**, Corollary 27.3]. The ring Z_{p^m} of integers modulo p^m has additive group $Z(p^m)$, and $Z(p^n)$ is a unital Z_{p^m}-module. Therefore G is not an SI-group by Lemmas 1 and 3, a contradiction. It may therefore be assumed that $G = D \oplus Z(p^n)$, n a non-negative integer, and it remains to be shown that if $D \neq 0$, then $n = 0$ or $n = 1$. If $D \neq 0$, then $H = Z(p^\infty) \oplus Z(p^n)$ is a direct summand of G, so by Lemma 3, it suffices to show that H is not an SI-group for $n > 1$. Let $Z(p^n) = (a)$, $|a| = p^n$, and let $d \in Z(p^\infty)$, $|d| = p^n$. Defining multiplication in H via $(x, ka)(y, ma) = (kmd, 0)$ for all $x, y \in Z(p^\infty)$, and all integers k, m, induces a ring structure R on H. The subring S of R generated by $x = \left(p^{n-1}d, p^{n-1}a\right)$ is $S = \{kx \mid 0 \leqslant k < p\}$. Since $x \cdot (0,a) = \left(p^{n-1}d, 0\right) \notin S$, it follows that S is not an ideal in R.

Conversely, if $G = Z(p^n)$, n a positive integer, then G is clearly an SI-group. If G is divisible, then G is a nil group, [**1**, Theorem 2.1.1], and hence an SI-group. It remains to show that $G = D \oplus Z(p)$, with D divisible, is an SI-group. Let R be a ring with $R^+ = G$, let S be a rubring of R, and let $x_0 = (0, a)$, with $|a| = p$. Since $RD = DR = 0$, [**1**, p.8, 1], it suffices to show that $x_0 S \subseteq S$, and that $Sx_0 \subseteq S$. Since D annihilates R, it follows that for $x_1 = (d_1, m_1 a)$, $x_2 = (d_2, m_2 a) \in R$, the product $x_1 x_2 = m_1 m_2 x_0^2$. Therefore if $x_0^2 = 0$, then $x_0 S = S x_0 = 0$, and S is an ideal in R. If $x_0^2 \neq 0$, then since $|a| = p$, it follows that $\left|x_0^2\right| = p$. Let $x = (d, ma) \in S$. If $m = 0$, then $x_0 \cdot x = x \cdot x_0 = 0$. It may therefore be assumed that $0 < m < p$. Now $x^2 = m^2 x_0^2$. Since $p \nmid m$ there exists an integer t such that $tm^2 = 1 \bmod p$. Therefore $x_0^2 = tx^2 \in S$, and so $x_0 \cdot x = x \cdot x_0 = m x_0^2 \in S$. ∎

It is well known, [**3**, Theorem 8.4], that a torsion group G is the direct sum of its

p-primary components, $G = \bigoplus_p G_p$. Similarly, a torsion ring R is the ring direct sum of its p-primary components, $R = \bigoplus_p R_p$, [**4**, p.278(B)]. If S is a subring of a torsion ring R, then $S_p = S \cap R_p$ for all primes p, and $R_q \cdot R_p = R_p \cdot R_q = 0$ for all primes $q \neq p$. It therefore follows that a torsion group G is an SI-group if and only if each of its p-primary components is an SI-group. This combined with Lemma 6 yields:

THEOREM 7. *A torsion group G is an SI-group if and only if $G = D \oplus \bigoplus_{p \in P} Z(p^{n_p})$, with D a divisible torsion group, P a set of distinct primes, n_p a non-negative integer for every $p \in P$, and $n_p \leqslant 1$ if $D_p \neq 0$.*

LEMMA 8. *Let G be an SI-group. Then G_p, the p-primary component of G, is a direct summand of G for every prime p.*

PROOF: Let D_p be the maximal divisible subgroup of G_p, then $G = D_p \oplus K$, [**3**, Theorem 21.2]. If $K_p \neq 0$ then, by [**3**, Corollary 27.3], $G = D_p \oplus Z(p^n) \oplus H$, with n a positive integer. If $H_p \neq 0$ then, again by [**3**, Corollary 27.3], there exists a positive integer m such that $Z(p^n) \oplus Z(p^m)$ is a direct summand of G. This contradicts Lemmas 3 and 6. Therefore $G_p = D_p \oplus Z(p^n)$ and so G_p is a direct summand of G. ∎

It clearly follows from Lemma 8 and Lemma 3, that the p-primary component of an SI-group is an SI-group, for every prime p. This combined with the remarks preceding Theorem 7 yields:

COROLLARY 9. *Let G be an SI-group. Then G_t, the torsion part of G, is an SI-group.*

THEOREM 10. *Let G be a mixed SI-group. Then $G_t = \bigoplus_{p \in P} Z(p^{n_p})$, with P a non empty set of distinct primes, and n_p a positive integer for all $p \in P$.*

PROOF: By Corollary 9, and Theorem 7, it suffices to show that D_p, the maximal divisible subgroup of G_p, is trivial for every prime p. If not, then $G = Z(p^\infty) \oplus H$ for some prime p. Let $0 \neq a \in H$ be torsion free, and let $d \in Z(p^\infty)$, with $|d| = p^2$. There exists a homomorphism $\phi : (a) \oplus (a) \to Z(p^\infty)$ satisfying $\phi(a \otimes a) = d$. Since $Z(p^\infty)$ is injective in the category of Abelian groups, [**3**, Theorem 21.1], ϕ extends to a homomorphism $\phi : H \otimes H \to Z(p^\infty)$. The multiplication $(d_1, h_1)(d_2, h_2) = (\phi(h_1 \otimes h_2), 0)$ for all $d_1, d_2 \in Z(p^\infty)$, and all $h_1, h_2 \in H$, induces a ring structure R on G. Since $(0, pa)(0, pa) = 0$, it follows that the subring S of R generated by $(0, pa)$ is $S = \{(0, npa) \mid (n \in \mathbb{Z})\}$. However $(0, a)(0, pa) = (pd, 0) \notin S$, and so S is not an ideal in R. ∎

COROLLARY 11. *Let G be an SI-group, and let p be a prime for which $G_p \neq 0$. Then $G = G_p \oplus H$, and H is p-divisible.*

PROOF: G_p is a direct summand of G by Lemma 8. It may be assumed by

Theorem 10, that $G_p = (a)$, with $|a| = p^n$. Suppose that H is not p-divisible. Then there exists $h \in H$ with zero p-height. $\overline{h} = h + pH$ has order p in the group $\overline{H} = H/pH$, and $\overline{H} = (\overline{h}) \oplus K$. Let $\phi : (a) \otimes \overline{H} \to (a)$ be the homomorphism induced by the maps $\phi[a \otimes (m\overline{h}, k)] = ma$, for all $0 \leqslant m < p$, and all $k \in K$. The multiplication $(m_1 a, h_1)(m_2 a, h_2) = (m_1\phi(a \otimes \overline{h}_2) + m_2\phi(a \otimes \overline{h}_1), 0)$ for all $0 \leqslant m_1, m_2 < p^n$, and all $h_1, h_2 \in H$, induces a ring structure R on G. The subring S of R generated by $(0, h)$ is $= \{(0, nh) \mid n \in \mathbb{Z}\}$, but $(a, 0)(0, h) = (a, 0) \notin S$, and so S is not an ideal in R. ∎

An immediate consequence of Corollary 11 is:

Corollary 12. *Let G be an SI-group, and let p be a prime such that $G_p \neq 0$. Then G/G_t is p-divisible.*

Lemma 13. *Let $G = \bigoplus_{i \in I} G_i$. If there exist $i, j, k \in I$ such that $i \neq j$, $i \neq k$, and if there exists $\phi \in \operatorname{Hom}(G_i \otimes G_j, G_k)$ such that $\phi \neq 0$, then G is not an SI-group.*

Proof: Let $0 \neq \phi \in \operatorname{Hom}(G_i \otimes G_j, G_k)$. There are two cases to consider:

(1) $j \neq k$, and

(2) $j = k$.

It may be assumed that $i = 1$, $j = 2$, and $k = 3$ in case (1), and that $i = 1$, $j = k = 2$ in case (2). By Lemma 3, it suffices to show that $H = G_1 \oplus G_2 \oplus G_3$ is not an SI-group in cae (1), and that $H = G_1 \oplus G_2$ is not an SI-group in case (2). Let $a_u \in G_u$, $u = 1, 2$, such that $\phi(a_1 \otimes a_2) \neq 0$. Let R be the ring with $R^+ = H$, and multiplication defined by $(b_1, b_2, b_3)(c_1, c_2, c_3) = (0, 0, \phi(b_1 \otimes c_2 + c_1 \otimes b_2))$ for all $b_u, c_u \in G_u$, $u = 1, 2, 3$ in case (1), and by $(b_1, b_2)(c_1, c_2) = (0, \phi(b_1 \otimes c_2 + c_1 \otimes b_2))$ for all $b_u, c_u \in G_u$, $u = 1, 2$ in case (2). Let S be the subring of R generated by $(a_1, 0, 0)$ in case (1), and by $(a_1, 0)$ in case (2). Then $(a_1, 0, 0)(0, a_2, 0) = (0, 0, \phi(a_1 \otimes a_2)) \notin S$ in case (1), and $(a_1, 0)(0, a_2) = (0, \phi(a_1 \otimes a_2)) \notin S$ in case (2). In either case, S is not an ideal in R. ∎

The following two results concerning nil rank 1, and completely decomposable torsion free groups will be useful in obtaining a description of the completely decomposable torsion free SI-groups.

Proposition 14. *Let G be a rank 1 torsion free group G. Then G is nil if and only if $t(G)$ is not idempotent. If G is not nil, then either $G \simeq \mathbb{Z}$, or G is the additive group of a ring isomorphic to a subring of $\mathbb{Q}$ which contains $\mathbb{Z}$ as a proper subring.*

Proof: [**1**, 1.4.8], and [**4**, Theorem 121.1]. ∎

Proposition 15. *Let $G = \bigoplus_{i \in I} G_i$ be completely decomposable, with each G_i*

a rank 1 torsion free group. Then G is nil if and only if $t(G_i) + t(G_j) \not\leqslant t(G_k)$ for all $i, j, k \in I$.

PROOF: [1, Corollary 2.1.3].

THEOREM 16. *Let $G = \bigoplus_{i \in I} G_i$ be a completely decomposable torsion free group, with each G_i a rank 1 group. Then G is an SI-group if and only if either $G \simeq \mathbb{Z}$, or G is nil.*

PROOF: Suppose that G is an SI-group. If G is a rank 1 group then Proposition 14 implies that either $G \simeq \mathbb{Z}$ or G is nil. It may therefore be assumed that G has rank greater than 1. It follows from Proposition 14, Corollary 2, and Lemma 3, that $t(G_i)$. is not idempotent, and so $t(G_i) + t(G_i) \not\leqslant t(G_i)$ for all $i \in I$. If there exist $i, j, k \in I$ such that $i \neq j$, $i \neq k$ and $t(G_i) + t(G_j) \leqslant t(G_k)$ then $\operatorname{Hom}(G_i \otimes G_j, G_k) \neq 0$, [4, Propositions 85.3 and 85.4]. Lemma 13 yields that G is not an SI-group, a contradiction. To show that G is nil it suffices to show, by Proposition 15, that $t(G_i) + t(G_i) \not\leqslant t(G_j)$ for all $i, j \in I$, $i \neq j$. It may be assumed that $i = 1$, $j = 2$, and that G_u is nil, $u = 1, 2$. Suppose that $t(G_1) + t(G_1) \leqslant t(G_2)$. By Lemma 3, it suffices to show $H = G_1 \oplus G_2$ is not an SI-group. Since $t(G_1)$ is not idempotent, Proposition 14, there exists $a_1 \in G_1$, and a prime p, such that the p-height if a_1 is 1. There exists $0 \neq \phi \in \operatorname{Hom}(G_1 \otimes G_1, G_2)$, [4, Propositions 85.3 and 85.4]. Let R be the ring with $R^+ = H$, and multiplication defined by $(b_1, b_2)(c_1, c_2) = (0, \phi(b_1 \otimes c_1))$ for all $b_u, c_u \in G_u$, $u = 1, 2$. Let S be the subring of R generated by $(a, 0)$, and let $a_0 \in G_1$ such that $pa_0 = a$. Then $(a_0, 0)(a, 0) = (0, \phi(a_0 \otimes a)) \notin S$, and so S is not an ideal in R.

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