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DOMINATION CONDITIONS UNDER WHICH A COMPACT SPACE IS METRISABLE

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Abstract

In this note we partially answer a question of Cascales, Orihuela and Tkachuk [‘Domination by second countable spaces and Lindelöf Σ-property’, *Topology Appl.* **158**(2) (2011), 204–214] by proving that under CH a compact space X is metrisable provided $X^2\backslash\Delta$ can be covered by a family of compact sets $\{K_f : f \in \omega^\omega\}$ such that $K_f \subset K_h$ whenever $f \leq h$ coordinatewise.

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1. Introduction

As in [10], we put $\mathbb{P} = \omega^\omega$, and for every $f, g \in \mathbb{P}$ we say that $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \omega$. A topological space that has a compact cover $\{K_f : f \in \mathbb{P}\}$ such that $K_f \subset K_g$ whenever $f \leq g$ is called (again as in [10]) a $\mathbb{P}$-dominated space, and if for every compact $K \subset X$ there is an $f \in \mathbb{P}$ such that $K \subset K_f$ then X is strongly $\mathbb{P}$-dominated. Other authors (for example, [7]) refer to (strongly) $\mathbb{P}$-dominated spaces as those having a compact resolution (swallowing compact sets).

It has been shown that studying $\mathbb{P}$-dominated spaces can be useful for obtaining conditions for metrisability in some cases. In particular, in [1] Cascales and Orihuela proved, using different terminology, that a compact space X is metrisable whenever $X^2\backslash\Delta$ is strongly $\mathbb{P}$-dominated. It is not yet clear if we can obtain the same result without assuming that the $\mathbb{P}$-domination of $X^2\backslash\Delta$ is strong. However, certain partial results in this direction have been obtained.

In [2], Cascales *et al.* proved in ZFC, among many other things, that if X is a compact space with countable tightness and $X^2\backslash\Delta$ is $\mathbb{P}$-dominated then X is metrisable. In the same paper, assuming $MA(\omega_1)$, they showed that if X is a compact space and $X^2\backslash\Delta$ is $\mathbb{P}$-dominated then X has a small diagonal and hence it is countably tight. Therefore $MA(\omega_1)$ implies that if X is a compact space and $X^2\backslash\Delta$ is $\mathbb{P}$-dominated then

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X is metrisable. They asked if the same conclusion can be obtained in ZFC for every compact space X such that $X^2 \backslash \Delta$ is $\mathbb{P}$-dominated.

According to [2], the problem reduces to showing that $\mathbb{P}$-domination of $X^2 \backslash \Delta$ implies small diagonal in compact spaces and hence countable tightness. As a consequence, it is not difficult to see that if there is a nonmetrisable compact space X such that $X^2 \backslash \Delta$ is $\mathbb{P}$-dominated then its weight cannot exceed ω_1 (see [5]).

We will show that, at least under CH, we have a positive result.

Our notation is standard. Every compact space in this note is assumed to be Hausdorff. Given a compact space X, the diagonal of X is the set $\Delta = \{(x, x) : x \in X\} \subset X^2$. Given a cardinal κ, the tightness of X is not greater than κ if for every $Y \subset X$ and every $y \in \overline{Y}$ there is a set $Z \subset Y$ such that $y \in \overline{Z}$ and $|Z| \leq \kappa$. A κ-sequence $\{x_\alpha : \alpha \leq \kappa\} \subset X$ is a free sequence of length κ if $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \leq \alpha\}} = \emptyset$ for every $\beta \in \kappa$. For a cardinal θ we let $H(\theta)$ denote the collection of all sets whose transitive closure has cardinality less than θ (see [8, Ch. IV]).

2. Spaces with $\mathbb{P}$-diagonal

Definition 2.1. A space X has a $\mathbb{P}$-diagonal if $X^2 \backslash \Delta$ is $\mathbb{P}$-dominated, that is, covered (dominated) by a family of compact sets $\{K_f : f \in \mathbb{P}\}$ satisfying $K_f \subset K_h$ whenever $f \leq h$.

Theorem 2.2. *CH implies that every compact space with a $\mathbb{P}$-diagonal is metrisable.*

As mentioned in the introduction, the following ZFC result is proved in [2]. We prove it here to introduce the ideas applied to obtain the later results.

Proposition 2.3. *Every countably tight compact space with $\mathbb{P}$-diagonal is metrisable.*

Proof. Let $\{K_f : f \in \mathbb{P}\}$ witness the $\mathbb{P}$-diagonal property of a countably tight compact space X. For each $t \in \omega^{<\omega}$, let $K(t) = \bigcup\{K_f : t \subset f\}$. Observe that if $s \geq t$, and $\mathrm{dom}(s) \subset \mathrm{dom}(t)$, then $K(s) \supset K(t)$. This is simply because if $t \subset f$, then $s \oplus f = s \cup f \restriction [\mathrm{dom}(s), \omega)$ satisfies $K_{s \oplus f} \supset K_f$.

For each $h \in \omega^\omega$, let $C(h) = \bigcap\{K(h \restriction n) : n \in \omega\}$. We show that the closure of $C(h)$ is disjoint from Δ. Since X^2 has countable tightness, it suffices to consider any sequence $\{y_n : n \in \omega\} \subset C(h)$. Recursively choose $\langle h_n : n \in \omega\rangle$ with $h = h_0 \leq h_1 \leq \cdots$ so that $h_n \restriction n \subset h_{n+1}$ and so that $y_n \in K_{h_{n+1}}$. To do so, observe that since $K(h_n \restriction n+1) \supset K(h \restriction n+1)$, the point $y_n \in K(h_n \restriction n+1)$. Therefore there is an h_{n+1} with $y_n \in K_{h_{n+1}}$ as required. Let $h_\omega = \bigcup_n h_n \restriction n$ and notice that $\{y_n\}_n \subset K_{h_\omega}$.

Now consider any open $U \subset X^2$ such that the closure of $C(h)$ is contained in U and $U \cap \Delta$ is empty. We claim there is an n such that the closure of $K(h \restriction n)$ is contained in U. Otherwise, perform a similar recursion, choosing $h_n \geq h \restriction n$ and $x_n \in K_{h_{n+1}} \backslash U$. For each n, let $h_\omega(n) = \max\{h_k(n) : k \leq n\}$ so that $\{x_n\}_n \subset K_{h_\omega} \backslash U$. More importantly, for each n, the set $\{x_k\}_{k>n} \subset K_{h \restriction n \oplus h_\omega}$ and all its limit points are contained in $K(h \restriction n)$. This yields a contradiction since U contains $C(h)$.

It now follows that X has a G_δ-diagonal, since $X^2 \backslash \Delta$ is covered by the collection of all $\overline{K(t)}$ which are disjoint from Δ. □

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It is easy to see that if $X = \omega_1 + 1$ then X does not have a $\mathbb{P}$-diagonal because $X^2 \backslash \Delta$ contains an uncountable closed discrete subset (see [5]).

Now suppose that X is a compact space with $\mathbb{P}$-diagonal and uncountable tightness. By [6], it contains a convergent free ω_1-sequence $\{x_\alpha : \alpha \in \omega_1\}$. We may assume that $\{x_\alpha : \alpha \in \omega_1\}$ is dense in X. This means that there is a continuous map from X onto $\omega_1 + 1$. We now show that X also maps continuously onto $[0, 1]^{\omega_1}$. To do so, we will apply some ideas from the investigations into the Moore–Mrowka problem, especially Eisworth's paper [4] on hereditary countable π-character.

Theorem 2.4. *Suppose that X is a compact space with $\mathbb{P}$-diagonal and φ maps X continuously onto $\omega_1 + 1$. Then X maps onto $[0, 1]^{\omega_1}$.*

Proof. Assume that X does not map onto $[0, 1]^{\omega_1}$. We will work in the subspace $Y = X \backslash \varphi^{-1}(\omega_1) = \varphi^{-1}([0, \omega_1))$. For a subset H of Y, define σH to be the $\aleph_0$-bounded closure of H, that is, $\sigma H = \bigcup \{\overline{H_0} : H_0 \in [H]^{\omega}\}$. Let $\mathcal{F}$ denote any maximal filter of $\aleph_0$-bounded sets such that the family $\{\varphi^{-1}([\alpha, \omega_1)) : \alpha \in \omega_1\}$ is contained in $\mathcal{F}$. Such a filter exists by Zorn's lemma. It is easy to verify that $\mathcal{F}$ is closed under countable intersections.

We say that $H \in \mathcal{F}^+$ provided $H \cap F$ is not empty for all $F \in \mathcal{F}$. Notice that if $H \in \mathcal{F}^+$, then $\sigma H \in \mathcal{F}$. We will now explore how the members of $\mathcal{F}$ interact with the family $\{K_f : f \in \omega^\omega\}$. Let π_2 denote the projection map from $Y \times Y$ onto the second coordinate; thus we will be focusing on the upper triangle in Y^2.

For $F \in \mathcal{F}$ and $t \in \omega^{<\omega}$, define

$$F(t) = \{x \in F : \sigma(\pi_2 \, [K(t) \cap (\{x\} \times F)]) \in \mathcal{F}\}.$$

For each $t \in \omega^{<\omega}$ choose, if possible, $F_t \in \mathcal{F}$ so that $F_t(t) \notin \mathcal{F}^+$. Let $F_0 \in \mathcal{F}$ be contained in each such F_t.

Now choose any countable elementary submodel $M \prec H(\theta)$, where θ is any sufficiently large regular cardinal and $H(\theta)$ denotes the family of sets which are hereditarily of cardinality less than θ. 'Sufficiently large' just means here that X is based on some ordinal λ and $|\mathcal{P}(\mathcal{P}(\lambda))| < \theta$. Of course, we want $\varphi, X, \mathcal{F}$ and $\{K_f : f \in \omega^\omega\}$ to be elements of M. One can assume that F_0 is also in M or simply carry out the selection of the F_t within M.

Now we define Z to be $\bigcap \{\overline{F \cap M} : F \in \mathcal{F} \cap M\}$.

Choose any $z \in Z$ and $y \in F_0 \cap M$. Notice that $z \notin M$ and so $(y, z) \in X^2 \backslash \Delta$. Choose any $h_0 \in \omega^\omega$ so that $(y, z) \in K_{h_0}$.

Choose any $t \geq h_0 \restriction \mathrm{dom}(t)$ (hence $(y, z) \in K(t)$). Let $H_y = \pi_2[K(t) \cap (\{y\} \times F)]$ and notice that H_y and σH_y are in M. If $\sigma H_y \notin \mathcal{F}$, then there is an $F_2 \in \mathcal{F} \cap M$ such that $\sigma H_y \cap F_2$ is empty. However, $z \in \sigma(F_2 \cap M) \subset F_2$ and also $z \in H_y$, which cannot happen. Thus $\sigma H_y \in \mathcal{F}$. Moreover, F_t does not exist. For otherwise, F_0 is contained in it and H_y is smaller than $\pi_2(K(t) \cap (\{y\} \times F_t))$ and so $\sigma(H_y)$ is not in $\mathcal{F}$, a contradiction.

We can say even more about $t \geq h_0 \restriction \mathrm{dom}(t)$. Choose any $F \in \mathcal{F} \cap M$ and any open $W \subset X$ such that $W \cap Z$ is not empty. Choose any $z_1 \in W \cap Z$. Then we claim that there is a $y_1 \in W \cap F \cap M$ such that $W \cap (H_{y_1} \cap M)$ is also not empty. As before, we may

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assume that $F \subset F_0$ and we know that F_t does not exist. This means that $F(t) \in \mathcal{F}^+$ and so $\sigma F(t)$ is in $\mathcal{F}$. Hence Z is contained in the closure of $M \cap \sigma F(t)$. By elementarity, $M \cap \sigma F(t)$ is contained in $\sigma(M \cap F(t))$. So we may choose some $y_1 \in W \cap M \cap F(t)$.

Again let $H_{y_1} = \pi_2[K(t) \cap (\{y_1\} \times F)]$ and it is easily shown that z is in the closure of $M \cap \sigma H_{y_1}$. But again, by elementarity, it follows that z_1 is in the closure of $M \cap H_{y_1}$, and the set $W \cap M \cap H_{y_1}$ is not empty, as required.

The conclusion is that if $t \geq h_0 \restriction \mathrm{dom}(t)$, $F \in \mathcal{F} \cap M$ and an open W meets Z, then there is a point $(y_1, y_2) \in K(t) \cap M \cap (W \cap F)^2$.

Since X does not map onto $[0,1]^{\omega_1}$ we may assume that every closed subset K of X contains a point which has countable π-character in K (see [9]).

Now choose a point $x \in Z$ which has countable π-character in Z. Let $\{U_n, W_n : n \in \omega\}$ be open subsets of X such that, for each n, $\overline{W_n} \subset U_n$ and $W_n \cap Z$ is nonempty, and such that the family $\{U_n \cap Z : n \in \omega\}$ is a local π-base for x in Z. For convenience, we assume that each pair U_n, W_n is listed infinitely many times.

Begin our (by now) standard recursive construction of a sequence of functions $\{h_n : n \in \omega\}$ so that $h_{n+1} \geq h_n$ and $h_{n+1} \supset h_n \restriction n$. Let $\{F_n : n \in \omega\}$ be an enumeration for a descending base for $M \cap \mathcal{F}$. Choose h_{n+1} so that there is a pair $(y_1^n, y_2^n) \in K(h_n \restriction n) \cap M \cap (W_n \cap F_n)^2$ as discussed above. Let $h_\omega = \bigcup_n h_n \restriction n$, so that $h_\omega \geq h_n$ for all n.

Consider a pair U_k, W_k which was listed infinitely often. Let $L_k = \{n : (U_n, W_n) = (U_k, W_k)\}$. The sequence $\{(y_1^n, y_2^n) : n \in L_k\}$ accumulates at some point (z_1^k, z_2^k) which is in $(\overline{W_n} \cap Z)^2$. To see this, it is enough to notice that every limit point of the entire set $\{y_1^n, y_2^n : n \in \omega\}$ is in Z because a cofinite subset of it is contained in $F_\ell \cap M$ for each ℓ. Notice then that $(z_1^k, z_2^k) \in (U_k \cap Z)^2$. Since the family $\{(U_k \cap Z)^2 : k \in \omega\}$ is a local π-base at (x, x), we see that (x, x) is in the closure.

But now we have a contradiction since $\{(y_1^n, y_2^n) : n \in \omega\}$ is contained in K_{h_ω}. □

We now prove that $\beta\omega$ does not have a $\mathbb{P}$-diagonal.

Theorem 2.5. *A compact space with a $\mathbb{P}$-diagonal must contain a nontrivial converging sequence.*

Proof. Suppose that we have a compact space X with no nontrivial converging sequences. Assume that $\{K_f : f \in \omega^\omega\}$ is a compact cover of $X^2 \backslash \Delta$.

First notice that for all $x \in X$ and infinite compact $J \subset X$, there is an f so that $K_f \cap (\{x\} \times J)$ is infinite. To see this, simply fix any uncountable $\{y_\alpha : \alpha \in \omega_1\} \subset J \backslash \{x\}$. For each α, choose f_α so that $(x, y_\alpha) \in K_{f_\alpha}$. There is an $h \in \omega^\omega$ so that for each n, there is an α_n such that $h \restriction n \subset f_{\alpha_n}$. Now define $f \in \omega^\omega$ so that for each n, $f(n) \geq \max\{f_{\alpha_k}(n) : k \leq n\}$. This means that $f_{\alpha_n} \leq f$ for all n, whence $(x, y_{\alpha_n}) \in K_f$ for all n.

Similarly (but now using the hypothesis), for each infinite compact $J \subset X$, there is an f so that K_f contains $J_0 \times J_1$ for some infinite compact $L_0, L_1 \subset J$.

To see this, let $\{x_\alpha : \alpha \in \omega_1\}$ be any subset of J (which must be uncountable because it has no nontrivial converging sequences). By recursion, we choose a descending sequence $\{J_\alpha : \alpha \in \omega_1\}$ of infinite compact sets with $J_0 = J$. We require that, for each α,

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there is an f_α so that K_{f_α} contains $\{x_\alpha\} \times J_{\alpha+1}$. If J_α is infinite compact, then the existence of f_α and infinite compact $J_{\alpha+1}$ follows from the first claim. For limit α, $J_\alpha = \bigcap\{J_\beta : \beta < \alpha\}$ is infinite because X contains no nontrivial converging sequences. Now again choose any f so that there is an infinite increasing sequence $\{\alpha_n : n \in \omega\}$ with $f_{\alpha_n} \leq f$ for all n. Let J_0 denote the (infinite) set of limit points of $\{x_{\alpha_n} : n \in \omega\}$, and let $L_1 = \bigcap\{J_{\alpha_n} : n \in \omega\}$. Note that $L_0 \times L_1$ is contained in K_f.

Now specify any indexing $\{t_k : k \in \omega\}$ of $\omega^{<\omega}$. We may assume that $t_k \subset t_j$ implies $k < j$. By a countable recursion, choose a descending sequence $\{J_k : k \in \omega\}$ of infinite closed subsets of X with $J_0 = X$. Having chosen J_k, we consider t_k. If there exists some infinite compact $J \subset J_k$ so that, for all $f \supset t_k$, K_f does not contain any product $J^0 \times J^1$ with J^0, J^1 infinite compact subsets of J, then choose J_{k+1} to be such a set. Otherwise, let $J_{k+1} = J_k$, and note that there is no such J contained in J_{k+1}.

When this recursion is complete, set $J = \bigcap_k J_k$, and again note that J is an infinite compact subset of X. Choose any h_0 so that K_{h_0} contains $L_0 \times L_1$ for infinite compact L_0, L_1 contained in J. We now know that for any $t_k \geq h_0 \restriction \mathrm{dom}(t_k)$, and every J_{k+1}, there is no $J \subset J_k$ such that, for all $f \supset t_k$, K_f does not contain any product $J^0 \times J^1$ with J^0, J^1 being infinite compact subsets of J. So we recursively choose $h_1 \geq h_0$ with $h_0 \restriction 1 \subset h_1$ and so that there are L_2, L_3 contained in L_0 with $L_2 \times L_3 \subset K_{h_1}$. Continue recursively with $L_{2k+2}, L_{2k+3} \subset L_{2k}$, $L_{2k+2} \times L_{2k+3} \subset K_{h_{k+1}}$ and $h_k \restriction k \subset h_{k+1}$. Choosing $h \geq h_k$ for all k, we show that K_h will hit Δ. For each k, choose $y_k \in L_{2k+1}$ and let y be any limit point of $\{y_k : k \in \omega\}$. Then $\{y_\ell : \ell > 2k\} \subset L_{2k}$ and so $y \in L_{2k}$ for all k. Similarly, $(y, y_{k+1}) \in L_{2k+2} \times L_{2k+3}$ for all k, which implies that $(y, y) \in K_h$. □

From the previous result, we conclude that a compact space with $\mathbb{P}$-diagonal cannot contain a copy of $\beta\omega$ and therefore it cannot be mapped continuously onto $[0, 1]^{\mathfrak{c}}$ (see [3]). We can now use CH and Theorem 2.4 to complete the proof of Theorem 2.2.

3. Open problems

The most important problem on the subject of this paper is to determine if every compact space with $\mathbb{P}$-diagonal is metrisable. We have already seen that $\omega_1 + 1$ is not a counterexample. However, if there is a counterexample it must map onto $\omega_1 + 1$. Moreover, any counterexample must contain the closure of a convergent free sequence of length ω_1 in a compact space of weight ω_1.

Question 3.1. Suppose that X is a compact space that contains a dense convergent free sequence of length ω_1. Must X contain a copy of $\omega_1 + 1$?

Question 3.2. Is every countably compact space with $\mathbb{P}$-diagonal metrisable?

Question 3.3. Is every pseudocompact space with $\mathbb{P}$-diagonal metrisable?

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