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Quasisymmetric harmonics of the exterior algebra

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Abstract. We study the ring of quasisymmetric polynomials in n anticommuting (fermionic) variables. Let R_n denote the ring of polynomials in n anticommuting variables. The main results of this paper show the following interesting facts about quasisymmetric polynomials in anticommuting variables:

(1) The quasisymmetric polynomials in R_n form a commutative subalgebra of R_n.
(2) There is a basis of the quotient of R_n by the ideal I_n generated by the quasisymmetric polynomials in R_n that is indexed by ballot sequences. The Hilbert series of the quotient is given by

$$\mathrm{Hilb}_{R_n/I_n}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{(n-k,k)} q^k ,$$

where $f^{(n-k,k)}$ is the number of standard tableaux of shape $(n-k,k)$.
(3) There is a basis of the ideal generated by quasisymmetric polynomials that is indexed by sequences that break the ballot condition.

1 Introduction

The study of coinvariants of groups dates back to Shephard—Todd and Chevalley [5, 27] and has fruitfully produced many connections between algebra, combinatorics, and physics. Motivated by recent developments in coinvariants of symmetric groups and symmetric functions theory incorporating fermionic variables, we study a coinvariant-like quotient of an exterior algebra obtained by the quotient of the ideal generated by quasisymmetric functions in fermionic variables. The quotient has a dimension that can be interpreted as the number of ballot sequences (or other interpretations; see, for instance, the OEIS [26] sequences A008315 and A001405).

A notable feature of many quotients similar to coinvariants is their amenability to combinatorial methods. One well-known example is the coinvariant ring of the symmetric group. It is the quotient of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in commuting variables by the ideal generated by the symmetric polynomials with no constant

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term. As an $\mathfrak{S}_n$ representation, this quotient is naturally graded and is well known to be isomorphic to the regular representation. Many useful bases of this space have been found by studying combinatorics related to permutations. For more details, see the nice surveys of [3, 13, 24, 25].

This line of inquiry inspired Garsia and Haiman [12, 18] to consider the ring of diagonal harmonics, a similar quotient in two sets of commuting variables as an $\mathfrak{S}_n$ module. Haiman's work [19] showed that the diagonal harmonics have a deep connection to the theory of Macdonald polynomials. A combinatorial expression for the Frobenius image of the diagonal harmonics known as the Shuffle Conjecture [16] showed that the module structure is closely related to the combinatorics of parking functions and can be described in terms of certain labeled Catalan paths. This connection relating the symmetric functions and the combinatorial expression was proved in [4] and is now known as the Shuffle Theorem.

The connection between the combinatorics and the symmetric function expressions of the Shuffle Theorem has been generalized [17] and proved [8] to an expression known as the Delta Conjecture. The last author with the group at the Fields Institute [31] proposed a deformation of diagonal harmonics to two sets of commuting variables and one set of anticommuting variables. In this case, the connection of representation theoretic interpretation to the symmetric function expression remains open. The symmetric function expressions and representation theoretic interpretation were extended further to include the quotient of two sets of commuting and two sets of anticommuting variables in [7] to what is known as the Theta Conjecture. At present, this also remains an open conjecture, but progress has been made on some special cases [21, 22, 29, 30].

The ring of quasisymmetric polynomials $QSym_n$ contains the ring of symmetric polynomials Sym_n. Many combinatorial structures of $QSym_n$ parallel that of Sym_n. Hivert described a Temperley–Lieb TL_n action on $\mathbb{Q}[x_1, \ldots, x_n]$ making $QSym_n$ exactly its trivial representation [20]. In 2003, Aval, F. Bergeron, and the first author studied $QSym$ coinvariant spaces obtained by replacing the ideal of nonconstant symmetric functions with the ideal of nonconstant quasisymmetric functions [1, 2]. Surprisingly, they found that dimensions of $QSym$ coinvariants are equal to the Catalan numbers. At the heart of their argument is a recursion built from Catalan paths. Li extended this argument to study some components of coinvariant spaces of diagonally quasisymmetric functions [23].

Motivated by physics, Desrosiers, Lapointe, and Mathieu [9, 10] introduced symmetric functions with one set of commuting and one set of anticommuting variables known as symmetric function in superspace. The commuting variables encode bosons, whereas the anticommuting ones encode fermions; hence, the anticommuting variables are sometimes referred to as "fermionic variables." The Hopf algebra structure of the ring of symmetric functions in superspace was extended to quasisymmetric functions in superspace [11] and so a natural question is to extend the study of coinvariants of polynomial rings with commuting and anticommuting variables to the quotients of these polynomial rings by the ideal generated by "super" quasisymmetric polynomials.

Parallel to the Delta Conjecture or Theta Conjecture, one ideally would like to understand quasisymmetric coinvariants in multiple sets of commuting and anticommuting variables. Our study of quasisymmetric coinvariant spaces in one set of anticommuting variables is a first step in that study. We denote polynomials in anticommuting variables by R_n. The main results of this paper show the following interesting facts about symmetric and quasisymmetric functions in anticommuting variables:

(1) The quasisymmetric polynomials in R_n form a commutative subalgebra of R_n (Proposition 2.3).
(2) That R_n is free over the ring of symmetric polynomials in R_n (Proposition 2.5).
(3) There is a basis of the quotient of R_n by the ideal I_n generated by the quasisymmetric polynomials in R_n that is indexed by ballot sequences (Proposition 2.10). The Hilbert series of the quotient is given by

$$\mathrm{Hilb}_{R_n/I_n}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{(n-k,k)} q^k ,$$

where $f^{(n-k,k)}$ is the number of standard tableaux of shape $(n-k,k)$ (Corollary 4.4).
(4) There is a basis of the ideal generated by quasisymmetric polynomials that is indexed by sequences that break the ballot condition (Theorem 4.2) and a minimal Gröbner basis that is a subset of this basis (Corollary 4.6).

2 Quasisymmetric invariants on the exterior algebra

Fix n a positive integer, and let $R_n = \mathbb{Q}[\theta_1, \theta_2, \dots, \theta_n]$ be the polynomial ring in anticommuting variables. The ring R_n is isomorphic to the exterior algebra of a vector space of dimension n. The variables of this ring satisfy the relations

$$\theta_i\theta_j = -\theta_j\theta_i \quad \text{if } 1 \le i \ne j \le n \quad \text{and} \quad \theta_i^2 = 0 \quad \text{for } 1 \le i \le n .$$

Since these conditions impose that a monomial in R_n has no repeated variables, the monomials are in bijection with subsets of $\{1, 2, \dots, n\}$ and the dimension of R_n is therefore equal to 2^n.

Denote $[n] := \{1, 2, \dots, n\}$, and let $A = \{a_1 < a_2 < \cdots < a_r\} \subseteq [n]$. We define $\theta_A := \theta_{a_1}\theta_{a_2}\cdots\theta_{a_r}$, then the set of monomials $\{\theta_A\}_{A\subseteq[n]}$ is a basis of R_n.

We define an action on monomials of R_n and extend this action linearly. For each integer $1 \le i < n$, let π_i be an operator on R_n that is defined by

$$\pi_i(\theta_A) = \begin{cases} \theta_A, & \text{if } i, i+1 \in A \text{ or } i, i+1 \notin A, \\ \theta_{A\cup\{i+1\}\setminus\{i\}}, & \text{if } i \in A \text{ and } i+1 \notin A, \\ \theta_{A\cup\{i\}\setminus\{i+1\}}, & \text{if } i+1 \in A \text{ and } i \notin A. \end{cases} \tag{2.1}$$

These operators, instead of exchanging an i for an $i+1$ like the symmetric group action, have the effect of shifting the indices of the variables (if possible). They are therefore known as quasisymmetric operators. They were studied in depth by Hivert [20]. The operators are not multiplicative on R_n in general since, for example,

$$\pi_1(\theta_1\theta_2) = \theta_1\theta_2 = -\pi_1(\theta_1)\pi_1(\theta_2) .$$

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They are also not multiplicative when they act on the polynomial ring in commuting variables.

A polynomial that is invariant under the action of quasisymmetric operators is said to be quasisymmetric invariant (or just "quasisymmetric"). The quasisymmetric invariants of R_n are linearly spanned by the elements:

$$F_{1^r}(\theta_1, \theta_2, \dots, \theta_n) := \sum_{\substack{A \subseteq [n] \\ |A| = r}} \theta_A \,. \tag{2.2}$$

The symbols F_{1^r} for the elements borrows the notation from the polynomial ring in commuting variable invariants known as the "fundamental quasisymmetric polynomials". The commuting polynomial quasisymmetric invariants are indexed by compositions.

Remark 2.1 As expressing polynomials with listing the variables (e.g., $p(\theta_1, \theta_2, \dots, \theta_n)$) can be notational cumbersome, there will be points where we will drop the variables in the expressions and this will indicate that the polynomials are in the variables $\theta_1, \theta_2, \dots, \theta_n$. There will also be expressions where some polynomials have fewer variables and there we will indicate this by listing the variables.

2.1 Quasisymmetric functions generate a commutative subalgebra

In [11], the authors showed that the quasisymmetric functions in one set of commuting variables and one set of anticommuting variables form a graded Hopf algebra. This implies that the quasisymmetric functions in one set of anticommuting variables are closed under multiplication and the space is spanned by one element at each nonnegative degree. It follows that for $r, s \geq 0$, there exists a (possibly 0) coefficient $a_{r,s}$ such that

$$F_{1^r} F_{1^s} = a_{r,s} F_{1^{r+s}} \,. \tag{2.3}$$

If $r + s > n$, then $F_{1^{r+s}} = 0$ by definition and so the only relevant coefficient $a_{r,s}$ is when $r + s \leq n$.

Remark 2.2 In the notation of [11], $F_{1^r} = M_{\dot{0}^r} = L_{\dot{0}^r}$ where $\dot{0}^r = (\dot{0}, \dot{0}, \dots, \dot{0})$ is a composition of length r. The " $\dot{}$ " over a part in [11] is to indicate a fermionic component. Therefore, the fermionic degree of F_{1^r} is exactly r. In [11], they show that $a_{r,s}$ exists and express it as a sum of ± 1, but they do not give an explicit formula. Furthermore, they indicate that $a_{r,s} = (-1)^{rs} a_{s,r}$. Here, we shall compute exactly $a_{r,s}$ and the formula shows that the subalgebra generated by the F_{1^r} is commutative.

Proposition 2.3 *The constants $a_{r,s}$ in equation (2.3) satisfy the following equation:*

$$a_{r,s} = \begin{cases} 0, & \text{if } r, s \text{ are both odd,} \\ \binom{\lfloor \frac{r+s}{2} \rfloor}{\lfloor \frac{r}{2} \rfloor}, & \text{otherwise.} \end{cases}$$

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A remark brought to our attention by D. Grinberg [14] shows that $a_{r,s}$ is equal to the q-binomial coefficient $\left[{r+s \atop r}\right]_q$ evaluated at $q \to -1$ [15, Equation (185) on page 291].

Proof For completeness, we give a proof not assuming any results of [11]. Using equation (2.2), we have

$$F_{1^r}F_{1^s} = \sum_{\substack{A\subseteq[n]\\|A|=r}}\sum_{\substack{B\subseteq[n]\\|B|=s}}\theta_A\theta_B = \sum_{\substack{C\subseteq[n]\\|C|=r+s}}\Bigg(\sum_{\substack{A\subseteq C\\|A|=r}}(-1)^{|\{b<a\,|\,a\in A,\,b\in C\setminus A\}|}\Bigg)\theta_C\,.$$

To see the second equality, we remark that the product $\theta_A\theta_B = 0$ if $A\cap B \neq \varnothing$. Furthermore, if $A\cap B = \varnothing$, then for $C = A\cup B$, we have $B = C\setminus A$ and $\theta_A\theta_B = (-1)^{|\{b<a\,|\,a\in A, b\in C\setminus A\}|}\theta_C$, where the sign is the number of interchanges needed to sort A followed by B into C. This does not depend on the values of the elements of C, but only on how A is chosen inside C. This shows that we get the same coefficient for all C of size $r+s$ and therefore $F_{1^r}F_{1^s} = a_{r,s}F_{1^{r+s}}$ with

$$a_{r,s} = \sum_{\substack{A\subseteq\{1,2,\ldots,r+s\}\\|A|=r}}(-1)^{|\{1\le b<a\le r+s\,|\,a\in A, b\notin A\}|} \tag{2.4}$$

by choosing $C = \{1,2,\ldots,r+s\}$.

Let $\binom{C}{r} = \{A\subseteq C, |A| = r\}$. We define a sign-reversing involution $\Phi\colon\binom{C}{r}\to\binom{C}{r}$ as follows. For $A\in\binom{C}{r}$, let $\gamma(A) = \gamma_1\gamma_2\cdots\gamma_{r+s}\in\{0,1\}^{r+s}$ be the sequence such that $\gamma_i = 1$ if $i\in A$, and $\gamma_i = 0$ otherwise. We look at the entries of $\gamma(A)$ two by two and find the smallest j (if it exists) such that the pair $\gamma_{2j-1}\gamma_{2j}$ is not 00 or 11. If there is no such pair, we let $\Phi(A) = A$. If we find such pair, we define the involution $\Phi(A) = A'$, where A' is such that $\gamma(A')$ is obtained from $\gamma(A)$ by interchanging $01\leftrightarrow 10$ in position $2j-1, 2j$. If r and s are both odd, then there must be at least one occurrence of 01 or 10 and there are no fixed points of this involution.

We let

$$\mathrm{Inv}(A) = \{1\le b<a\le r+s\,|\,a\in A, b\notin A\} = \{1\le \ell<t\le r+s\,|\,\gamma_\ell = 0, \gamma_t = 1\},$$

where $\gamma(A) = \gamma_1\gamma_2\cdots\gamma_{r+s}$. As long as $(t,\ell)\neq(2j-1,2j)$, there is a bijection between $(t,\ell)\in\mathrm{Inv}(A)$ and $(t',\ell)\in\mathrm{Inv}(A')$ interchanging the 1 and 0 in positions $2j-1$ and $2j$. The pair $(2j-1,2j)$ is in only one of $\mathrm{Inv}(A)$ or $\mathrm{Inv}(A')$ but not the other. Therefore,

$$(-1)^{|\{1\le b<a\le r+s\,|\,a\in A, b\notin A\}|} = -(-1)^{|\{1\le b<a\le r+s\,|\,a\in A', b\notin A'\}|}\,.$$

If $\Phi(A) = A$, we have that $|\mathrm{Inv}(A)|$ is even since we can match the pairs two by two. If r is odd and s is even, then the only $A\in\binom{C}{r}$ have $r+s\in A$ and $|\mathrm{Inv}(A)| = |\mathrm{Inv}(A\backslash\{r+s\})| + s$. Therefore, Φ is a sign reversing involution and all fixed points contribute in equation (2.4) with a +1. Therefore,

$$a_{r,s} = \left|\left\{A\in\binom{C}{r}\,\middle|\,\Phi(A) = A\right\}\right| = \binom{\lfloor\frac{r+s}{2}\rfloor}{\lfloor\frac{r}{2}\rfloor}\,,$$

since there are a total of $\lfloor\frac{r+s}{2}\rfloor$ pairs $2j-1, 2j$ in a sequence of length $r+s$ and we must have $\lfloor\frac{r}{2}\rfloor$ of them equal to 11 and all others equal to 00 in order to get $\Phi(A) = A$. ∎

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The generating series for the coefficients $F(x,y) = \sum_{r,s\geq 0} a_{r,s}x^r y^s$ is equal to $\frac{1+x+y}{1-x^2-y^2}$, and the OEIS [26] sequence number is A051159. This can be derived from Proposition 2.3 using standard techniques of generating functions.

One consequence of Proposition 2.3 is that $a_{r,s} = a_{s,r}$ for all $r,s \geq 0$. Remark that this does not contradict the fermionic law stating that $a_{r,s} = (-1)^{rs}a_{s,r}$ since $a_{r,s} = 0$ when both r,s are odd. Therefore, we have shown the following corollary.

Corollary 2.4 *The subalgebra generated by quasisymmetric invariants* $\{F_{1^r} | r \geq 0\}$ *is commutative.*

2.2 The ideal generated by symmetric invariants

The symmetric invariants Sym_{R_n} of R_n are very small since a basis consists of only two elements 1 and $F_1(\theta_1,\theta_2,\dots,\theta_n)$. Therefore, the ideal generated by the invariants of nonzero degree, which we shall denote J_n, is generated by a single element F_1. We begin by considering the symmetric coinvariants of R_n, the quotient ring R_n/J_n. Because the ideal J_n is principal, we can understand this quotient with much more detail. This quotient ring is a special case of the ring recently studied in [21, 22].

Recall that $\dim R_n = 2^n$, and if we consider the quotient R_n/J_n, it is isomorphic to R_{n-1} since in this algebra $\theta_n = -\theta_1 - \theta_2 - \cdots - \theta_{n-1}$. Let $A \subseteq [n-1]$ and $A' = A \cup \{n\}$, then the map which sends $\theta_{A'}$ to

$$-\theta_A(\theta_1 + \theta_2 + \cdots + \theta_{n-1}) \otimes 1 + \theta_A \otimes F_1 \quad \in R_n/J_n \otimes \mathrm{Sym}_{R_n}$$

and θ_A to

$$\theta_A \otimes 1 \quad \in R_n/J_n \otimes \mathrm{Sym}_{R_n}$$

is an algebra isomorphism. Since this map describes the image for each monomial in R_n, we have the following proposition.

Proposition 2.5 *For each* $n \geq 1$,

$$R_n \cong R_n/J_n \otimes \mathrm{Sym}_{R_n},$$

as an algebra. That is, R_n *is free over* Sym_{R_n}.

2.3 The ideal generated by the quasisymmetric invariants

Define an ideal of R_n generated by the quasisymmetric invariants as

$$I_n := \langle F_{1^r}(\theta_1,\theta_2,\dots,\theta_n) : 1 \leq r \leq n \rangle .$$

Remark 2.6 Note that since the operators π_i are not multiplicative, it is unlikely to be the case that I_n as an ideal is invariant under the action of the π_i. Indeed, we find that for $n = 4$,

$$\theta_2 F_1(\theta_1,\theta_2,\theta_3,\theta_4) = -\theta_1\theta_2 + \theta_2\theta_3 + \theta_2\theta_4.$$

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If we apply π_1 to this element, we obtain

$$\pi_1(\theta_2 F_1(\theta_1,\theta_2,\theta_3,\theta_4)) = -\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4.$$

This is not in I_4.

The *exterior quasisymmetric coinvariants* are defined to be

$$EQC_n := R_n/I_n.$$

We borrow the name "coinvariant" space even though the generators, and not the whole ideal, are invariant under the quasisymmetric operators.

2.4 Differential operators on the exterior algebra

We can define a set of differential operators on R_n which will permit us to define the orthogonal complement to the ideal and a notion of quasisymmetric harmonics.

The operators ∂_{θ_i} act on monomials in R_n by

$$\partial_{\theta_i}(\theta_A) = \begin{cases} (-1)^{\#\{j\in A: j<i\}}\theta_{A\setminus\{i\}}, & \text{if } i \in A,\\ 0, & \text{if } i \notin A.\end{cases}$$

The operators can equally be characterized by the action that $\partial_{\theta_i}(1) = 0$ and the commutation relations

$$\partial_{\theta_i}\partial_{\theta_j} = -\partial_{\theta_j}\partial_{\theta_i} \quad \text{if } 1 \le i \ne j \le n \quad \text{and} \quad \partial^2_{\theta_i} = 0 \quad \text{for } 1 \le i \le n,$$

$$\partial_{\theta_i}\theta_j = -\theta_j\partial_{\theta_i} \quad \text{if } 1 \le i \ne j \le n \quad \text{and} \quad \partial_{\theta_i}\theta_i = 1 \quad \text{for } 1 \le i \le n.$$

For a monomial $\theta_A = \theta_{a_1}\theta_{a_2}\cdots\theta_{a_r}$, let $\overline{\theta_A} = \theta_{a_r}\theta_{a_{r-1}}\cdots\theta_{a_1}$ represent reversing the order of the variables in the monomial. Extend this notation to both differential operators and polynomials (and polynomials of differential operators) by extending the notation linearly.

We can define an inner product on R_n by setting for $p, q \in R_n$.

$$\langle p, q\rangle = \overline{p(\partial_{\theta_1},\partial_{\theta_2},\dots,\partial_{\theta_n})}q(\theta_1,\theta_2,\dots,\theta_n)|_{\theta_1=\theta_2=\cdots=\theta_n=0}.$$

The monomials of R_n form an orthonormal basis of the space with respect to this inner product.

Define the orthogonal complement to I_n with respect to the inner product as the set

$$EQH_n : = \{q \in R_n : \langle p, q\rangle = 0 \text{ for all } p \in I_n\} \tag{2.5}$$

$$= \{q \in R_n : p(\partial_{\theta_1},\partial_{\theta_2},\dots,\partial_{\theta_n})q = 0 \text{ for all } p \in I_n\}. \tag{2.6}$$

The second equality follows from the fact that I_n is an ideal and shows that EQH_n is also the solution space of a system of differential equations. The containment of the set in equation (2.6) inside the set in equation (2.5) is clear. For the reverse inclusion, take

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an element q which is not in the set in equation (2.6), and we assume for some $p \in I_n$ that $p(\partial_\theta)q = c\theta^\alpha$ plus possibly some other terms, but then $\overline{p\theta^\alpha} \in I_n$ and $\langle \overline{p\theta^\alpha}, q \rangle = c$, which implies that q is not in the set in equation (2.5).

We refer to EQH_n as the *exterior quasisymmetric harmonics*. The harmonics and diagonal harmonics borrow the name from the physics literature because the harmonic operator $\partial_1^2 + \partial_2^2 + \cdots + \partial_n^2$ is symmetric in the differential operators. In the case of the exterior algebra, this operator acts as zero and yet we persist by borrowing the name from the analogous spaces of commuting variables.

It is clear that the monomials of R_n form an orthonormal basis of the space with respect to the inner product; hence, the inner product is positive-definite. It follows that since EQH_n is the orthogonal complement of the ideal I_n in R_n, then the following result must hold.

Proposition 2.7 *For all $n \geq 1$, as graded vector spaces,*

$$EQC_n \simeq EQH_n.$$

We will conclude this section by constructing a set of linearly independent elements inside EQH_n, which will give us a lower bound on the dimension of EQC_n. In Section 4, we will see that this is also an upper bound, thus concluding that our set is in fact a basis. To compute EQH_n, we need to solve the differential equations in equation (2.6). Remark first that since I_n is an ideal, we do not need to take all $p \in I_n$, but it is enough to solve for the generators $p = F_{1^r}$ for $1 \leq r \leq n$. We can reduce that further using Proposition 2.3 as noted in the following lemma.

Lemma 2.8 *For $n \geq 2$, we have that I_n is the ideal generated by F_1 and F_{1^2}.*

Proof Clearly, we have that the ideal generated by F_1, F_{1^2} is contained in I_n. For the converse, we note that for each $k \geq 1$, there are nonzero coefficients a and a' such that

$$aF_{1^{2k}} = (F_{1^2})^k \qquad \text{and} \qquad a'F_{1^{2k+1}} = (F_{1^2})^k F_1;$$

hence, all of the generators of I_n are contained in the ideal generated by F_1, F_{1^2}. ∎

From this, we conclude that

$$EQH_n = \left\{ q \in R_n : \sum_{1 \leq i \leq n} \partial_{\theta_i} q = 0 \quad \text{and} \quad \sum_{1 \leq i < j \leq n} \partial_{\theta_j} \partial_{\theta_i} q = 0 \right\}. \tag{2.7}$$

Given $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, a noncrossing pairing of length k is a list $(C_1, C_2, \ldots, C_k)$ with

$C_r = (i_r, j_r)$ for $1 \leq i_r < j_r \leq n$ for each $1 \leq r \leq k$ and,
either $i_r < j_r < i_s < j_s$ or $i_s < i_r < j_r < j_s$ for any $1 \leq r < s \leq k$.

Given a noncrossing pairing $C = (C_1, C_2, \ldots C_k)$, we define

$$\Delta_C = (\theta_{j_1} - \theta_{i_1})(\theta_{j_2} - \theta_{i_2}) \cdots (\theta_{j_k} - \theta_{i_k}). \tag{2.8}$$

Here, $\Delta_C = 1$ if $k = 0$. Remark that $j_1 < j_2 < \cdots < j_k$. The following proposition shows that there is a relationship between the noncrossing pairing condition and the differential equations from equation (2.7).

Proposition 2.9 *The set*

$$\mathcal{D}'_n = \left\{ \Delta_C : C = (C_1, C_2, \dots, C_k) \text{ noncrossing pairing and } 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

is contained in EQH_n.

Proof To show that Δ_C is contained in EQH_n, we fix C. We need to show that Δ_C satisfies the differential equation conditions in equation (2.7).

For the first defining equation of EQH_n, we have

$$\begin{aligned}\sum_{1\le i\le n} \partial_{\theta_i}\Delta_C &= \sum_{1\le r\le k} (\partial_{\theta_{i_r}} + \partial_{\theta_{j_r}})\Delta_C + \sum_{i\notin\bigcup_{r=1}^k C_r} \partial_{\theta_i}\Delta_C\\ &= \sum_{1\le r\le k} (\partial_{\theta_{i_r}} + \partial_{\theta_{j_r}})\Delta_C = 0\,.\end{aligned}$$

For the last equality, fix $1 \le r \le k$ and note that $\Delta_C = (-1)^{r-1}(\theta_{j_r} - \theta_{i_r})q$ for some polynomial q and so for each r,

$$(\partial_{\theta_{i_r}} + \partial_{\theta_{j_r}})\Delta_C = (-1)^{r-1}(\partial_{\theta_{i_r}} + \partial_{\theta_{j_r}})(\theta_{j_r} - \theta_{i_r})q = 0\,.$$

For the second defining equation of EQH_n, we decompose the sum over pairs $1 \le i < j \le n$ according to whether (a) $|\{i, j\} \cap \bigcup_{r=1}^k C_r| < 2$, (b) $C_r = (i, j)$ for some r, or (c) i, j appear in two different C_r, C_s.

In case (a), if $|\{i, j\} \cap \bigcup C| < 2$, then one of θ_i or θ_j does not appear in Δ_C and we have $\partial_{\theta_j}\partial_{\theta_i}\Delta_C = 0$.

In case (b), we have that the product $\theta_{j_r}\theta_{i_r}$ does not appear in Δ_C and we also have $\partial_{\theta_{j_r}}\partial_{\theta_{i_r}}\Delta_C = 0$.

Thus, we know that only case (c) contributes to the sum and we can thus write

$$\sum_{1\le i<j\le n} \partial_{\theta_j}\partial_{\theta_i}\Delta_C = \sum_{1\le r<s\le k} \sum_{\substack{i\in C_r\\ j\in C_s}} \pm\partial_{\theta_j}\partial_{\theta_i}\Delta_C\,.$$

In the second sum on the right-hand side, we have to be careful as when we pick $i \in C_r$ and $j \in C_s$ we are not guaranteed that $i < j$, so a sign may be needed in order to keep the equality. We will make a careful study of all possibilities for fixed $1 \le r < s \le k$. First, we rearrange the terms of Δ_C in equation (2.8) to bring the terms $(\theta_{j_r} - \theta_{i_r})(\theta_{j_s} - \theta_{i_s})$ in front performing $(r-1) + (s-2)$ anticommutations, we have

$$\Delta_C = (-1)^{r+s-1}(\theta_{j_r} - \theta_{i_r})(\theta_{j_s} - \theta_{i_s})q$$

for some polynomial q. Remark that i_r, j_r, i_s, j_s satisfy either the inequalities

$$i_r < j_r < i_s < j_s \qquad \text{or} \qquad i_s < i_r < j_r < j_s.$$

The only concern is their relative order, and we can thus assume that we have the numbers $1, 2, 3, 4$. There are two possibilities: $((i_r, j_r), (i_s, j_s))$ is equal to $((1, 2), (3, 4))$

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$n=1$	1				
$n=2$	1	1			
$n=3$	1	2			
$n=4$	1	3	2		
$n=5$	1	4	5		
$n=6$	1	5	9	5	
$n=7$	1	6	14	14	
$n=8$	1	7	20	28	14
$n=9$	1	8	27	48	42

Figure 1: The number of ballot sequences of length n with exactly k1s with $1 \le n \le 9$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$. These will be shown to be the graded dimensions of $EQH_n \simeq EQC_n$.

or $((2,3),(1,4))$. In the first case, we have

$$\left(\partial_{\theta_3}\partial_{\theta_1} + \partial_{\theta_3}\partial_{\theta_2} + \partial_{\theta_4}\partial_{\theta_1} + \partial_{\theta_4}\partial_{\theta_2}\right)(\theta_2-\theta_1)(\theta_4-\theta_3) = 0,$$

and in the second case, we get

$$\left(\partial_{\theta_2}\partial_{\theta_1} + \partial_{\theta_4}\partial_{\theta_2} + \partial_{\theta_3}\partial_{\theta_1} + \partial_{\theta_4}\partial_{\theta_3}\right)(\theta_3-\theta_2)(\theta_4-\theta_1) = 0.$$

Furthermore, this shows that $\Delta_C \in EQH_n$ for all noncrossing pairings C. ■

The set $\mathcal{D}'_n$ is not linearly independent, for example, for $n=3$ and $k=1$, we have the following three noncrossing pairings: $((1,2))$, $((1,3))$, and $((2,3))$, but

$$\Delta_{((1,2))} - \Delta_{((1,3))} + \Delta_{((2,3))} = 0.$$

We want to select a linearly independent subset of $\mathcal{D}'_n$. We proceed as follows: consider a sequence $\alpha = (a_1, a_2, \dots, a_n) \in \{0,1\}^n$ such that $\sum_{i=1}^r a_i \le r/2$ for all $1 \le r \le n$. Such sequences are known as *ballot sequences*. If ever it is the case that $\sum_{i=1}^r a_i > r/2$, then we say that α *breaks the ballot condition* at position r.

Given a ballot sequence α, we build a noncrossing pairing $C(\alpha)$ by first replacing all 0s by open parentheses $0 \mapsto$"(," and all 1s by close parentheses $1 \mapsto$")," and then do the natural maximal pairing of parenthesis. The positions of the pairings give us in lexicographic order a noncrossing pairing which we shall denote $C(\alpha)$. Since α is a ballot sequence, every closed parenthesis is matched and some open parentheses might remain unpaired. The natural pairing of parenthesis guarantees that the result will be noncrossing. For example,

$$\alpha = 0010001101 \quad \mapsto \quad (()((())() \quad \mapsto \quad C(\alpha) = ((2,3),(6,7),(5,8),(9,10)).$$

The total number of ballot sequences of size n is equal to $\binom{n}{\lfloor n/2 \rfloor}$ (see [26, A001405]). The number of ballot sequences graded by the number of 1s in the sequence (see [26, A008315]) is given in Figure 1.

Given this construction, we have the following proposition.

Proposition 2.10 *The set*

$$\mathcal{D}_n = \left\{\Delta_{C(\alpha)} : \alpha \in \{0,1\}^n \textit{ a ballot sequence}\right\}$$

is contained in EQH_n and is linearly independent.

Proof The first statement follows from Proposition 2.9 since $\mathcal{D}_n \subseteq \mathcal{D}'_n \subseteq EQH_n$. To show the linear independence, fix α a ballot sequence and let $C(\alpha) = ((i_1, j_1), \dots, (i_k, j_k))$ be its noncrossing pairing. We remark that the sequence of numbers $j_1 < j_2 < \cdots < j_k$ corresponds to the position of the 1s in α. Using the monomial ordering described in Section 3 and by inspection of the product in equation (2.8), we observe that the term $\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}$ is the smallest lexicographic monomial in $\Delta_{C(\alpha)}$. For different ballot sequences α, we get different positions of the 1s in α and thus different smallest lexicographic monomials, which shows the independence of $\mathcal{D}_n$. ∎

Remark 2.11 For a fixed $0 \le k \le \lfloor \frac{n}{2} \rfloor$, the set

$$\mathcal{D}_n^{(k)} = \left\{\Delta_{C(\alpha)} : \alpha \in \{0,1\}^n \text{ a ballot sequence with } k \text{ 1s}\right\}$$

spans a subspace of R_n of degree k. It is known that the ballot sequences with k 1s are in bijection with standard tableaux of shape $(n-k, k)$. If the variables θ were commutative, the space spanned by $\mathcal{D}_n^{(k)}$ would be the same as the space spanned by the Specht polynomials indexed by standard tableaux and therefore would be an irreducible symmetric group module. Here, the situation appears to be related, but is in fact quite different.

A small example is informative. Consider $n = 4$ and $k = 2$. There are two ballot sequences 0101 and 0011. The associated two noncrossing pairings are $((1,2),(3,4))$ and $((2,3),(1,4))$, and we have

$$\Delta_{C(0101)} = (\theta_2 - \theta_1)(\theta_4 - \theta_3) \quad \text{and} \quad \Delta_{C(0011)} = (\theta_3 - \theta_2)(\theta_4 - \theta_1).$$

On the other hand, the two standard tableaux associated with 0101 and 0011 are

$$T_1 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array} \qquad \text{and} \qquad T_2 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}.$$

A standard construction of the symmetric group irreducible of shape $(2,2)$ from the tableaux T_1 and T_2 is to use the Garnir polynomials

$$\Delta_{T_1} = (\theta_2 - \theta_1)(\theta_4 - \theta_3) = \Delta_{C(0101)},$$
$$\Delta_{T_2} = (\theta_3 - \theta_1)(\theta_4 - \theta_2).$$

Unfortunately, $\Delta_{T_2} \notin EQH_n$. In commutative variables, the span of the $\{\Delta_{T_1}, \Delta_{T_2}\}$ (an irreducible module) would be the same as the span of $\{\Delta_{C(0101)}, \Delta_{C(0011)}\}$. However, for anticommutative variables, it is a different story.[1]

3 A linear basis of the ring

Again, let n be a fixed positive integer and $R_n = \mathbb{Q}[\theta_1, \theta_2, \dots, \theta_n]$. We have thus far represented the basis for R_n as the elements θ_A with $A \subseteq [n]$. Define $\alpha(A) \in \{0,1\}^n$

[1] A more correct construction would be to apply the Young idempotent associated with T_2 to the monomial associated with T_2 using Hivert's action. In this case, we get $\Delta'_{T_2} = \theta_3\theta_4 - \theta_1\theta_4 - \theta_2\theta_3 + \theta_1\theta_2 \notin EQH_n$. The span $\{\Delta_{T_1}, \Delta'_{T_2}\}$ is a symmetric group irreducible module, but is not fully contained in EQH_n.

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to be the sequence $a_1 a_2 a_3 \cdots a_n$ with $a_i = 1$ if $i \in A$ and $a_i = 0$ if $i \notin A$ so that

$$\theta_A = \theta_1^{a_1} \theta_2^{a_2} \cdots \theta_n^{a_n} := \theta^{\alpha(A)} .$$

For such a sequence $\alpha \in \{0,1\}^n$, let $m_1(\alpha) := \sum_{i=1}^n a_i$ represent the number of 1s in the string. This will also be the degree of the monomial θ^α.

For sequences $\alpha \in \{0,1\}^n$, define elements G_α by

$$G_{1^s 0^{n-s}} = F_{1^s} \tag{3.1}$$

and if $\alpha \neq 1^s 0^{n-s}$, then α is of the form $u01^s 0^{n-k-s}$ for some string u of length $k-1$ and we recursively define

$$G_{u01^s 0^{n-k-s}} = G_{u1^s 0^{n-k-s+1}} - (-1)^{m_1(u)} \theta_k G_{u1^{s-1} 0^{n-k-s+2}} . \tag{3.2}$$

We will show below that the recurrence for the G_α is defined so that they are S-polynomials [6] for elements of the ideal I_n. In commutative variables, similar polynomials were defined by Aval–Bergeron–Bergeron [1, 2] as a (complete) subset of S-polynomials needed to compute all possible S-polynomials in the Buchburger algorithm for a Gröbner basis. It is not given that one can easily describe such a set of S-polynomials and here we have adapted the definition for working in the exterior algebra.

Example 3.1 For $\alpha = 010110$ and $\beta = 001100$, we compute the elements G_α and G_β using the definition.

$$\begin{aligned} G_{010110} &= G_{011100} + \theta_3 G_{011000} = (G_{111000} - \theta_1 G_{110000}) + \theta_3 (G_{110000} - \theta_1 G_{100000}) \\ &= \theta_2\theta_4\theta_5 + \theta_2\theta_4\theta_6 + \theta_2\theta_5\theta_6 + 2\theta_3\theta_4\theta_5 + 2\theta_3\theta_4\theta_6 + 2\theta_3\theta_5\theta_6 + \theta_4\theta_5\theta_6, \end{aligned}$$

and we have that

$$\begin{aligned} G_{001100} &= G_{011000} - \theta_2 G_{010000} = (G_{110000} - \theta_1 G_{100000}) - \theta_2 (G_{100000} - \theta_1 G_{000000}) \\ &= \theta_3\theta_4 + \theta_3\theta_5 + \theta_3\theta_6 + \theta_4\theta_5 + \theta_4\theta_6 + \theta_5\theta_6. \end{aligned}$$

We follow [6] for the convention of lexicographical ordering on monomials. Given vectors u, v with nonnegative integer entries, we say that $u < v$ lexicographically if there exists an index $j \geq 1$ such that $u_i = v_i$ for all $1 \leq i < j$, but $u_j < v_j$. Monomials of R_n are ordered by their exponent vectors. More precisely, $\theta_A < \theta_B$ if $\alpha(A) < \alpha(B)$ lexicographically. For example, we have $\theta_1 > \cdots > \theta_n$ and the lexicographically largest monomial in the above example G_{001100} is $\theta_3\theta_4$. The latter demonstrates an important property of these elements stated in the following proposition.

Proposition 3.2 *The largest lexicographic term in G_α is θ^α.*

The proof of Proposition 3.2 follows by induction on the length of α and from a lemma that is analogous to Lemma 3.3 of [2]. The recursion in this result is really the origin of the definition of G_α because equation (3.2) was adapted so that this lemma holds. It follows that the set $\{G_\alpha\}_{\alpha \in \{0,1\}^n}$ is a basis for R_n.

The argument for the proposition is elementary (chasing the largest lexicographic term in (3.3) and (3.4)) and so we do not include it; however, the proof of the following

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result comes from a careful analysis of the terms arising in the recursive definition of the G_α.

Lemma 3.3 *Let $\alpha \in \{0,1\}^{n-1}$, then*

$$G_{0\alpha} = G_\alpha(\theta_2, \theta_3, \ldots, \theta_n) \text{ and } \tag{3.3}$$

$$G_{1\alpha} = \theta_1 G_{0\alpha} + P_\alpha(\theta_2, \theta_3, \ldots, \theta_n) \tag{3.4}$$

for some polynomial $P_\alpha(\theta_2, \theta_3, \ldots, \theta_n) \in \mathbb{Q}[\theta_2, \theta_3, \ldots, \theta_n]$.

Remark 3.4 By convention, the length of the index for our polynomials indicates in which polynomial space we are. For example, if $\beta \in \{0,1\}^n$, then $G_\beta \in R_n$. For $\alpha \in \{0,1\}^{n-1}$ in Lemma 3.3, when we write $G_\alpha(\theta_2, \theta_3, \ldots, \theta_n)$, we mean $G_\alpha \in R_{n-1}$ embedded in R_n with the substitution $\theta_i := \theta_{i+1}$. A similar convention will be followed for P_α.

Proof of Lemma 3.3 The proof will proceed by induction on $n - i$ where i is the number of trailing 0s in α. The base case $0 = n - n$ with n zeros is $0\alpha = 0^n$, and we have

$$G_{0^n} = F_{1^0}(\theta_1, \theta_2, \ldots, \theta_n) = 1 = G_{0^{n-1}}(\theta_2, \theta_3, \ldots, \theta_n)\,.$$

We then consider the case $0\alpha = 01^s0^{n-s-1}$. The polynomials F_{1^s} satisfy the following identity:

$$F_{1^s}(\theta_1, \theta_2, \ldots, \theta_n) = \theta_1 F_{1^{s-1}}(\theta_2, \ldots, \theta_n) + F_{1^s}(\theta_2, \theta_3, \ldots, \theta_n). \tag{3.5}$$

This follows directly from the definition (2.2) where we split the sum in two parts depending if $1 \in A$ or not. The definition of $G_{01^s0^{n-s-1}}$ gives us

$$\begin{aligned} G_{01^s0^{n-s-1}} &= G_{1^s0^{n-s}} - \theta_1 G_{1^{s-1}0^{n-s+1}} = F_{1^s}(\theta_1, \theta_2, \ldots, \theta_n) - \theta_1 F_{1^{s-1}}(\theta_2, \ldots, \theta_n) \\ &= F_{1^s}(\theta_2, \theta_3, \ldots, \theta_n) = G_\alpha(\theta_2, \theta_3, \ldots, \theta_n). \end{aligned}$$

To finish the proof of equation (3.3) by induction, let us assume that α is not of the form 01^s0^{n-s-1} for some $s > 0$. Instead, we have $0\alpha = 0w01^s0^{n-k-s}$ for some $s > 0$ and some string w of length $k - 2$. For $0\alpha = 0w01^s0^{n-k-s}$, we have $n - k - s$ trailing zeros. Remark that for $0w1^s0^{n-k-s+1}$ and $0w1^{s-1}0^{n-k-s+2}$, we have more trailing zeros than that of 0α and we will use the induction hypothesis with (3.3) in the equality (3.6).

$$G_{0w01^s0^{n-k-s}} = G_{0w1^s0^{n-k-s+1}} - (-1)^{m_1(0w)}\theta_k G_{0w1^{s-1}0^{n-k-s+2}}$$

$$= G_{w1^s0^{n-k-s+1}}(\theta_2, \theta_3, \ldots, \theta_n) - (-1)^{m_1(0w)}\theta_k G_{w1^{s-1}0^{n-k-s+2}}(\theta_2, \theta_3, \ldots, \theta_n) \tag{3.6}$$

$$= \left[G_{w1^s0^{n-k-s+1}} - (-1)^{m_1(w)}\theta_{k-1} G_{w1^{s-1}0^{n-k-s+2}}\right](\theta_2, \theta_3, \ldots, \theta_n) \tag{3.7}$$

$$= G_{w01^s0^{n-k-s}}(\theta_2, \theta_3, \ldots, \theta_n) = G_\alpha(\theta_2, \theta_3, \ldots, \theta_n). \tag{3.8}$$

In (3.7), the expression inside the square bracket $[\cdots]$ is treated as a polynomial in the variables $\theta_1, \ldots, \theta_{n-1}$ in R_{n-1} (see Remark 3.4). Hence, the variable θ_k from (3.6) must be replaced by θ_{k-1} in (3.7). Furthermore, $m_1(0w) = m_1(w)$. The expression we get is exactly the definition of $G_{w01^s0^{n-k-s}} \in R_{n-1}$ and equation (3.8) follows. This concludes the proof of (3.3).

We next prove equation (3.4) by induction. The base case is if $1\alpha = 1^{s+1}0^{n-s-1}$, then using equation (3.5) we have

$$\begin{aligned} G_{1^{s+1}0^{n-s-1}} &= F_{1^{s+1}}(\theta_1, \theta_2, \ldots, \theta_n) \\ &= \theta_1 F_{1^s}(\theta_2, \theta_3, \ldots, \theta_n) + F_{1^{s+1}}(\theta_2, \theta_3, \ldots, \theta_n) \\ &= \theta_1 G_{01^s 0^{n-s-1}} + F_{1^{s+1}}(\theta_2, \theta_3, \ldots, \theta_n)\,. \end{aligned} \tag{3.9}$$

In (3.9), we use (3.3) with $G_{01^s0^{n-s-1}} = G_{1^s0^{n-s-1}}(\theta_2, \ldots, \theta_n) = F_{1^s}(\theta_2, \ldots, \theta_n)$. We then let $P_{1^s0^{n-s-1}} = F_{1^{s+1}}(\theta_1, \ldots, \theta_{n-1})$, and this shows that (3.4) holds in this case.

We now assume that $1\alpha \neq 1^{s+1}0^{n-s-1}$. Therefore, $1\alpha = 1w01^s0^{n-k-s}$ for some string w of length $k-2$. We have

$$\begin{aligned} G_{1w01^s0^{n-k-s}} &= G_{1w1^s0^{n-k-s+1}} - (-1)^{m_1(1w)}\theta_k G_{1w1^{s-1}0^{n-k-s+2}} \\ &= \big(\theta_1 G_{0w1^s0^{n-k-s+1}} + P_{w1^s0^{n-k-s+1}}(\theta_2, \theta_3, \ldots, \theta_n)\big) \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\quad - (-1)^{m_1(1w)}\theta_k\big(\theta_1 G_{0w1^{s-1}0^{n-k-s+2}} + P_{w1^{s-1}0^{n-k-s+2}}(\theta_2, \theta_3, \ldots, \theta_n)\big) \\ &= \theta_1\big(G_{0w1^s0^{n-k-s+1}} - (-1)^{m_1(w)}\theta_k G_{0w1^{s-1}0^{n-k-s+2}}\big) \\ &\quad + \big[P_{w1^s0^{n-k-s+1}} - (-1)^{m_1(1w)}\theta_{k-1}P_{w1^{s-1}0^{n-k-s+2}}\big](\theta_2, \theta_3, \ldots, \theta_n). \end{aligned} \tag{3.11}$$

In (3.10), we have used the induction hypothesis of (3.4) on both terms. In (3.11), we group together the terms with θ_1 in front, using the identity $(-1)^{m_1(1w)}\theta_k\theta_1 = (-1)^{m_1(1w)+1}\theta_1\theta_k = (-1)^{m_1(w)}\theta_1\theta_k$. The term with θ_1 in (3.11) is the definition of $G_{0\alpha}$. The expression inside the square bracket is a polynomial in R_{n-1} that we take as the definition for P_α. This shows by induction that (3.4) holds in all cases and concludes the proof of the lemma. ∎

4 A basis for the quotient

The elements G_α are defined so that we could use them to identify a nice basis of the ideal I_n. Our first result establishes that the G_α such that α is not a ballot sequence are in the ideal. The slightly more difficult step is to show that these elements also span the ideal.

Proposition 4.1 *If $\alpha \in \{0,1\}^n$ is not a ballot sequence, then $G_\alpha \in I_n$.*

Proof A sequence $\alpha \in \{0,1\}^n$ is either of the form $\alpha = 1^s0^{n-s}$ for some $s > 0$ or $\alpha = u01^s0^{n-s-k}$ for some $s > 0$ and some $u \in \{0,1\}^{k-1}$.

In the first case, α breaks the ballot condition in position 1 and by equation (3.1), $G_{1^s0^{n-s}} = F_{1^s}$ is in the ideal I_n.

Now, the other case is established by induction on the position of the last 1 in α. We assume that $\alpha = u01^s0^{n-s-k}$ and, by equation (3.2), G_α is in I_n if both $G_{u1^s0^{n-k-s+1}}$ and $G_{u1^{s-1}0^{n-k-s+2}}$ are elements of I_n.

Assume that $u01^s0^{n-s-k}$ breaks the ballot condition for the first time at position r. If $r < k$, then $u1^s0^{n-k-s+1}$ and $u1^{s-1}0^{n-k-s+2}$ both break the ballot condition also at position r. Since $\alpha_k = 0$, α does not break the ballot condition for the first time at $r = k$, so the other possibility is that $r > k$. In this case, $u01^{r-k}$ with $r - k \leq s$ breaks

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the ballot condition for the first time and therefore so does $u1^{r-k-1}$ and so do both $u1^s0^{n-k-s+1}$ and $u1^{s-1}0^{n-k-s+2}$. By our inductive hypothesis, this implies that $G_\alpha \in I_n$.

Therefore, by induction, α breaking the ballot condition implies that $G_\alpha \in I_n$ for all $\alpha \in \{0,1\}^n$. ∎

We will show that the ideal lies in the span of the G_α such that α breaks the ballot condition, therefore establishing our main theorem.

Theorem 4.2 *The set $A_n := \left\{G_\alpha : \alpha \in \{0,1\}^n breakstheballotcondition\right\}$ is a $\mathbb{Q}$-linear basis of the ideal I_n.*

The proof of this theorem uses our understanding of the harmonic space $EQH_n \cong EQC_n$. In Proposition 2.10, we found that $\dim(EQH_n) = \dim(EQC_n)$ is at least the number of ballot sequences. We first establish a small lemma about a spanning set for the quotient EQC_n showing that the dimension is at most the number of ballot sequences. Therefore, we have equality and the set $\mathcal{D}_n$ in Proposition 2.10 is in fact a basis of EQH_n.

Lemma 4.3 *The set $B_n = \left\{\theta^\beta : \beta \in \{0,1\}^n \text{is a ballot sequence}\right\}$ $\mathbb{Q}$-spans the quotient R_n/I_n.*

Proof Order the monomials lexicographically, and let θ^γ be the smallest monomial that is not in the $\mathbb{Q}$-span of B_n (modulo I_n). We must have that γ breaks the ballot condition, since otherwise $\theta^\gamma \in B_n$. Therefore, Proposition 4.1 tells us that $G_\gamma \in I_n$. Proposition 3.2 says that $G_\gamma = \theta^\gamma + \sum_{\beta<\gamma} c_\beta \theta^\beta$. Hence, modulo I_n, we have

$$\theta^\gamma \equiv \theta^\gamma - G_\gamma = -\sum_{\beta<\gamma} c_\beta \theta^\beta.$$

The right-hand side is a linear combination of monomials strictly smaller than θ^γ. By the choice of θ^γ, all such monomials are in the $\mathbb{Q}$-span of B_n. Therefore, θ^γ is also in the $\mathbb{Q}$-span of B_n, a contradiction. We must conclude that there are no such θ^γ and all monomials are in the $\mathbb{Q}$-span of B_n modulo the ideal I_n. ∎

Proof of (Theorem 4.2) Let d_n be the number of ballot sequences of size n. We have

$$d_n \le \dim EQH_n = \dim EQC_n \le d_n,$$

where the first inequality follows from Proposition 2.10 and the second follows from Lemma 4.3. By Proposition 2.7, we have $d_n = \dim EQC_n$. Let $\mathbb{Q}A_n$ be the $\mathbb{Q}$-span of the elements of A_n. Similarly, let $\mathbb{Q}B'_n$ be the $\mathbb{Q}$-span of the set $\left\{G_\beta : \beta \in \{0,1\}^n \textit{is a ballot sequence}\right\}$. Using Proposition 3.2, we have that

$$R_n = \mathbb{Q}B'_n \oplus \mathbb{Q}A_n.$$

Since $\mathbb{Q}A_n \subseteq I_n$ and $\dim \mathbb{Q}B'_n = d_n$, we conclude that $\mathbb{Q}A_n = I_n$. ∎

There are several straightforward consequences of this theorem which we state here.

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Corollary 4.4 *The number of ballot sequences of length n with k entries* 1 *is known* [28] *to be equal to the number* $f^{(n-k,k)}$ *of standard tableaux of shape* $(n-k,k)$. *Therefore, we have that the Hilbert series of* EQH_n *is*

$$Hilb_{EQH_n}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{(n-k,k)} q^k .$$

Corollary 4.5 *The set* $\mathcal{D}_n$ *is a basis of* EQH_n *and of* EQC_n.

Corollary 4.6 *The set* A_n *is a (nonreduced, nonminimal) Gröbner basis of* I_n. *A minimal Gröbner basis for* I_n *is given by*

$$\left\{ G_\alpha : \alpha \in \{0,1\}^n \textit{breaks the ballot condition only at the rightmost 1 of } \alpha \right\}.$$

Remark 4.7 In this paper, we have adopted the language of ballot sequences. An alternative (as in [2, 3]) is to use northeast lattice paths in the first quadrant from $(0,0)$ with a north step for every 0 and an east step for every 1 as we read in a 0–1 sequence. In such representation, a sequence is ballot if and only if it remains above the diagonal.

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