J. Plasma Phys. (2024), *vol.* 90, 905900502
Published by Cambridge University Press
doi:10.1017/S0022377824001168

A necessary condition for breakeven in dipole-confined plasmas

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(Received 3 June 2024; revised 30 August 2024; accepted 2 September 2024)

We have derived a necessary condition for the achievement of breakeven in axisymmetric plasmas with zero toroidal field and confined by a dipole magnetic field (B. Lehnert, *Nature*, vol. 181, 1958, p. 4605; A. Hasegawa, *Comments Plasma Phys. Control. Fusion*, vol. 11, no. 3, 1987). Excellent MHD stability, high values of β (up to 26 %) and good confinement properties awaken the interest of private investors after years of neglect due to lack of public funding and competing alternative lines of research like the tokamak. Starting from a requirement of self-consistency between the balances of momentum and energy in a dipole-confined, two-species plasma and assuming a Maxwellian distribution function for ions and electrons, we derive a necessary condition for breakeven. This condition is more stringent than the Lawson criterion because of the lack of a stabilizing toroidal field. For a given current flowing across the toroidal coil internal to the plasma, the crucial factor at stake is the ratio between the radius of the main toroidal coil and the radius of the vacuum chamber.

Keywords: fusion plasma, plasma confinement, dipole-confined plasma

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Nomenclature

a	linear size of the plasma
A_ϕ	toroidal component of the magnetic potential vector
b	$k_f(1+1/Q)/2(1+w)$
B	intensity of the magnetic field
B_N	$\pi B_e R_c/\mu_0 I$
$\boldsymbol{B}$	magnetic field
B_e	vertical magnetic field due to external coils
B_ϕ	toroidal component of the magnetic field
ECRH	electron cyclotron radiofrequency heating
F	B_ϕ/r
I	electric current flowing in the internal coil
k_f	proportionality constant between the density of fusion power and p^2
MCF	fusion with magnetic confinement
n	particle density
$n_{\max}$	maximum value of n compatible with stability
N	total number of particles

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NBI	neutral beam injection
p	plasma pressure
$p_{\max}$	maximum plasma pressure
p_*	λp^2
P_a	non-fusion heating power
P_c	power lost through conduction
P_f	fusion heating power
P_r	power lost through radiation
Q	P_f/P_a
r	radial coordinate
R_c	radius of the internal coil at $z = 0$
R_p	value of r where $p = p_{\max}$ at $z = 0$
R_w	radius of the internal wall at $z = 0$
T	temperature
T_e	electron temperature
T_i	ion temperature
T_{opt}	optimum temperature
u	internal energy density
v	$\mathrm{d}V/\mathrm{d}\psi$
V	volume contained within a $\psi =$ const. surface
w	P_r/P_c
W_{int}	internal energy
W_m	magnetic energy
W_n	energy of neutrons
W_α	energy of α particles
z	vertical coordinate

Greek Symbols

β	ratio of plasma pressure and magnetic pressure
γ	specific heat ratio
η	stability parameter
λ	Lagrange multiplier
μ_0	$4\pi \cdot 10^{-7}\ \mathrm{T \cdot A^{-1} \cdot m}$
τ_E	energy confinement time
τ_*	threshold on τ_E
ϕ	toroidal coordinate
ψ	$(2\pi)^{-1}\cdot$ poloidal flux
$\psi_{\max}$	maximum value of ψ
$\psi_{\min}$	minimum value of ψ

1. The problem

Magnetic confinement of a toroidal axisymmetric plasma with vanishing toroidal magnetic field and dipole-like poloidal field is a decade-old idea in the controlled fusion research with magnetically confined plasmas (MCFs) (Lehnert 1958, 1968*a*,*b*; Hasegawa 1987). This plasma is the terrestrial analogue of the ionospheric plasma which is confined by the dipole magnetic field of the Earth. The field is produced by a toroidal loop of current; this current flows across a circular, toroidal coil.

In spite of decades spent in MCF research since the proposal of Lehnert (1958), research on fusion-relevant dipole-confined plasmas is somehow still in its infancy. To start with,

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the main toroidal coil whose current sustains the confining magnetic field is internal to the plasma, rather than external to it; this fact provides the basis for excellent MHD stability (Freidberg 2014) but then applications on Earth (unlike applications to space propulsion Teller *et al.* 1992) require a reliable coil support system. In an early experiment, a copper coil was kept in place by retractile supports and then allowed to fall for 0.02 s, a time interval during which experiments were performed (Anderson *et al.* 1968). If unmagnetized rods and wires support a superconducting coil, then losses towards the supports are far from negligible (Freeman *et al.* 1969); this remains true also if a toroidal field is added (Breton & Ya'akobi 1973). In the proposal of Hasegawa (1987), the coil is superconducting, and levitates magnetically in an external field. A superconducting coil has been successfully levitated for time intervals $\approx O(10^4 s)$ and an electric current $\approx O(10^6 A)$ has been induced in such a coil in the experiments Levitron (Skellett 1975), RT-1 (Saitoh *et al.* 2010) and LDX (Garnier *et al.* 2006; Kesner & Mauel 2013). An alternative layout, where the toroidal coil is in contact with material leads which are screened against the impinging plasma particles by the very magnetic field flowing across them, has been proposed in Lehnert (1958, 1968*a*,*b*) and has never been tested to date, to the best of the author's knowledge at least. Moreover, the formidable technological issues involved in the close proximity of a thermonuclear plasma to the main toroidal coil nudged public funding towards allegedly less technologically demanding approaches, like e.g. the tokamak. As a consequence, today (2024) the performances of dipole-confined plasmas are far from being of thermonuclear interest, and order-of-magnitude worse than the corresponding results of much more investigated devices like tokamaks: peak electron density 10^{18} m^{-3}, peak electron temperature 0.5 KeV, energy confinement time 0.028 s (Kesner & Mauel 2013) (even if much better results have been claimed Yoshida *et al.* 2013).

The potential advantages of the dipole approach to MCF are nevertheless relevant, and the results of both theory and experiments are quite promising. For example, dipole-confined plasmas enjoy excellent MHD stability. The internal coil of the dipole acts as the internal, rigid current-carrying conductor of a toroidal hardcore Z-pinch, which is MHD stable and achieves values of β much larger than in tokamaks (Freidberg 2014), i.e. up to 26 % (Mauel 2008). Qualitatively at least, the dipole can be thought of as a toroidal mirror, much more similar to the Earth's field than the linear system in a traditional magnetic mirror trap (Dolan 1982). Particles in the toroidal area around the outside of the central magnet that move up or down see increasing magnetic density and tend to move back towards the equator area again. This gives the system some level of natural stability. Particles with higher energy, the ones that would escape a traditional mirror trap, instead follow the field lines through the hollow centre of the magnet, recirculating back into the equatorial area again. The poloidal current density vanishes everywhere; then, the only destabilizing term in the extended energy principle of MHD (Freidberg 2014) arises from the curvature of the magnetic field lines and can be compensated either by raising B or by making the system larger (thus reducing the curvature), ensuring stability against both interchange and ballooning modes (Garnier, Kesner & Mauel 1999; Krashenninikov, Catto & Hazeltine 1999; Simakov 2001).

Moreover, the magnetic geometry is quite simple, an important technological advantage. For a given amount of fusion power, *the simpler dipole is expected to be much less massive than a tokamak of comparable power and therefore able to produce greater specific power* (Teller *et al.* 1992). Additional toroidal coils – external to the plasma – may help in shaping the plasma at MHD equilibrium, e.g. by providing separatrices for impurity control. The simplicity of the coil set and its lack of interlocking coils allows easy access to the coils for routine maintenance. Large values of β and lack of toroidal field allow a dramatic reduction of mechanical stresses for given volume-averaged pressure of the confined plasma. The fact that the plasma is confined outside of the coil allows for very good field usage and mitigates issues related to heat dissipation in the divertor since the geometry allows for a large, cooling expansion of magnetic flux.

Furthermore, unfavourable scaling laws notoriously plague MCF research. Undesired diffusion of energy and particles far from the high-pressure regions where fusion occurs hampers the achievement of a favourable energy balance. For example, tokamak plasmas have a 'neoclassical' degradation of transport that derives from the drifts of particles off of the flux surfaces. In a dipole the drifts are toroidal and define the flux surfaces; the plasma carries only a diamagnetic, toroidal current, and is created with the help of auxiliary heating (typically electron cyclotron radiofrequency heating, or ECRH) rather than through ohmic heating (Kesner & Mauel 1997), and is not prone to major disruptions. Steady states have been experimentally sustained up to 20 s through ECRH (Kesner & Mauel 2013). The Lorentz force due to the diamagnetic current and the poloidal field counteracts the pressure gradient and confines the plasma. Then, only cross-field transport occurs as far as contact with the unscreened material wall is prevented. In tokamaks, the energy confinement time degrades with increasing auxiliary heating. In a dipole, no such degradation occurs (Kesner & Mauel 2013), the particle confinement time is 3–10 times smaller than the value predicted by classical diffusion but with the same favourable scaling with $B^2\sqrt{T}/n$ and particle transport can be reduced through control of the magnetic shear, just like in other MCF devices (Edlington *et al.* 1980).

Finally, in other MCF experiments small fluctuations can cause significant energy loss. In a dipolar magnetic field fluctuations tend to compress the plasma, without energy loss. The spontaneous peaking (Kobayashi, Rogers & Dorland 2010) of particle density and pressure (in opposition to the usual direction of diffusion) near the axis, where the dipole field lines followed by the particles are densely packed, is dubbed 'turbulent pinch' (Boxer *et al.* 2010). These peaked profiles keep the plasma far from the material walls and raise the fusion power density locally, potentially facilitating the achievement of breakeven and ignition.

Thus, in today's climate of renewed widespread interest in nuclear fusion, a start-up, OpenStar, puts forward the proposal of a levitating superconducting coil made of the newest high temperature superconductors (Berry, Mataira-Cole & Simpson 2023), also used in RT-1. Another start-up, Deutelio, follows the alternative strategy (Lehnert 1968*b*) of magnetically isolated material leads sustaining a water-cooled coil made of copper and puts forward the proposal of a dedicated experiment, the POLOMAC (Elio 2014).

Information on the prospect of success of these (or similar future) proposals is highly desirable. Admittedly, no data concerning the behaviour of dipole-confined plasmas near breakeven are available – yet. Moreover, our knowledge of energy transport across the plasma is incomplete, to say the least; this makes any independent reliable prediction impossible. The aim of the present work is to write down a necessary condition for the achievement of breakeven in dipole fusion. Our goal is more modest than a prediction; correspondingly, we need fewer assumptions.

We assume steady-state, non-rotating, axisymmetric MHD equilibrium with zero toroidal field. We limit ourselves to a two-species plasma in the MHD approximation, neglect plasma rotation, assume the temperatures of all species to be equal (unless stated otherwise), invoke no assumption concerning the actual physical mechanism ruling non-fusion plasma heating and energy transport in the plasma, assume that the same peaked profiles of particle density and pressure observed so far occur also at breakeven, make the customary assumption (Dolan 1982) that the fusion heating power of the plasma

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is proportional to the square of the pressure, assume that the pressure is isotropic and neglect the impact of non-Maxwellian particles. Admittedly, the latter assumption is well satisfied for levitated coils only (Saitoh *et al.* 2010; Kesner & Mauel 2013; Berry *et al.* 2023); contact with material leads has been associated with non-thermal particles and degraded confinement (Mauel 2008). Thus, our discussion applies to POLOMAC, where material leads are in contact with the central coil (Elio 2014), only if the prediction of screening due to the magnetic field generated by the currents flowing in the leads themselves turns out to be correct (Lehnert 1968*b*). Finally, our discussion does not apply to another plasma with vanishing toroidal field, the field reversed configuration (FRC) with both neutral beam injection (NBI) and a confinement barrier sustained by radial electric fields (Binderbauer *et al.* 2015), because the stability of this FRC relies on fast particles due to NBI, which our discussion neglects.

We present a short review of the properties of the peaked profiles of density and pressure hinted at above in § 2, as these properties play a crucial role in the following discussion. Section 3 displays the relevant balances of momentum and energy. We show in § 4 that self-consistency between such balances under the assumptions listed above leads to a necessary condition on the energy confinement time, which can be more stringent than the familiar Lawson criterion because of the constraint of zero toroidal field. We derive useful expressions for this condition in a dipole-confined plasma in § 5 and obtain numerical results in § 6. Conclusions are drawn in § 7. SI units are used, unless stated otherwise.

2. Invariant profiles

Spontaneous relaxation towards well-peaked profiles of pressure p and particle density n has been predicted (Hasegawa 1987; Kobayashi *et al.* 2010) and experimentally confirmed (Saitoh *et al.* 2011; Kesner & Mauel 2013; Yoshida *et al.* 2013) when the superconducting current loop is levitated (so that particle losses along the field lines are negligible); for a thermodynamic description of the relaxation, see the Appendix. The relaxed profile of p is marginally stable against the ideal MHD interchange mode, i.e. any interchange of position between adjacent tubes in this profile induces a perturbation in the pressure profile which satisfies

$$\delta(pv^{\gamma}) = 0, \tag{2.1}$$

where $\gamma = \frac{5}{3}$ for three-dimensional motion, $v = \mathrm{d}V/\mathrm{d}\psi$, ψ being the poloidal flux. Interchange motion leaves the ('invariant') pressure profile unaffected; as (2.1) holds, the value of pv^{γ} is the same for all flux tubes. Analogously, if each flux tube contains the same number of particles

$$\delta(nv) = 0, \tag{2.2}$$

(we assume temperature $T \propto p/n$) then the profile of n is invariant. Remarkably, invariant profiles have flat gradients in magnetic-flux space, not real space. Flattening of such gradients is highly beneficial to plasma stability and transport. In principle, the plasma may still be unstable to both pressure gradient driven MHD modes, and density or temperature gradient driven drift modes. Theoretical studies have shown that stability to the MHD interchange mode is sufficient for stability to MHD ballooning modes (Simakov *et al.* 2000). Studies of drift modes (Kesner 1997; Kesner & Hastie 2002) have shown that the critical parameter for stability is $\eta = \mathrm{d}\ln T/\mathrm{d}\ln n = \gamma - 1$, and the most stable operating point is $\eta = \frac{2}{3}$, which, combined with (2.1), corresponds to the invariant profiles of p and n.

Invariant profiles are tantalizing because flat gradients in the magnetic-flux space are extremely peaked in real space, for a dipole at least. Let $N = \int n\,\mathrm{d}V \propto nr^3$ be

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the total number of particles. In a dipole field $\psi \propto 1/r$ at large r. (This scaling is exact at $\beta = 0$ and is a common tool in dipole research. If $\beta > 0$, the profile $\psi \propto 1/r$ approximately becomes $\psi \propto r^{-1+502\beta/1001}$ Krashenninikov *et al.* 1999.) Then, (2.2) implies $nr^3 \propto \int nv\,\mathrm{d}\psi = nv \int \mathrm{d}\psi = \text{const.} \int \mathrm{d}\psi \propto \psi \propto 1/r$, i.e. $n \propto 1/r^4$; as (2.2) gives $nv =$ const., we get $v \propto 1/n \propto r^4 \propto \psi^{-4}$, and (2.1) gives $p \propto 1/r^{4\gamma} \propto \psi^{4\gamma}$. Peaked profiles make nuclear fusion easier at low r while limiting the heat load on the material wall, as the fusion power density is quadratic in n and is a strongly increasing function of temperature $T \propto p/n \propto 1/r^{4(\gamma-1)}$. Moreover, the ratios of peak p to the scrape-off layer (SOL) value p, peak n to the SOL n and peak T to the SOL T are $(R_w/R_p)^{4\gamma}$, $(R_w/R_p)^4$ and $(R_w/R_p)^{4(\gamma-1)}$ respectively, where R_w and R_p are the radial coordinates of the wall and the peak of $p \propto nT$, respectively. Thus, the basic design for a dipole reactor has a small coil levitated inside a large vacuum chamber.

3. Balances of energy and momentum

The balance of Lorenz force $(1/\mu_0)(\nabla \wedge \boldsymbol{B}) \wedge \boldsymbol{B}$ and pressure gradient ∇p allows mechanical equilibrium in magnetically confined plasmas described by MHD ($\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot A^{-1} \cdot m}$, $\boldsymbol{B}$ magnetic field). The Grad–Shafranov equation

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 r^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi}, \tag{3.1}$$

describes such equilibrium in cylindrical coordinates (r, ϕ, z). Here, $\psi = \psi(r, z)$ is $(2\pi)^{-1}$ times the poloidal flux and is such that the radial and the vertical component of the poloidal magnetic field are $-(1/r)(\partial\psi/\partial z)$ and $(1/r)(\partial\psi/\partial r)$ respectively. Moreover, $p = p(\psi)$, $F = F(\psi) \equiv B_\phi/r$ and B_ϕ is the toroidal component of $\boldsymbol{B}$.

We focus on problems where ψ is fixed at the plasma boundary here and below. Then, (3.1) is the Euler–Lagrange equation in ψ of the variational principle (Lao, Hirshman & Wieland 1981)

$$\int \mathrm{d}V \left[\frac{|\nabla\psi|^2}{2\mu_0 r^2} - \frac{F^2(\psi)}{2\mu_0 r^2} - p(\psi)\right] = \text{extremum}, \tag{3.2}$$

where the domain of integration and $\mathrm{d}V$ are the plasma volume $\propto O(r^3)$ and the volume element respectively. (We specify if the extremum is a minimum or a maximum below.) In particular, if $B_\phi \equiv 0$ then

$$F \equiv 0, \tag{3.3}$$

and (3.2) simplifies to

$$\int \mathrm{d}V \left[\frac{|\nabla\psi|^2}{2\mu_0 r^2} - p(\psi)\right] = \text{extremum}. \tag{3.4}$$

As for the energy balance, plasma heating compensates for energy losses in steady state, hence

$$P_f + P_a = P_c + P_r, \tag{3.5}$$

where P_f, P_a, P_c and P_r are the fusion plasma heating power, the non-fusion plasma heating power (ohmic, RF...), the power lost through conduction and the power lost through radiation, respectively.

As for P_f, we neglect the impact of fast, non-Maxwellian particles, follow the well-known, approximate treatment of Maxwellian-averaged fusion cross-section of section 2D of Dolan (1982) and write $P_f = \int k_f p^2\,\mathrm{d}V$.

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As for P_a, we write $P_a = P_f/Q$, Q being the fusion plasma heating power; $Q = 1$ at breakeven. In the following, our results rely on no detailed description of the actual mechanism underlying auxiliary heating.

As for P_c, we write $P_l = W_{\mathrm{int}}/\tau_E$, with $\tau_E, u = \frac{3}{2}p$ and $W_{\mathrm{int}} \equiv \int u\,\mathrm{d}V$ energy confinement time, internal energy density and internal energy of the plasma, respectively.

As for P_r, for our purposes it is useful to define the dimensionless quantity $w \equiv P_r/P_c$; we are going to discuss cases with different values of w in the following.

4. A necessary condition

If we want to assess the feasibility of a steady-state, $B_\phi = 0$, axisymmetric plasma at breakeven (dubbed 'our plasma', below), then the usual approach needs computing of τ_E starting either from first principles or from dimensional analysis or both, a far-from-trivial task because the available models of plasma energy transport are far from achieving general consensus.

Here, we adopt an alternative strategy: firstly, we postulate that our plasma exists, then we derive the minimum value τ_* that τ_E must have in our plasma in order to achieve fusion at a given Q. Here, we focus on breakeven; we drop this assumption below. If $\tau_E < \tau_*$, then the energy confinement is unable to sustain steady-state fusion; this inequality $\tau_E < \tau_*$ is a sufficient condition for failure and describes implicitly a range of parameters to be avoided. Thus, its violation

$$\tau_E \geq \tau_*, \tag{4.1}$$

is the necessary condition for achieving fusion at a given Q. If it turns out that τ_* is unrealistically high (say, 1000 s), then fusion is definitely impossible. Generally speaking, it is desirable to lower τ_* as much as possible.

We are going to compute τ_* with the help of three physically independent conditions - namely, (3.1), (3.3) and (3.5) for $\tau_E = \tau_*$

$$\int \mathrm{d}V \left[\frac{3}{4\tau_* b}p - p^2\right] = 0; \quad b \equiv \frac{k_f\left(1 + \dfrac{1}{Q}\right)}{2(1+w)}, \tag{4.2a,b}$$

for three quantities ψ, F and p in dipole fusion devices; $b \to k_f$ for $Q = 1$ and $w = 0$ (negligible P_r). Our plasma satisfies both momentum and energy balance simultaneously, i.e. it solves the extremum problem (3.4) with the constraint (4.2*a*,*b*). Accordingly, we write

$$\int \mathrm{d}V \left[\frac{|\nabla\psi|^2}{2\mu_0 r^2} - p(\psi)\right] + \lambda \int \mathrm{d}V \left[\frac{3}{4\tau_* b}p(\psi) - p^2(\psi)\right] = \text{extremum}, \tag{4.3}$$

with λ the Lagrange multiplier. Remarkably, λ/b is a time. But there is only one physically meaningful quantity with the dimension of a time in a system described by (3.1), (3.3) and (4.2*a*,*b*), namely τ_*. Self-consistency requires that we solve this conundrum; to this purpose, we invoke the fact that any rescaling of the Lagrange multiplier leaves the solution of the problem unaffected. (A well-known example is the thermodynamic equilibrium in Gibbs' statistical mechanics. Such an equilibrium, indeed, corresponds to a maximum of entropy constrained by a given amount of total energy with $1/T$ as the Lagrange multiplier; of course, its nature does not change if we rescale T, e.g. by changing the scale of the thermometer.) Analogously, the nature of a steady state remains unaffected if we rescale time.

Below, we justify the following choice of time rescaling:

$$\frac{\lambda}{b} = \frac{4\tau_*}{3}. \tag{4.4}$$

Then, (4.3) reduces to

$$\int \mathrm{d}V \left[\frac{|\nabla \psi|^2}{2\mu_0 r^2} - p_*(\psi) \right] = \text{extremum}, \tag{4.5}$$

where $p_*(\psi) \equiv \lambda p^2(\psi)$. With the rescaling (4.4), it is clear that the extremum in (3.2)–(4.5) is a minimum. In this case, indeed, the facts that $B = |\nabla \psi|/r$ and $\int \mathrm{d}V p_* \propto \int \mathrm{d}V p^2 \propto P_f$ imply that solving (4.5) means finding a minimum of magnetic energy $W_m \equiv \int \mathrm{d}V(B^2/2\mu_0)$ for a given P_f. But this is equivalent to finding a maximum of P_f for a given W_m, according to a lemma of variational calculus, the reciprocity principle for isoperimetric problems (Elsgolts 1981). In particular, if $W_m = 0$ (i.e. if there is no confining magnetic field) then the maximum allowable value of P_f is zero (i.e. no fusion occurs, as expected). Now, maximization of P_f for a given W_m implies minimization of $W_m/P_f = (Q+1)W_m/Q(P_f + P_a) = (Q+1)W_m/Q(P_c + P_r) = (Q+1)W_{\text{int}}/Q\beta(P_c + P_r) = (Q+1)/Q\beta(1+w)\tau_E$, hence of τ_E for plasmas with given values of Q, w and β. In other words, (4.4) allows the value of τ_* in (4.5) to be precisely the required lower bound on the values of τ_E compatible with fusion. This is why we adopt the rescaling (4.4).

Comparison of (3.4) and (4.5) shows that the inclusion of the energy balance constraint (4.2*a*,*b*) in the momentum balance of our plasma leads back to the familiar, unconstrained momentum balance of our plasma with the rescaling $p \rightarrow p_*$; i.e. p_* is the pressure field which solves (3.1)–(3.3) in our plasma where $\tau_E = \tau_*$.

The definition of p_* and (4.2*a*,*b*) give $p = \sqrt{p_*/\lambda}$ and $\lambda = \int p\,\mathrm{d}V / \int p^2\,\mathrm{d}V$ respectively. Then, $\lambda = (\int \sqrt{p_*}\,\mathrm{d}V / \int p_*\,\mathrm{d}V)^2$, (4.1) and the definition of λ provide us with the following necessary condition for our plasma to achieve fusion at a given Q:

$$\tau_E \geq \tau_* = \frac{3}{4b} \left(\frac{\displaystyle\int \sqrt{p_*}\,\mathrm{d}V}{\displaystyle\int p_*\,\mathrm{d}V} \right)^2. \tag{4.6}$$

5. Thresholds

In order to apply (4.6) to dipole fusion, we drop the subscript $_*$ below and recall that $\mathrm{d}V = v\,\mathrm{d}\psi$, $v = v_0 \psi^{-4}$, $\gamma = \frac{5}{3}$ and $p = p_0 \psi^{4\gamma}$ in present experiments, where v_0 and p_0 are positive proportionality constant quantities. With no loss of generality we assume also that $\psi_{\min} \leq \psi \leq \psi_{\max}$ with $\psi_{\min} \ll \psi_{\max}$. Then, (4.6) gives

$$\tau_E \geq \tau_* = \frac{363}{4} \frac{1}{b p_{\max}}, \tag{5.1}$$

where $p_{\max} = p(\psi = \psi_{\max})$, $p_0 = p_0(p_{\max})$ and v_0 cancels out. In turn, if $p = p_{\max}$ at a radial position R_p and is minimum near the wall located at $r = R_w$ then for an invariant profile $p \propto \psi^{4\gamma} \propto r^{-20/3}$ we write $p = p_{\max}(R_p/r)^{20/3}$ and the volume average of p is $\langle p \rangle = \int_{R_p}^{R_w} 4\pi p r^2\,\mathrm{d}r / \frac{4}{3}\pi(R_w^3 - R_p^3) \approx \frac{9}{11} p_{\max}(R_p/R_w)^3$ for $R_p \ll R_w$, i.e. $p_{\max} = \frac{11}{9}(R_w/R_p)^3 \langle p \rangle$.

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Substitution in (5.1) gives the main result of this paper

$$\tau_E \geq \tau_* = \frac{297}{4}\frac{1}{b\langle p\rangle}\left(\frac{R_c}{R_w}\right)^3\left(\frac{R_p}{R_c}\right)^3, \tag{5.2}$$

as $(R_c/R_w)(R_p/R_c) = R_p/R_w$, where R_c is the radius of the toroidal coil internal to the plasma. The smaller b is, the larger τ_* is, the more likely the violation of (4.1) is. Just as expected, if there is no fusion then $b \to 0$, $\tau_* \to \infty$ and (4.1) is always violated; we may also raise τ_* by increasing w (i.e. in case of larger radiation losses) and ignition ($Q = \infty$) corresponds to a larger τ_* than breakeven ($Q = 1$).

Qualitatively speaking, we may reasonably say that the fact that (4.6) – which (5.2) is based upon – is an additional constraint on τ_E beyond Lawson's criterion itself corresponds to the fact that (3.3) is an additional restriction on MHD equilibrium, which forces the steady state to give up the beneficial stabilizing effect of the toroidal field. By the way, we may note that, in his original paper, Lawson assumed *that the conduction loss is zero* (Lawson 1957); this seems to agree with the fact that the necessary condition (5.2) is physically independent of Lawson's criterion.

The fact that $\tau_* \propto 1/b\langle p\rangle$ has an intuitive physical meaning. It has been shown (Taylor 1997) that the picture of a plasma with linear size a made of filamentary tube fluxes continuously swapping their mutual position while leaving the global profiles of n and T unaffected, i.e. – in the dipole slang – of a plasma with invariant profiles, leads to the so-called 'Bohm scaling' $\tau_E \propto Ba^2/T$ (Connor & Taylor 1977). This scaling is often observed in turbulent plasmas, like the central, turbulent pinch region near the axis of a dipole plasma (Boxer *et al.* 2010) where density peaking occurs and nuclear fusion reactions are more likely to occur; it is also routinely observed in tokamaks (Perkins *et al.* 1993) and is a common tool in the design of future experiments (Romanelli & Orsitto 2021). As $p \propto nT$, we rewrite Bohm scaling as $\tau_E \propto nBa^2/p$. Now, when solving (4.5) we aim at finding a maximum value P_{max} for P_f in stable equilibrium; accordingly, we write $\tau_* = (\tau_E)_{P_f = P_{max}}$. We obtain the maximum of P_f at an optimal temperature T_{opt} dictated by Lawson's criterion and at the maximum value n_{max} of n compatible with the stability of a toroidal plasma. Conservatively, we assume that $n_{max} \propto I/a^2 \propto B/a$ (Greenwald 2002) where $I \propto aB$ has the dimension of an electric current. Accordingly, $P_{max} \propto bn_{max}^2 T_{opt}^2 a^3$ and $\tau_* = (\tau_E)_{n=n_{max}, T=T_{opt}}$. Bohm scaling gives therefore: $\tau_* \propto a^2(n_{max}B/p) \propto n_{max}^2 a^3/p = bn_{max}^2 T_{opt}^2 a^3/bpT_{opt}^2 = (1/bp)(P_{max}/T_{opt}^2)$. Since both P_{max} and T_{opt} are fixed, we obtain $\tau_* \propto 1/bp$, as in (5.2).

As for breakeven, Bohm scaling is a more pessimistic model of energy transport than both the classical diffusion assumed in Kesner & Mauel (1997) and the scaling experimentally observed in levitating dipoles (Edlington *et al.* 1980). Since fusion reactions occur mainly where p is maximum, i.e. not too far from the $r = 0$ axis where the turbulent pinch is located, it is a conservative but reasonable assumption that Bohm scaling applies in such a region. On the other hand, it is only natural to invoke a pessimistic scaling for energy confinement, Bohm scaling, when estimating a lower bound on the values of τ_E allowed for breakeven to occur.

The fact that $\tau_* \propto (R_p/R_w)^3$ reflects the positive impact of peaked profiles of p, as τ_* shrinks as $R_p \ll R_w$. Moreover, (5.2) shows the advantage in having $R_c \ll R_w$, i.e. a vacuum chamber much larger than the internal coil. It remains to compute R_p/R_c. To this purpose, we recall that $p = p_{max}$, where $\psi = \psi_{max}$ and that $\psi = rA_\phi$ with A_ϕ being the toroidal component of the magnetic potential vector (Freidberg 2014), focus on the horizontal symmetry plane $z = 0$, follow the $\beta = 0$ analytical approximation of Kesner & Mauel (1997) in vacuum (i.e. as $R_w \to \infty$ for a conducting wall) including the contribution

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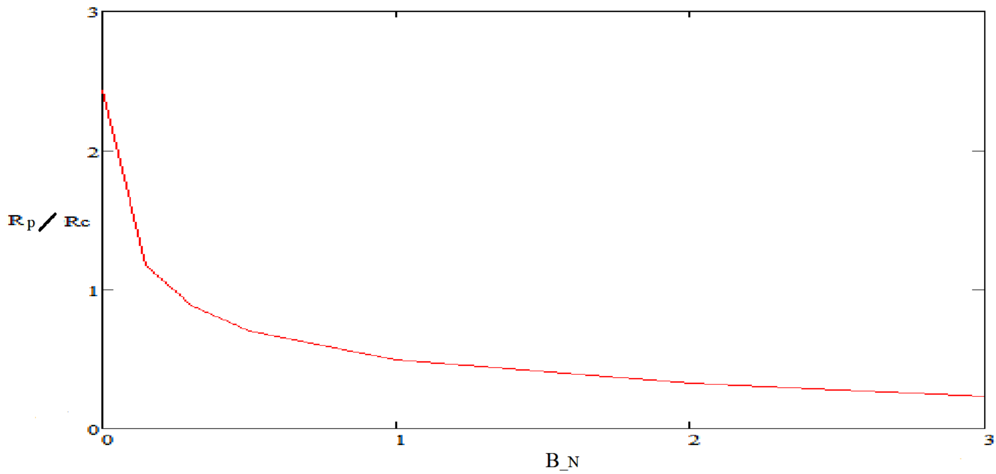

FIGURE 1. Value of R_p/R_c vs. normalized vertical field $B_N \equiv \pi B_e R_c/\mu_0 I$.

of coils external to the plasma and compute R_p as the solution of

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)_{r=R_p} = 0; \quad \psi = rA_\phi; \quad A_\phi = \frac{\mu_0 I R_c}{\pi \kappa r}\left[\left(1 - \frac{\kappa^2}{2}\right)K(\kappa^2) - E(\kappa^2)\right] - \frac{B_e r}{2}, \tag{5.3}$$

$$\kappa = \kappa(r) \equiv \sqrt{\frac{4R_c r}{(R_c + r)^2}}, \tag{5.4}$$

where I is the current flowing in the coil internal to the plasma, K and E are complete elliptic integrals of the first and second kind respectively, and B_e is an additional, vertical magnetic field due to additional toroidal coils external to the plasma, which can give birth to magnetic separatrices (Kesner & Mauel 1997). It turns out that R_p/R_c is a decreasing function of the dimensionless parameter $B_N \equiv \pi B_e R_c/\mu_0 I$ – see figure 1. For given values of R_c, R_w and I, therefore, the necessary condition (5.2) reduces to a lower bound on the external vertical field. If $R_p/R_c < 1 (> 1)$ then the peak of pressure lies inside (outside) the coil internal to the plasma. Not surprisingly, external coils help in compressing the plasma towards $r = 0$. Of course, a description of ψ more detailed than (5.3) should take into account either the presence of a conductive wall at finite R_w or the fact that $\beta > 0$; the former and the latter (Krasheninnikov *et al.* 1999) may provide lower and larger values of R_p/R_c respectively.

6. Numerical results

Our discussion has referred to no particular nuclear fusion reaction up to this point. In order to obtain numerical results, however, we must specify the value of k_f, which in turn depends on the reaction of interest. Let us start with DT fusion, where (Dolan 1982) $k_f = 2.9 \cdot 10^{-5} \cdot (W_\alpha/(W_\alpha + W_n)) \cdot (1 + T_e/T_i)^{-2}\ \mathrm{W \cdot m^{-3} \cdot Pa^{-2}} = 5.8 \cdot 10^{-6} \cdot (1 + T_e/T_i)^{-2}\ \mathrm{W \cdot m^{-3} \cdot Pa^{-2}}$, with T_e the electron temperature, T_i the ion temperature, $W_\alpha = 3.52$ MeV and $W_n = 14.06$ MeV the energy of the α's and of the neutrons produced by the reaction, respectively. We assume $T_e/T_i = 1$ is uniform across the plasma for simplicity.

A typical value for $\langle p \rangle$ in a DT reactor for $n = 10^{20}\ \mathrm{m^{-3}}$, $T = 10$ KeV is $1.6 \cdot 10^5$ Pa. Correspondingly, Lawson's criterion requires $\tau_E \approx 3$ s. As for τ_*, let us start

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with $Q = 1$ and negligible $P_r(w = 0)$; we shall drop both assumptions below. Then, $\tau_* = (6.8 \cdot R_p/R_w)^3$ in (5.2). Breakeven is achievable if $\tau_* \leq \tau_E$, i.e. for $R_p/R_w \leq 0.2$. For example, in POLOMAC, $R_w = 7.9$ m and $R_c = 5.4$ m. Then, $R_p/R_c = (R_p/R_w)(R_w/R_c) \leq 0.2 \cdot (R_w/R_c) = 0.2 \cdot (7.9/5.4)1.46 = 0.29$, corresponding to $B_N \geq 1.5$. A more detailed analysis which takes into account both the actual shape of a conducting wall at finite distance from the internal coil on one hand and the impact of $\beta > 0$ on the other hand is likely to modify this lower bound somehow, as the magnetic surfaces are likely to be bent towards $r = 0$ by the wall while a finite β hints at a deformation of the same surfaces outwards, i.e. away from $r = 0$.

The cubic dependence of τ_* on R_c/R_w is due to the peakedness of the invariant profile of p and has relevant consequence when removing the assumptions $Q = 1$ and $w = 0$. If we require ignition rather than breakeven, i.e. if we replace $Q = 1$ with $Q = \infty$, then b gets halved; the same occurs if we drop the assumption $w = 0$ of negligible radiation losses and replace it with the more pessimistic assumption $P_r = P_c$, i.e. $w = 1$. In both cases (5.2) leads to $R_p/R_c \leq 0.29$ so that $B_N \geq 1.6$. As for P_f, our results depend on Q only weakly: basically, if a dipole is able to attain breakeven it is also likely to be able to attain ignition. As for P_r, our results agree with the fact that both plasma temperature and particle density are particularly low near the wall (where radiation processes are dominant) for the very peaked, invariant profiles of a dipole-confined plasma.

When it comes to catalysed DD fusion, according to Kesner *et al.* (2004), *approximately 94 % of the power is generated in energetic ions and a substantial fraction of the plasma energy leaves the plasma as bremsstrahlung radiation*. Then, we write (Dolan 1982) $k_f = \frac{94}{100} \cdot 6.9 \cdot 10^{-7} \cdot (1 + T_e/T_i)^{-2}\sqrt{30/T_i(\text{KeV})}$ $\text{W} \cdot \text{m}^{-3} \cdot \text{Pa}^{-2}$. We assume T_e/T_i uniform across the plasma for simplicity.

Kesner *et al.* (2004) discusses two scenarios of ignition in catalysed DD fusion: A ($P_a = 0, P_f = 610$ MW, $P_r = 430$ MW, $p_{\max} = 5.4 \cdot 10^6$ Pa, $T_i = 41$ KeV, $T_e = 30$ KeV) and B ($P_a = 0, P_f = 384$ MW, $P_r = 362$ MW , $p_{\max} = 4.1 \cdot 10^6$ Pa, $T_i = 37$ KeV, $T_e = 30$ KeV). The values of $w = P_r/P_c = P_r/(P_f - P_r)$, b and τ_* in A and B are $w = 430/(610 - 430) = 2.4$, $b = 2.7 \cdot 10^{-8}$ $\text{W} \cdot \text{m}^{-3} \cdot \text{Pa}^{-2}$, $\tau_* = 622$ s and $w = 362/(384 - 362) = 16.5$, $b = 5.0 \cdot 10^{-9}$ $\text{W} \cdot \text{m}^{-3} \cdot \text{Pa}^{-2}$ and $\tau_* = 4426$ s, respectively. As for τ_E, according to Kesner *et al.* (2004) (and in agreement with Lawson's criterion for ignition in this range of temperatures) $\tau_E = 5.1$ s. Then (5.1) tells us that both A and B are too optimistic. (It is possible to show that a similar conclusion holds also for the DT ignition scenario outlined in Kesner *et al.* 2004.) Remarkably, and in contrast with Kesner *et al.* (2004), our results do not rely on *the assumption that turbulent transport does not substantially degrade confinement*.

However, the peakedness of the pressure profile – and, in particular, the fact that $p_{\max} = \frac{11}{9}(R_w/R_p)^3\langle p\rangle$ – suggests that we may still achieve ignition in catalysed DD fusion – e.g. in scenario A – by raising R_w/R_p by a factor $(\frac{622}{5.1})^{1/3} \approx 5$ while leaving $\langle p\rangle$ unaffected. A detailed discussion lies outside the scope of the present work; all the same, our discussion highlights the beneficial role played by the peaked profiles of pressure, which provide us with a wide choice of available operational parameters for a reactor. As far as these profiles are invariant (Davis 2013), moreover, if larger values of p at the SOL near the wall are allowed then larger values of $\langle p\rangle$ can be obtained, thus facilitating the achievement of large values of Q.

7. Conclusions

After years of neglect, and in spite of considerable technological issues arising in a reactor-relevant environment, the excellent MHD stability properties and the promising

https://doi.org/10.1017/S0022377824001168 Published online by Cambridge University Press

scaling laws for energy confinement have aroused new, significant interest (Elio 2014; Berry *et al.* 2023) in the approach to controlled nuclear fusion based on plasmas confined by a dipole magnetic field (Lehnert 1958, 1968*a*,*b*; Hasegawa 1987; Teller *et al.* 1992).

The profiles of both particle density and pressure in a dipole-confined plasma can be well peaked, as a result of a spontaneous relaxation process (Hasegawa 1987; Kobayashi *et al.* 2010; Davis 2013; Yoshida *et al.* 2013). Relaxation leads to a spontaneous raise of density and pressure near the axis of the dipole, the so-called 'turbulent pinch' (Boxer *et al.* 2010), where the fusion rate is maximum. As a whole, moreover, the relaxed dipole-confined plasma enjoys excellent MHD stability, is inherently steady state, achieves high values of β, is intrinsically free from disruptions, satisfies promising scaling laws as for the particle confinement time, shows no degradation of the energy confinement time with increasing ECRH power sustaining the plasma and is confined by toroidal coils with very simple geometry (Kesner & Mauel 2013). Crucially, however, one of these coils is internal to the plasma itself. The resulting technological issues have been coped with to date through levitation of a superconducting coil with no contact with material leads and supports (Skellett 1975; Hasegawa 1987). Admittedly, this approach – with its lack of any umbilical connection to the dipole coil – may be considered fragile. In the years following the shutdown of the major experiment on dipole fusion – LDX (Kesner & Mauel 2013) – in 2011, the topic has been basically neglected almost completely. An experimental campaign with high-temperature superconductors (Saitoh *et al.* 2010) triggered the proposal of an experiment equipped with such superconductors and focussed on a reactor regime (Berry *et al.* 2023). An alternative (and older) approach is to allow contact between the coil and material leads, provided that the latter are magnetically screened (Lehnert 1968*a*,*b*). This is the rationale of the proposal of a new experiment with a dipole-confined plasma, POLOMAC (Elio 2014).

Admittedly, when assessing the prospects of these recent proposals (Elio 2014; Berry *et al.* 2023) from the point of view of net energy production both insufficient available data and incomplete knowledge of physical processes ruling energy transport in plasmas still make any independent extrapolation of present performances (Saitoh *et al.* 2010; Kesner & Mauel 2013) to a reactor scarcely reliable.

We have written down a necessary condition for the achievement of breakeven in dipole fusion. This necessary condition is distinct from and more stringent than the Lawson criterion; this is justified by the fact that, in dipole-confined plasmas, we are subject to the constraint of zero toroidal magnetic field, and we are therefore bound to give up the stabilizing impact of this field on the plasma. Our goal is more modest than a prediction; correspondingly, we need fewer assumptions, and rely on no detailed model of energy transport across the plasma.

We describe our dipole-confined plasma as an axisymmetric, non-rotating, steady-state plasma with no fast particles, isotropic pressure and zero toroidal magnetic field at breakeven. We make also the customary assumption (Dolan 1982) that the fusion heating power of the plasma is proportional to the square of the pressure. (The fact that we rule out fast particles prevents us from applying the present discussion to NBI-assisted FRCs; Binderbauer *et al.* 2015.) Both the Grad–Shafranov equation and the energy balance depend crucially on the *same* pressure profile, which must satisfy both of them simultaneously. When looking for such a profile, we may cast the two equations in the form of a constrained variational principle. This principle takes a particularly simple form when the toroidal magnetic field is zero. The Lagrange multiplier of the latter is proportional to a time constant. But the only time constant relevant to the system is the energy confinement time, which rules power losses in the energy balance and appears explicitly in the variational principle. We remove this apparent contradiction just by taking advantage of the fact that no rescaling of time may ever affect the physics of steady states. We choose therefore that rescaling which makes the pressure profile identified by the variational principle to provide us with the minimum value of the energy confinement time compatible with the assumptions. Remarkably, the rationale of this lower bound relies on no particular assumption concerning power losses – like e.g. the relevance of bremsstrahlung radiation in the proof of Lawson criterion. The existence of this minimum value acts therefore as a new, independent condition, due to the vanishing of the stabilizing toroidal field. Breakeven is therefore only possible if this minimum value is equal or smaller than the value required by Lawson criterion. If this is not true, then breakeven is impossible.

Together, the necessary condition on breakeven hinted at above and the relaxed profiles of a dipole-confined plasma link the ratio between the radius of the vacuum vessel and the radius of the main toroidal coil on one hand and the (suitably adimensionalized) ratio between the vertical field provided by external shaping coils and the current flowing across the toroidal coil internal to the plasma on the other hand. For breakeven to be possible, once one of these ratios is given the other cannot be too small. As for DT fusion, numerical values are not unreasonably far from those available in recent proposals (Elio 2014). A constraint is also obtained on ignition in catalysed DD fusion (Kesner *et al.* 2004).

Remarkably, our results rely on no assumption on energy transport; on the contrary, it turns out that the scaling law which fits our results in the most natural way is Bohm scaling. This fact makes sense in the turbulent pinch near the dipole axis at least, but strongly suggests that our estimates are likely to be too conservative. Despite this, we conclude therefore that a dipole-confined plasma contained in a vacuum vessel with given radius and with a given current flowing across the toroidal coil internal to the plasma may achieve breakeven in DT fusion provided that we suitably choose both the radius of the internal coil and the value of the external vertical field.

Of course, this conclusion does not mean that this plasma will actually achieve the breakeven: such prediction is impossible without a detailed model of energy transport across the plasma, all the more so because the crucial issue of the interaction between the thermonuclear plasma and the toroidal coil internal to this plasma remains unanswered to date. Our conclusion says only that the energy transport does not prevent the plasma from achieving breakeven provided that enough auxiliary heating of the plasma is available (e.g. via ECRH), that the internal coil withstands the proximity with the thermonuclear plasma, etc.

Generalization of this discussion to ignition is straightforward. Our results depend on the fusion gain factor only weakly: in other words, if a dipole is able to attain breakeven it is also likely to be able to attain ignition.

Admittedly, our discussion is rather qualitative. A rigorous discussion requires a detailed computation of MHD equilibria in a realistic geometry. All the same, it seems worthwhile to stress the point that the attractiveness of dipole confinement as a roadmap to steady-state nuclear fusion in a reactor with relatively simple layout stems out from a minimum set of physical assumptions, well grounded in both theory and experiments.

Acknowledgements

Motivation and encouragement of Dr F. Elio of DEUTELIO, Gavirate, Italy are gratefully acknowledged.

Editor Cary Forest thanks the referees for their advice in evaluating this article.

Declaration of interest

The authors report no conflict of interest.

Funding

This research received no specific grant from any funding agency, commercial or not-for-profit sectors.

Appendix

As suggested in Yoshida *et al.* (2013) – where a maximization of entropy is postulated with no further justification – the stability of the relaxed state allows thermodynamic interpretation. Experiments (Garnier *et al.* 2009; Davis 2013; Kesner & Mauel 2013) show that the levitation of the coil corresponds to a reduced fraction of non-thermal electrons (due e.g. to the interaction of the electrons with ECRH waves Davis 2013). In this case, (15) and (16) of Rogister & Li (1992) – a consequence of the Fokker–Planck equation – show that the typical time scales of thermalization (= relaxation to a Maxwellian distribution function) of ions and electrons in a weakly collisional, axisymmetric, toroidal, low-β turbulent plasma region between adjacent magnetic surfaces are $\ll$ the typical time scales of particle and energy transport in the direction parallel to $\nabla\psi$. This constraint holds at all times during the plasma lifetime even for weakly collisional plasmas and regardless of the detailed mechanism underlying transport; thus, it implies that we may safely assume that the distribution function of ions and electrons are locally Maxwellian, where by 'locally' we mean: 'as far as we are interested in phenomena occurring between adjacent magnetic surfaces on a time scale $\ll$ the typical time scales related to the transport of energy and particles'. If the distribution function is locally Maxwellian then the familiar thermodynamical relationships apply and we may identify the $pv^{\gamma} = \text{const.}$ condition equivalent to (2.1) with the condition of constant entropy inside the flux tube. In other words, if the profile of p is invariant then no net amount of entropy is ever produced inside the flux tube. Note that entropy is actually always produced in the bulk of a non-ideal plasma – e.g. by Joule dissipation of the electric currents induced during the interchange motion of flux tubes; it is just carried away by transport in steady state. In a stable steady state this entropy production is a minimum, regardless of the detailed mechanism of heating (e.g. fusion); as for Joule dissipation, this is a property clearly described by Kirchhoff (Hermann 1986). Its role in the self-organization of plasmas (Yoshida *et al.* 2013) is discussed in Di Vita (2022). In particular, in the framework of Hall MHD (which is likely to hold in the outer regions where the particle density is lower and the ion collisionless skin depth correspondingly larger) Saitoh *et al.* (2011) describes relaxed high-β dipole-confined plasmas as double Beltrami states; the latter states are also the outcome of the minimization of entropy production in the plasma bulk (Di Vita 2009).

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