III. Les triangles semblables PMQ et RBQ donnent

$$\frac{BR}{PM}=\frac{QR}{QP}=\frac{QB}{QM};$$

on, puisque BR = PA,

$$\frac{PA}{PM}=\frac{QR}{QP}.$$

Le dernier rapport est constant, puisque les points Q et R sont fixes. Le point P étant fixe, il résulte de cette égalité que le point M est sur une circonférence homothétique à la circonférence donnée par rapport au point P, qui est centre d' homothétie directe.

On a aussi $$\frac{QB}{QM}=\frac{QR}{QP}.$$

Cette égalité montre que le point Q est le centre d' homothétie inverse des deux circonférences.

Ces résultats se rapportent à la figure 45; ils seraient renversés pour le cas de la figure 46. Dans tous les cas P et Q sont les centres d' homothétie directe et inverse de la circonférence donnée O, et de la circonférence lieu du point M, que nous appellerons circonférence O′. Considérons par exemple la figure 51. Les points M et A′ sont anti-homologues. La tangente en A′ au cercle O et la tangente en M à la circonférence O′ font donc avec MA′ des angles égaux, et se coupent en I sur l' axe radical des deux cercles. Pour la même raison, les points M et B′ étant anti-homologues, la tangente en B′ au cercle O rencontre MI sur l' axe radical, c'est-à-dire au point I, et les angles IMB′ et IB′M sont égaux. Le point I est donc situé sur les perpendiculaires élevées au milieu de MA′ et au milieu de MB′; c' est donc le centre du cercle circonscrit au triangle MA′B′. Il résulte des constructions précédentes que ce point est toujours sur l' axe radical des circonférences O et O′. Donc le lieu demandé est cet axe radical.

The Equilateral and the Equiangular Polygon.

By R. E. Allardice, M.A.

The Equilateral Polygon.

Since an n-gon is determined by $2n-3$ conditions, and $n-1$ conditions are involved in its being equilateral, there are still in the case of an equilateral n-gon $n-2$ conditions to be determined. These $n-2$ conditions cannot all be given in terms of the angles, since a

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infinite number of n-gons may always be described similar to, out not necessarily congruent with, any given equilateral n-gon. Hence only $n-3$ of the angles of an equilateral n-gon may be assigned arbitrarily, and there must therefore be 3 independent relations connecting the angles of any equilateral n-gon. These three conditions may be obtained by projecting the perimeter of the n-gon on any three lines.

Let a be the common length of the sides; A, B, C......L, the exterior angles of the n-gon.

Projection on one of the sides adjacent to the angle A gives the relation

$$a\cos A + a\cos(A+B) + a\cos(A+B+C) + \ldots\ldots = 0;$$
$$\cos A + \cos(A+B) + \cos(A+B+C) + \ldots\ldots = 0. \qquad (1).$$

Similarly projection on the other sides gives

$$\cos B + \cos(B+C) + \cos(B+C+D) + \ldots\ldots = 0, \qquad (2),$$
$$\cos C + \cos(C+D) + \cos(C+D+E) + \ldots\ldots = 0; \text{ \&c.}; \qquad (3),$$

and projection on lines perpendicular to these gives

$$\sin A + \sin(A+B) + \sin(A+B+C) + \ldots\ldots = 0, \qquad (4),$$
$$\sin B + \sin(B+C) + \sin(B+C+D) + \ldots\ldots = 0. \qquad (5).$$

There thus arise $2n$ equations; but, as has been seen, only three of these are independent. This may be proved analytically in the following manner.

Assume equations (1), (2), and (4) above, and assume also

$$\sin B + \sin(B+C) + \sin(B+C+D) + \ldots\ldots = p. \qquad (6).$$

Let $\cos A + i\sin A = e^{iA} = \alpha$, $\cos B + i\sin B = e^{iB} = \beta$, &c.;

then $\cos(A+B) + i\sin(A+B) = e^{i(A+B)} = e^{iA}e^{iB} = \alpha\beta$.

Hence equations (1), (2), (4), and (6) are equivalent to

$$\alpha + \alpha\beta + \alpha\beta\gamma + \ldots\ldots + (\alpha\beta\gamma\ldots\ldots\lambda) = 0,$$
$$\beta + \beta\gamma + \beta\gamma\delta + \ldots\ldots\ldots(\beta\gamma\delta\ldots\ldots\alpha) = pi$$

$\therefore$ $$\alpha - (\alpha^2\beta\gamma\ldots\ldots\lambda) = -\alpha pi;$$

but $\alpha = \cos A + i\sin A$ is not equal to 0,

$\therefore$ $$1 - (\alpha\beta\gamma\ldots\ldots\lambda) = -pi,$$

$\therefore$ $$\cos(A+B+\ldots\ldots L) + i\sin(A+B+\ldots\ldots L) = 1 + pi,$$

$\therefore$ $$\cos(A+B+\ldots\ldots L) = 1, \ \sin(A+B+\ldots\ldots L) = p,$$

$\therefore$ $$A+B+C\ldots\ldots L = 2n\pi, \ p = 0, \text{ and } \alpha\beta\ldots\ldots\lambda = 1.$$

Hence $$\beta + \beta\gamma + \beta\gamma\delta + \ldots\ldots(\beta\gamma\delta\ldots\ldots\lambda\alpha) = 0$$

Hence also, since β is not equal to 0,

$$1 + \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha) = 0,$$

that is, $$(\alpha\beta\gamma\ldots\ldots\lambda) + \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha) = 0,$$
$$\gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha\beta) = 0;$$

and in the same way the truth of all the other relations may be established.

These equations may also be written in the form

$$1+\beta+\beta\gamma+\ldots\ldots(\beta\gamma\ldots\ldots\lambda)=0, \text{ \&c.}$$

In order now to prove that the equations do in general involve three independent conditions, it will be sufficient to show that in the case where n is 3, they are sufficient to determine the actual values of A, B and C.

The equations may be assumed in the form

$$\cos A+\cos(A+B)+\cos(A+B+C)=0,$$
$$\cos B+\cos(B+C)+\cos(B+C+A)=0,$$
$$A+B+C=2n\pi;$$

which lead at once to the equations

$$1+\cos A+\cos C=0,$$
$$1+\cos B+\cos A=0;$$

whence also

$$1+\cos C+\cos B=0.$$

These equations give $\cos A=\cos B=\cos C=-\frac{1}{2}$; the only solution of which lying between 0 and π is $A=B=C=2\pi/3$.

From the equations in the case of the quadrilateral may be deduced the equations $\cos A=\cos C$ and $\cos B=\cos D$. The only equilateral quadrilateral in the ordinary sense is of course the rhombus; but the above equations include the limiting cases of two straight lines which meet at a point taken twice over, and a single straight line taken four times over.

It should be noticed that in general it is not allowable to take the angles in an arbitrary order. This is exemplified in the case of the quadrilateral; but it may be shown that in general two angles cannot change places.

Suppose that an equilateral polygon may be formed with certain angles taken in the order ABCDEF......and also with the same angles taken in the order EBCDAF......

Then

$$\alpha+\alpha\beta+\beta\gamma+\alpha\beta\gamma\delta+\alpha\beta\gamma\delta\epsilon+\ldots\ldots=0,$$

and

$$\epsilon+\epsilon\beta+\epsilon\beta\gamma+\epsilon\beta\gamma\delta+\epsilon\beta\gamma\delta\alpha+\ldots\ldots=0$$
$$(\alpha-\epsilon)(1+\beta+\beta\gamma+\beta\gamma\delta)=0.$$

Hence the angles cannot change places unless they are equal, or the intervening angles satisfy the relation $1+\beta+\beta\gamma+\beta\gamma\delta=0$. This condition involves that the sides between the two vertices considered, themselves form a closed polygon; for it implies that the sum of the projections of these sides on each of two straight lines at right angles

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to one another is zero. This equation which is equivalent to two conditions, since it involves the imaginary unit, only contains $(n-1)$ of the angles in the general case, and is therefore not in itself sufficient to determine that the n angles may be the angles of an equilateral polygon.

It may be noted further that three conditions of the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$

are not sufficient; for they may be satisfied by angles whose sum is not a multiple of 2π. For let n equal lines P_1P_2, P_2P_3......P_nP_{n+1}, be drawn making angles A, B, C......K with one another (the first $n-1$ of the angles); then the first condition will be satisfied if P_{n+1} lies in the perpendicular to P_1P_2 through the point P_1. Let now another straight line $P_{n+1}P_{n+2}$ be drawn making with P_nP_{n+1} an angle L; then the second condition will be satisfied if the point P_{n+2} lies in the perpendicular to P_2P_3 through the point P_2; and so on for a third, fourth, etc. condition, the next line making with that last considered an angle A, the next again an angle B, and so on. Thus in the case where n is 3, the conditions are satisfied by the angles of a square, that is, by the values $A=B=C=\pi/2$; in the case where n is 4, the four conditions of the form considered are satisfied by the angles of a regular hexagon, that is, by the values, $A=B=C=D=\pi/3$; and, in the general case, the n conditions are satisfied by the angles of a regular polygon of $2(n-1)$ sides, that is, by the values $A=B=C=\ldots\ldots=\pi/(n-1)$.

The three conditions may be given in the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$
$$1+\cos B+\cos(B+C)+\ldots\ldots=0$$
$$A+B+C+\ldots\ldots=2\pi.$$

Assume
$$1+\sin A+\sin(A+B)+\ldots\ldots=p$$
$$1+\sin B+\sin(B+C)+\ldots\ldots=q.$$

Then
$$1+\alpha+\alpha\beta+\ldots\ldots=pi$$
$$1+\beta+\beta\gamma+\ldots\ldots=qi$$
$$\alpha\beta\gamma\ldots\ldots=1.$$

$$\therefore \quad pi-qi\alpha=0;$$
$$\therefore \quad pi-qi(\cos A+i\sin A)=0;$$
$$\therefore \quad q\sin A+(p-q\cos A)i=0.$$

$\therefore$ if sin A is not equal to 0, $q=0$ and $p=0$.

If the equations are assumed in the form

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$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$
$$1+\sin A+\sin(A+B)+\ldots\ldots=0$$
$$A+B+C+\ldots\ldots=2\pi,$$

or in the equivalent form

$$1+\alpha+\alpha\beta+\ldots\ldots=0$$
$$\alpha\beta\gamma\ldots\ldots=1,$$

the truth of all the other equations is at once obvious.

From the above it is seen that when the conditions are given in any of the forms discussed here, either two of the three conditions must be derived from the projection of the sides of the polygon on two different straight lines, or one of them must be that the sum of the exterior angles is $2m\pi$, where m is some integer.

If n angles be found which may be made the angles of an equilateral polygon, then these angles may be combined in various ways so as to form the angles of other equilateral polygons. Suppose, for example, that n is 5; then the conditions connecting the angles may be written

$$1+\alpha+\alpha\beta+\alpha\beta\gamma+\alpha\beta\gamma\delta=0$$
$$\alpha\beta\gamma\delta\epsilon=1;$$

which may be written in the form

$$1+(\alpha\beta)+(\alpha\beta)(\gamma\delta)+(\alpha\beta)(\gamma\delta)(\epsilon\alpha)+(\alpha\beta)(\gamma\delta)(\epsilon\alpha)(\beta\gamma)=0$$
$$(\alpha\beta)(\gamma\delta)(\epsilon\alpha)(\beta\gamma)(\delta\epsilon)=1;$$

and these equations show that an equilateral polygon may be made of which the exterior angles are, A + B, C + D, E + A, B + C, D + E. In the same way, if n is not divisible by 3, the angles may be added together in sets of three to form the angles of a new equilateral polygon. And it may easily be seen that the angles may be combined in various ways. For let a radius vector rotate from the position OX through an angle A into the position OA, then through an angle B into the position OB, and so on. Now if the radius vector be conceived to start from OX and to rotate into the position of any of the lines just considered, OP say, then form OP into another position, OQ say, and so on, the different angles through which it rotates may be made the angles of an equilateral polygon, it being supposed that the radius vector always rotates in the same direction and never stops in the same position twice. This may be proved in the same way as the particular cases given above; but it is seen even more easily as a consequence of the laws of the composition of vectors.

The Equiangular Polygon.

In the fact that an n-gon is equiangular $(n-1)$ conditions are involved; and since the other $(n-2)$ conditions may all be given in terms of the sides, it follows that there must be two relations connecting the sides of an equiangular polygon.

The exterior angle of a regular n-gon is $2\pi/n$ or $2m\pi/n = A$, say, where A may be assumed not to be a multiple of π; and hence α is not real and cannot vanish, where

$$\alpha = e^{Ai} = \cos A + i\sin A.$$

Let the sides be denoted by $a, b, c \ldots\ldots k$; and let the perimeter be projected on the side a and on a line perpendicular to a.

Then
$$a + b\cos A + c\cos 2A + \ldots\ldots + k\cos(n-1)A = 0;$$
$$b\sin A + c\sin 2A + \ldots\ldots + k\sin(n-1)A = 0.$$

$$\therefore \quad a + b\alpha + c\alpha^2 + \ldots\ldots + k\alpha^{n-1} = 0; \qquad (1)$$
$$\alpha^n = 1.$$

Multiply equation (1) by α^{n-1}; then

$$b + c\alpha + d\alpha^2 + \ldots\ldots a\alpha^{n-1} = 0;$$

and in the same way all the other similar equations may be deduced.

It may easily be shown that the two equations may be assumed in the form

$$a + b\cos A + c\cos 2A + \ldots\ldots = 0;$$
$$b + c\cos A + d\cos 2A + \ldots\ldots = 0.$$

The two conditions may be expressed in terms of the sides alone as follows:—

If the sides are represented by $a_1, a_2, a_3 \ldots\ldots a_n$, the conditions are

$$a_1 + a_2\alpha + a_3\alpha^2 + \ldots\ldots + a_n\alpha^{n-1} = 0$$
$$a_2 + a_3\alpha + a_4\alpha^2 + \ldots\ldots + a_1\alpha^{n-1} = 0$$
$$\ldots\ldots\ldots\ldots$$
$$a_n + a_1\alpha + a_2\alpha^2 + \ldots\ldots + a_{n-1}\alpha^{n-1} = 0.$$

Hence

$$\alpha = -\begin{vmatrix} a_1 & a_3 & \ldots\ldots & a_n \\ a_2 & a_4 & \ldots\ldots & a_1 \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & a_1 & \ldots\ldots & a_{n-2} \end{vmatrix} \div \begin{vmatrix} a_2 & a_3 & \ldots\ldots & a_n \\ a_3 & a_4 & \ldots\ldots & a_1 \\ \ldots & \ldots & \ldots & \ldots \\ a_n & a_1 & \ldots\ldots & a_{n-2} \end{vmatrix}.$$

But α is not real, while $a_1, a_2 \ldots\ldots a_n$, are all real; and therefore both numerator and denominator in the value of α must vanish. It is obvious that all the other first minors of the circulant determinant $(a_1 a_2 \ldots\ldots a_n)$ must also vanish. The conditions may be expressed in the form—

$$\begin{Vmatrix} a_2 & a_3 & \ldots\ldots & a_n & a_1 \\ a_3 & a_4 & \ldots\ldots & a_1 & a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_1 & \ldots\ldots & a_{n-2} & a_{n-1} \end{Vmatrix} = 0.$$

Conversely, if the condition

$$a_1 + a_2\alpha + a_3\alpha^2 + \ldots\ldots a_n\alpha^{n-1} = 0$$

be given, where $\alpha^n = 1$, it may be shown that an equiangular polygon can be formed with the sides a_1, $a_2 \ldots\ldots a_n$. For if a polygon be formed with these sides and with $(n-1)$ exterior angles, each equal to $2\pi/n$, the above condition shows that the polygon will be a closed one, and the condition $\alpha^n = 1$ that the last exterior angle will also be $2\pi/n$.

In the reasoning of last paragraph α, instead of being $\cos 2\pi/n + i\sin 2\pi/n$, may have any one of the values $\cos 2k\pi/n + i\sin 2k\pi/n$, where k has any integral value from 0 to $(n-1)$. It would thus appear that there are n distinct species of equiangular n-gons, distinguished by the magnitude of the exterior angle; but some of these are degenerate cases and others are not distinct.

Thus if $$\alpha = \cos 0 + i\sin 0$$
then $$a_1 + a_2 + \ldots\ldots + a_n = 0\,;$$
and since none of the sides are negative each must be zero.

Again if n is even and

$$\alpha = \cos\pi + i\sin\pi,$$

the condition becomes

$$a_1 - a_2 + \ldots\ldots + a_{n-1} - a_n = 0\,;$$

and the polygon is a flat one, such as ABCDEFGHA (fig. 52). Further the two values

$$\alpha = \cos 2r\pi/n + i\sin 2r\pi/n,$$
$$\alpha = \cos 2(n-r)\pi/n + i\sin 2(n-r)\pi/n,$$

give the same polygon, the angles regarded as the exterior angles in the two cases being the conjugates of one another.

All the other polygons are distinct; and hence it follows that if n is odd there are $\frac{1}{2}(n-1)$, and if n is even $\frac{1}{2}(n-2)$, equiangular n-gons of different species; all of which, of course, with the exception of one, are crossed polygons.

A particular case of the above is that in which the polygons are regular. In this case, however, if n is not prime, some of the polygons consist of those with a smaller number of sides taken several times over. Thus one of the regular octagons consists of a square taken twice over. These may be called degenerate cases; but there are always certain regular n-gons which are not degenerate cases; one such being the ordinary regular non-crossed n-gon. The number of non-degenerate regular polygons is easily seen to be half the number of special roots of the equation $x^n - 1 = 0$. Now if $p, q, r \ldots\ldots$ are primes, and if $n = p^\alpha q^\beta r^\gamma \ldots\ldots$, the number of special roots of this equation is

$$n(1 - 1/p)(1 - 1/q)\ldots\ldots$$

Hence the number of regular n-gons is

$$\tfrac{1}{2}n(1 - 1/p)(1 - 1/q)\ldots\ldots$$

where $p, q, r \ldots\ldots$ are the prime factors of n, including n if n be prime.

The number of regular n-gons may also be seen very easily by consideration of a circle divided into n equal parts. Each point may be joined to the next, or to the next but one, or to the next but two, and so on; and a regular n-gon will be formed in each case provided every point of division is included.

If an equiangular n-gon can be formed with the sides, a_1, $a_2 \ldots\ldots a_n$ taken in a definite order, then an equiangular polygon of any other species may be formed by taking the same sides in a different order, namely in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}} \ldots$ provided neither of the polygons be of a species which degenerates when it becomes regular. For if an equiangular polygon of the kind considered can be formed with the sides a_1, $a_2 \ldots\ldots a_n$, taken in that order, then

$$a_1 + a_2\alpha + \ldots\ldots + a_n\alpha^{n-1} = 0,$$

where α is a special root of the equation $x^n - 1 = 0$. If β is any other special root of this equation, then

$$\beta = \alpha^r,\ \beta^2 = \alpha^{2r},\ \&c.,$$

$$\therefore \qquad a_1 + a_{r+1}\beta + a_{2r+1}\beta^2 + \ldots\ldots = 0;$$

which shows that an equiangular polygon may be formed with the same sides taken in the order, a_1, a_{r+1}, a_{2r+1} ... Figs. 53, 54, 55, represent the three species of equiangular nonagons which can be formed by taking the same lines in different orders. The first is the ordinary non-crossed nonagon the exterior angle of which is $2\pi/9$, the second is formed from the first by taking every second side, and has $4\pi/9$ for its exterior angle; and the third is formed from the first by taking every fourth side, and has $8\pi/9$ for its exterior angle. It is in general impossible to form an equiangular nonagon of the remaining (third) species with the same sides, as this species degenerates when it becomes regular, becoming in that case an equilateral triangle taken thrice over. The same thing is indicated by the fact that it is impossible to get all the sides of a nonagon by going round it and taking every third side.

The propositions of last paragraph may also be proved by means of the laws of composition of vectors. For if vectors be drawn through any point parallel to the sides of an equiangular polygon, these vectors will make equal angles with one another, and if they be compounded in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$...... they will again form an equiangular polygon, provided all the lines are included, when they are compounded in this way, that is, provided n be not a multiple of r.

It may now be shown that conversely an equiangular polygon may be formed with the sides a_1, a_2......a_n, if the matrix of the circulant $C(a_1a_2......a_n)$ vanishes.

It is sufficient to show that if this matrix vanishes, then

$$a_1 + a_2\alpha + ... + a_n\alpha^{n-1} = 0, \text{ where } \alpha^n = 1.$$

Now if the above matrix vanishes, then the circulant $C(a_1a_2......a_n)$ or this divided by $(a_1 + a_2 + ... + a_n)$ also vanishes.

But $$C(a_1a_2......a_n) = \Pi(a_1 + a_2\alpha_r + ... + a_n\alpha_r^{n-1}),$$

where α_r is a root of $x^n - 1 = 0$, and r has every value from 1 to n.

Hence $$a_1 + a_2\alpha + ... + a_n\alpha^{n-1} = 0,$$

where α is some root of $x^n - 1 = 0$; and therefore an equilateral triangle may be formed with the sides taken in the above order, the exterior angle being one of the values of $2k\pi/n$.

It is obvious that the sides may not be taken in any arbitrary order and that in general only one of the angles $2k\pi/n$ may be taken. As a matter of fact, if the above-mentioned matrix vanishes, at least two of the factors $(a_1 + a_2\alpha_r + ... + a_n\alpha_r^{n-1})$ must vanish; namely those in which the quantities α_r are conjugate imaginaries. A possible case when n is even, included in the above, is that of the flat polygon mentioned before. This case arises when α is equal to π and the factor which vanishes is then $(a_1 - a_2 + a_3 - ......)$.

The condition that a_1, a_2......a_n be the sides of an equiangular polygon may be represented as follows:—

$$\text{Mat.}C(a_1a_2......a_n) = 0.$$

Now, if the sides taken in the order a_1, a_{r+1}, a_{2r+1}......, form an equiangular polygon, which they will do if r is not a factor of n, then

$$\text{Mat.}C(a_1a_{r+1}a_{2r+1}......) = 0.$$

Hence if one of these matrices vanishes so must the other. In fact it may easily be proved by means of the identity

$$\mathrm{C}(a_1 a_2 \ldots\ldots a_n) = \Pi(a_1 + a_2 a + a_3 a^2 \ldots\ldots)$$

that the two circulants are equal; that is the circulant $\mathrm{C}(a_1 a_2 \ldots\ldots a_n)$ is not altered if the letters be written in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$ &c., if r be not a factor of n.

It follows from the above that if

$$\mathrm{C}(a_1 a_2 \ldots\ldots a_n) = 0, \text{ and } a_1 + a_2 + \ldots\ldots + a_n \text{ is not equal to } 0,$$

then $$\mathrm{Mat.C}(a_1 a_2 \ldots\ldots a_n) = 0.$$

For in this case, for some value of a

$$a_1 + a_2 a + \ldots\ldots + a_n a^{n-1} = 0;$$

and therefore, as has been seen before, $\mathrm{Mat.C}(a_1 a_2 \ldots\ldots a_n) = 0$. It is assumed that if n is even $a_1 - a_2 + a_3 - \ldots\ldots$ is not equal to 0; and that $a_1, a_2 \ldots\ldots a_n$ are all real.

Again, if three consecutive sides be increased by x, $-2x\cos\mathrm{A}$, x, the polygon still remains equiangular, A being the exterior angle of the polygon. Hence if the first matrix of a circulant vanish and if three consecutive letters p, q, r, be changed to $p+x$, $p-2x\cos\mathrm{A}$, $r+x$, where x is arbitrary and A is a certain one of the angles $2k\pi/n$, the circulant will still vanish.

This may also be proved analytically; for let the factor of $\mathrm{C}(a_1 a_2 \ldots\ldots a_n)$ which vanishes be

$$a_1 + a_2 a + \ldots\ldots a_n a^{n-1}$$

where $$a = \cos 2p\pi/n + i\sin 2p\pi/n.$$

Then $$a_{r+1} a^r + a_{r+2} a^{r+1} + a_{r+3} a^{r+2}$$

becomes $$a_{r+1} a^r + a_{r+2} a^{r+1} + a_{r+3} a^{r+2} + a^r(x - 2xa\cos 2p\pi/n + xa^2);$$

and $$x - 2xa\cos 2p\pi/n + xa^2$$

$$= x\{1 - 2(\cos 2p\pi/n + i\sin 2p\pi/n)\cos 2p\pi/n + \cos 4p\pi/n + i\sin 4p\pi/n\}$$
$$= x\{(1 - 2\cos^2 2p\pi/n + \cos 4p\pi/n) + i(\sin 4p\pi/n - \sin 4p\pi/n)\}$$
$$= 0.$$

Conversely this transformation will indicate which of the factors of $\mathrm{C}(a_1 a_2 \ldots\ldots a_n)$ vanishes when the circulant itself vanishes.

Particular cases. In the simpler cases the general conditions given above reduce to the following:—

Triangle $$a_1 = a_2 = a_3.$$

Quadrilateral $$a_1 = a_3;\ a_2 = a_4.$$

Pentagon. $$\{4a - (b + c + d + e) + \sqrt{5}(b - c - d + e)\} = 0;$$
$$\{(b - e)^2 + (d - c)^2\} + \sqrt{5}\{(b - e)^2 - (d - c)^2\} = 0.$$

By means of these equations it may easily be seen that an equiangular pentagon can only have its sides commensurable if it be regular.

Hexagon. $$a_1 + a_2 = a_4 + a_5;$$
$$a_2 + a_3 = a_5 + a_6.$$

Octagon, $$(a_2 - a_4 - a_6 + a_8) + \sqrt{2}(a_1 - a_5) = 0;$$
$$(a_2 + a_4 - a_6 - a_8) + \sqrt{2}(a_3 - a_7) = 0.$$

In an equiangular octagon with commensurable sides, opposite sides are equal.

Decagon, $$4(a_1 - a_6) + (a_2 - a_3 + a_4 - a_5 - a_7 + a_8 - a_9 + a_{10})$$
$$+ \sqrt{5}(a_2 + a_3 - a_4 - a_5 - a_7 - a_8 + a_9 + a_{10}) = 0,$$
$$(a_2 + a_5 - a_7 - a_{10})^2 + (a_3 + a_4 - a_8 - a_9)^2$$
$$+ \sqrt{5}\{(a_3 + a_4 - a_8 - a_9)^2 - (a_2 + a_5 - a_7 - a_{10})^2\} = 0.$$

In an equiangular decagon with commensurable sides,
$$a_1 - a_6 = a_7 - a_2 = a_3 - a_8 = a_9 - a_4 = a_5 - a_{10};$$
where a_1 and a_6, a_2 and a_7, &c., are opposite sides.

Dodecagon. $$2(a_1 - a_7) + (a_3 - a_5 - a_9 + a_{11}) + \sqrt{3}(a_2 - a_6 - a_8 + a_{12}) = 0;$$
$$2(a_4 - a_{10}) + (a_2 + a_6 - a_8 - a_{12}) + \sqrt{3}(a_3 + a_5 - a_9 - a_{11}) = 0.$$

In an equiangular dodecagon with commensurable sides,
$$a_2 - a_8 = a_{10} - a_4 = a_6 - a_{12};\ a_1 - a_7 = a_9 - a_3 = a_5 - a_{11};$$
where a_2 and a_8, a_4 and a_{10}, &c., are opposite sides.

Third Meeting, January 14th, 1887.

W. J. MACDONALD, Esq., M.A., Vice-President, in the Chair.

On Certain Inverse Roulette Problems.

By PROFESSOR CHRYSTAL.

The problem of designing cams or centrodes to produce any given motion in one plane is one of some practical importance; and it seems worth while to illustrate by examples some simple methods by which the solution can in certain cases be arrived at. These methods are founded, for the most part, on the use of the so-called Pedal Equation (or *p-r*-equation), which has great advantages in the present investigation, inasmuch as it depends on the form but not on the position of the curve which it represents.