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# EXTENSIONS OF JENSEN'S INEQUALITY ON THE UNIT CIRCLE

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The purpose of this note is to give extensions of an inequality of Jensen for analytic functions in an unit disc. We investigate functions which satisfy equalities in the inequalities.

Suppose $f \in H^1$ is a nonzero function such that $f(z) = \sum_{j=0}^{\infty} f(j)z^j$ $(|z| < 1)$, where $H^1$ denotes the Hardy space. Then

$$|f(0)| \leq \exp \int_0^{2\pi} \log|f(e^{i\theta})|d\theta/2\pi.$$

Equality holds in the inequality above if and only if $f$ is an outer function, that is,

$$f(z) = \exp\{ \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})|dt/2\pi + i\alpha \} \quad (|z| < 1)$$

for some real $\alpha$. Jensen's inequaltiy may be shown as a consequence of Szegö's theorem by the Helson-Lowdenslager approach (see [1,p51]). We

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shall give extensions of Jensen's inequality using a technique used to prove a theorem in the paper [2] of the author and K. Takahashi.

If $f$ is analytic in $|z| < 1$ and $\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ is bounded for $0 \leq r < 1$, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on the unit circle. If $\lim_{r\to 1}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta$, then $f$ is said to be of class $N^+$. The set of all boundary functions in $N^+$ is denoted by $N^+$ again. For $0 < p \leq \infty$, the Hardy space $H^p$, is defined by $N^+ \cap L^p$ where $L^p = L^p(d\theta/2\pi)$. We call $q$ in $N^+$ an inner function if $|q(e^{i\theta})| = 1$ almost everywhere. If $\log|f| \sim \sum_{j=-\infty}^{\infty} a_j e^{ij\theta}$ for nonzero $f$ in $N^+$, set

$$A_n(|f|) = \Sigma' \, z^{m_1+\dots+m_n} \frac{a_1^{m_1}\cdots a_n^{m_n}}{m_1!\cdots m_n!}$$

where $\Sigma'$ is the summation over all permutations of nonnegative integers $m_1, m_2, \dots, m_j$ with $m_1 + 2m_2 + \dots + jm_j = j$. Then

$$A_0(|f|) = 1,\ A_1(|f|) = 2a_1 ,$$

$$A_2(|f|) = \frac{a_1^2}{2} + a_2 .$$

THEOREM 1. *If $f$ is a nonzero function in $N^+$ and $\log|f| \sim \sum_{j=-\infty}^{\infty} a_j e^{ij\theta}$, then*

$$\sum_{j=0}^{n} |\hat{f}(j)|^2 \leq \exp \int \log|f|^2 d\theta/2\pi \times \{1 + \sum_{j=1}^{n} |A_j(|f|)|^2\} .$$

*If $f$ is an outer function then equality holds in the inequality above.*

Proof. Suppose $f$ is an outer function in $H^2$. By [1, p.61] $h(z) = \exp(a_0 + 2\sum_{j=1}^{\infty} a_j z^j)$ $(|z| < 1)$. We decompose $h$ into $h_1 h_2$ where

$h_1(z) = \exp(a_0 + 2 \sum_{j=1}^{n} a_j z^j)$ and $h_2(z) = \exp(2 \sum_{j=n+1}^{\infty} a_j z^j)$ $(|z| < 1)$.

Then $\sum_{j=0}^{n} |\hat{h}(j)|^2 = \operatorname{dist}(h, e^{i(n+1)\theta} H^2) = \operatorname{dist}(h_1, e^{i(n+1)\theta} H^2)$ since $h = h_1 + e^{i(n+1)\theta} h_1 k$ for some $k \in H^2$. Also we have

$$h_1(e^{i\theta}) = e^{a_0} \prod_{j=1}^{n} \exp(2a_j e^{ij\theta})$$

$$= e^{a_0} \frac{(2a_1 e^{i\theta})^{m_1} \cdots (2a_n e^{in\theta})^{m_n}}{m_1! \cdots m_n!}$$

where $m_j (1 \le j \le n)$ ranges independently over nonnegative integers.

Suppose $f$ is an outer function in $N^+$. Set $f_r(z) = f(rz)$ $(|z| < 1)$ for $0 \le r \le 1$, then $\hat{f}_r(j) = r^j \hat{f}(j)$, $f_r \in H^2$ and $f_r$ is an outer function. By what was just proved above, $\sum_{j=0}^{n} r^{2j} |\hat{f}(j)|^2$ $= e^{2a(r)_0}\{1 + \sum_{j=1}^{n} |A_j(|f_r|)|^2\}$ where $\log|f_r| \sim \sum_{j=-\infty}^{\infty} a(r)_j e^{ij\theta}$. $f \in N^+$ implies $a(r)_j \to a_j$ as $r \to 1$. This implies $\sum_{j=0}^{n} |\hat{f}(j)|^2$ $= e^{2a_0} \{1 + \sum_{j=1}^{n} |A_j(|f|)|^2\}$.

In general, if $f \in N^+$ and $h$ is the outer part then $\sum_{j=0}^{n} |\hat{f}(j)|^2 \le \sum_{j=0}^{n} |\hat{h}(j)|^2$ (see,[3 , p.305]). This implies the theorem.

COROLLARY 1. *If $f$ is an outer function in $N^+$ then*

$$|\hat{f}(n)| = \exp \int \log|f| d\theta/2\pi \times |A_n(|f|)|$$

*for any* $n \ge 0$.

$|\hat{f}(0)| = \exp \int \log|f| d\theta/2\pi$ if and only if $f$ is an outer function.

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The following theorem determines the functions for which equality holds in Theorem 1.

THEOREM 2. *Suppose* $f = qh$ *is a nonzero function in* $H^2$ *where* $q$ *is an inner function and* $h$ *is an outer function.*

$$(1) \qquad \sum_{j=0}^{n} |\hat{f}(j)|^2 = \exp \int \log|f|^2 d\theta/2\pi \times \{1 + \sum_{j=1}^{n} |A_j(|f|)|^2\}$$

*if and only if* $\sum_{j=0}^{n} \hat{f}(j)e^{ij\theta} = q \sum_{j=0}^{n} \hat{h}(j)e^{ij\theta}$.

$$(2) \qquad \sum_{j=0}^{n} |f(j)|^2 = \exp \int \log|f|^2 d\theta/2\pi \times \{1 + \sum_{j=1}^{n} |A_j(|f|)|^2\}$$

*for any* $n \geq n_0 \geq 0$ *if and only if* $f$ *is an outer function or* $f$ *is an analytic polynomial of degree* $n_0$.

Proof. $\sum_{j=0}^{n} |\hat{f}(j)|^2 = \exp \int \log|f|^2 d\theta/2\pi \times \{1 + \sum_{j=1}^{n} |A_j(|f|)|^2\}$ if and only if $\sum_{j=0}^{n} |\hat{f}(j)|^2 = \sum_{j=0}^{n} |\hat{h}(j)|^2$ if and only if

$\mathrm{dist}(f, e^{i(n+1)\theta}H^2) = \mathrm{dist}(f, e^{i(n+1)\theta}qH^2)$. Relative to the decomposition $H^2 = (H^2 \ominus e^{i(n+1)\theta}H^2) \oplus e^{i(n+1)\theta}(H^2 \ominus qH^2) \oplus e^{i(n+1)\theta}qH^2$, $f = f_1 \oplus f_2 \oplus f_3$, where '$\ominus$' is the orthogonal complement and '$\oplus$' is the orthogonal direct sum. Hence $\mathrm{dist}(f, e^{i(n+1)\theta}H^2) = \mathrm{dist}(f, e^{i(n+1)\theta}qH^2)$ if and only if $\int |f - (f_2 + f_3)|^2 d\theta/2\pi = \int |f - f_3|^2 d\theta/2\pi$ if and only if $f_2 = 0$. Moreover $f_2 = 0$ if and only if $f - \sum_{j=0}^{n} \hat{f}(j)e^{ij} \in e^{i(n+1)\theta}qH^2$ because $f_1 = \sum_{j=0}^{n} \hat{f}(j)e^{ij\theta}$.

(1) If $f - \sum_{j=0}^{n} \hat{f}(j)e^{ij\theta} \in e^{i(n+1)\theta}qH^2$ then $\sum_{j \geq n+1} \hat{f}(j)e^{ij\theta} = e^{i(n+1)\theta}qh_2$ and $\sum_{j=0}^{n} \hat{f}(j)e^{ij} = qh_1$ for some $h_1, h_2 \in H^2$. It is easy to see that

$h_1$ is an analytic polynomial of degree less than $n$. Hence
$\sum_{j=0}^{n} \hat{h}(j)e^{ij\theta} = h_1$ because $f = qh = q(h_1 + e^{in\theta}h_2)$. This implies the 'only if' part.

Conversely if $\sum_{j=0}^{n} \hat{f}(j)e^{ij\theta} = q \sum_{j=0}^{n} \hat{h}(j)e^{ij\theta}$ then

$$f - \sum_{j=0}^{n} \hat{f}(j)e^{ij\theta} = q(h - \sum_{j=0}^{n} \hat{h}(j)e^{ij\theta}) \in e^{i(n+1)\theta}qH^2 .$$

Hence $f_2 = 0$ and this implies the 'if' part.

(2) If $f - \sum_{j=0}^{n} \hat{f}(j)e^{ij\theta}$ and $f - \sum_{j=0}^{n+1} \hat{f}(j)e^{ij\theta}$ belong to $qH^2$ then $\hat{f}(n+1)e^{i(n+1)\theta} \in qH^2$. Hence $\hat{f}(n+1) = 0$ or $q$ is a constant. This implies the 'only if' part. The 'if' part is clear.

COROLLARY 2. Suppose $f = qh$ is a nonzero function in $H^2$ where $q$ is an inner function and $h$ is an outer function.

(1) $|\hat{f}(0)|^2 + |\hat{f}(1)|^2 \le \exp \int \log|f|^2 d\theta/2\pi \times \{1 + |\int e^{-i\theta} \times \log|f|^2 d\theta/2\pi|^2\}$ and equality holds if and only if $\hat{f}(0) + \hat{f}(1)e^{i\theta} = q(\hat{h}(0) + \hat{h}(1)e^{i\theta})$,

(2) $|\hat{h}(0)| = \exp \int \log|h| d\theta/2\pi$ and $|\hat{h}(1)| = \exp \int \log|h| d\theta/2\pi \times |\int e^{i} \log|h| d\theta/2\pi|$,

(3) If $\hat{f}(0) + \hat{f}(1)\, e^{i\theta} = q(\hat{h}(0) + \hat{h}(1)\, e^{i\theta})$ then $\log|\hat{f}(0) + \hat{f}(1)e^{i\theta}| d\theta/2\pi = \int \log|f| d\theta/2\pi + \log^+ |\int e^{-i\theta} \log|f| d\theta/2\pi|$.

Proof. (1) follows from Theorems 1 and 2 and (2) follows from Corollary 1.

(3) If $\hat{f}(0) + \hat{f}(1)e^{i\theta} = q(\hat{h}(0) + \hat{h}(1)e^{i\theta})$ then we may assume that $f = h$ since $|\hat{f}(0) + \hat{f}(1)e^{i\theta}| = |\hat{h}(0) + \hat{h}(1)e^{i\theta}|$.

$$\int \log|\hat{h}(0) + \hat{h}(1)e^{i\theta}| d\theta/2\pi = \log \max(|\hat{h}(0)|, \hat{h}(1)|)$$

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and hence if $|\hat{h}(0)| \geq |\hat{h}(1)|$ then $\log|\hat{h}(0)| \geq \log|\hat{h}(1)| = a_0 + \log|a_1|$ and $\log|\hat{h}(0)| = a_0$ by (2), where $\int \log|h|d\theta/2\pi \sim \sum_{j=0}^{\infty} a_j e^{ij\theta}$. So $\log|a_1| \leq 0$ and $\log|\hat{h}(0)| = a_0 + \log^+|a_1|$. If $|\hat{h}(0)| \leq |\hat{h}(1)|$ then we can show $\log|\hat{h}(1)| = a_0 + \log^+|a_1|$ similarly. Hence

$$\int \log|\hat{h}(0) + \hat{h}(1)e^{i\theta}|d\theta/2\pi = \int \log|h|d\theta/2\pi + \log^+ 2\left|\int e^{-i\theta} \log|h|d\theta/2\pi\right| .$$

If $f = qh$ is a nonzero function in $H^2$ where $q$ is an inner function and $h$ is an outer function, then $\sum_{j=0}^{n} |\hat{f}(j)|^2 \leq \sum_{j=0}^{n} |\hat{h}(j)|^2$ for any $n$ (see [3, p.305]). Theorem 2 determines the functions for which equality holds in the above inequality, using Theorem 1. The $q$ in (1) of Theorem 2 is clearly a finite Blashke product.

## References

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