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A NOTE ON UMD SPACES AND TRANSFERENCE IN VECTOR-VALUED FUNCTION SPACES*

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A Banach space X is called an HT space if the Hilbert transform is bounded from $L^p(X)$ into $L^p(X)$, where $1 < p < \infty$. We introduce the notion of an ACF Banach space, that is, a Banach space X for which we have an abstract M. Riesz Theorem for conjugate functions in $L^p(X)$, $1 < p < \infty$. Berkson, Gillespie and Muhly [5] showed that $X \in \mathrm{HT} \Rightarrow X \in \mathrm{ACF}$. In this note, we will show that $X \in \mathrm{ACF} \Rightarrow X \in \mathrm{UMD}$, thus providing a new proof of Bourgain's result $X \in \mathrm{HT} \Rightarrow X \in \mathrm{UMD}$.

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1. Introduction

Recent structure theorems of orders and results in abstract harmonic analysis on groups with ordered dual groups have shown that abstract harmonic analysis on such groups captures martingale theory and classical harmonic analysis (see [4]). Our goal in this note is to illustrate this further by obtaining Bourgain's result [6] as a consequence of a generalized version of M. Riesz's theorem due to Berkson, Gillespie and Muhly [5].

We use the usual notation $L^p(\Omega, \mu, X) = L^p(X)$ to denote the set of all strongly measurable functions on the measure space $(\Omega, \mathcal{F}, \mu)$ with values in the Banach space X such that $\int_\Omega \|f(\omega)\|^p d\mu(\omega) < \infty$. Recall that X is a UMD space if, for some (equivalently, all) $p \in (1, \infty)$, there exists a constant $C_p(X)$ such that for all $n \in \mathbb{N}$,

$$\|\sum_{j=1}^{n} \epsilon_j d_j\|_{L^p(X)} \leq C_p(X)\|\sum_{j=1}^{n} d_j\|_{L^p(X)} \tag{1.1}$$

for every X-valued martingale difference sequence (d_j) and for every $(\epsilon_j) \in \{-1, 1\}^{\mathbb{N}}$. (For further background regarding this property, see [7], [8] and [9].)

We say that X is an HT space if for some $p \in (1, \infty)$, there exists a constant $N_p(X)$ such that for all $n \in \mathbb{N}$, we have $\|H_n f\|_{L^p(X)} \leq N_p(X)\|f\|_{L^p(X)}$ for all $f \in L^p(\mathbb{R}, X)$, where

$$H_n f(t) = \frac{1}{\pi}\int_{1/n \leq |s| \leq n} \frac{f(t-s)}{s}\, ds.$$

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Let G be a compact abelian group with dual group Γ. Let $P \subset \Gamma$ be an order on Γ, that is, $P + P \subset P$, $P \cap (-P) = \{0\}$, and $P \cup (-P) = \Gamma$. Define a signum function sgn_P on Γ by $\mathrm{sgn}_P(\chi) = 1, 0$, or -1 according as $\chi \in P \backslash \{0\}$, $\chi = 0$, or $\chi \in (-P) \backslash \{0\}$. Define a conjugate function operator T_P on the X-valued trigonometric polynomials by

$$T_P\left(\sum_{\chi \in \Gamma} a_\chi \chi\right) = -i \sum_{\chi \in \Gamma} \mathrm{sgn}_P(\chi) a_\chi \chi. \tag{1.2}$$

Then we will say that a Banach space X has the ACF (abstract conjugate function) property if, for some $p \in (1, \infty)$, there is a constant $A_p(X)$ such that for all compact abelian groups G with ordered dual groups, the operator T_P extends to $L^p(G, X)$, and for all $f \in L^p(G, X)$, $\|T_P f\|_{L^p(X)} \leq A_p(X) \|f\|_{L^p(X)}$.

In **[8]**, Burkholder showed that if X is a UMD space, it is an HT space. He also conjectured that the converse was true, and this was soon answered affirmatively by Bourgain **[6]**. We will show the same result by first showing that every HT space is an ACF space, and then showing that every ACF space is a UMD space.

This new concept of ACF provides a natural bridge between the concepts of UMD and HT, thus demonstrating how the study of functions on abstract abelian groups enhances and solidifies the connections between harmonic analysis and martingale theory.

2. Transference and the ACF Property

The following generalized version of M. Riesz's theorem follows from **[5**, Theorem 4.1] (see also **[2**, Theorem 6.3]). We will sketch its proof to show the role of transference and the HT property.

Theorem 2.1. *If X has the HT property, then it has the ACF property. Furthermore, for each $p \in (1, \infty)$, we have that $A_p(X) \leq N_p(X)$.*

We follow the proof of **[5**, Theorem 4.1]. We will need the following separation theorem for discrete groups (see **[3**, Theorem 5.14]). As shown recently, this result follows easily from the basic background required for the proof of Hahn's Embedding Theorem for orders (**[12**, Theorem 16, p. 16]). For details, see **[4]**.

Theorem 2.2. *Let P be an order on a discrete abelian group Γ. Given a finite subset $F \subset \Gamma$, there is a homomorphism $\psi : \Gamma \to \mathbb{R}$ such that for all $\chi \in F$,*

$$sgn_P(\chi) = sgn(\psi(\chi)). \tag{2.2.1}$$

Proof of Theorem 2.1. For f an X-valued trigonometric polynomial, let F be a finite subset of Γ such that $f = \sum_{\chi \in F} a_\chi \chi$ where $a_\chi \in X$. Apply Theorem 2.2 and obtain a real-valued homomorphism ψ such that $\mathrm{sgn}(\psi(\chi)) = \mathrm{sgn}_P(\chi)$ for all $\chi \in F$. With this choice of ψ, $T_P(f) = -i \sum_{\chi \in F} \mathrm{sgn}(\psi(\chi)) a_\chi \chi$.

Let $\phi : \mathbb{R} \to G$ be the adjoint homomorphism of ψ which is defined by the relation $\psi(\chi)(t) = \chi(\phi(t))$ for all $t \in \mathbb{R}$ and all $\chi \in \Gamma$. It is easy to see that for any $\chi \in \Gamma$, we have

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{1/n\le|t|\le n} \frac{\chi(x-\phi(t))}{t}\,dt = -i\,\mathrm{sgn}(\psi(\chi))\chi(x) \tag{2.3.1}$$

for all $x \in G$. Consequently, for an X-valued trigonometric polynomial, we have

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{1/n\le|t|\le n} \frac{f(x-\phi(t))}{t}\,dt = T_P(f)(x) \tag{2.3.2}$$

for all $x \in G$. Thus, it is enough to show that, for all $n \in \mathbb{N}$,

$$\left\| \frac{1}{\pi} \int_{1/n\le|t|\le n} \frac{f(x-\phi(t))}{t}\,dt \right\|_{L^p(G,X)} \le N_p(X)\|f\|_{L^p(G,X)}. \tag{2.3.3}$$

This last inequality follows by adapting the transference argument of Calderón **[10]**, Coifman and Weiss **[11]**, to the setting of vector-valued functions, and requires the HT property of X. We omit its proof and refer the reader to the proof of Theorem 2.8 in **[5]** for details. □

3. Proof of Bourgain's result

As noted in **[9]**, to show that $X \in$ UMD, it is enough to consider dyadic martingale difference sequences defined on $[0,1]$ using the Rademacher functions $(r_n)_{n=1}^{\infty}$. (A proof of this reduction can be obtained by adapting Remarque 3 in Maurey **[14]** to the vector-valued setting.) Also, as noted by Burkholder in **[9]**, it suffices to consider martingale difference sequences such that $d_1 = 0$. In **[6]**, Bourgain implicitly uses the fact that to prove $X \in$ UMD, it is enough to consider dyadic martingale difference sequences on the infinite dimensional torus, $\mathbb{T}^{\mathbb{N}}$. We will give a precise version of this, setting the notation in the process.

Let $\mathbb{Z}^{\infty *} = \prod_{n\in\mathbb{N}}{}^{*}\,\mathbb{Z}$ denote the weak direct product of $\mathbb{N}$ copies of $\mathbb{Z}$. When endowed with the discrete topology, $\mathbb{Z}^{\infty *}$ is topologically isomorphic to the dual group of $\mathbb{T}^{\mathbb{N}}$. For each $J = (j_n) \in \mathbb{Z}^{\infty *}$, we denote the corresponding character by χ_J, that is, $\chi_J(\theta_1, \theta_2, \ldots) = \prod_{n=1}^{\infty} e^{ij_n\theta_n}$ where, except for finitely many factors, $e^{ij_n\theta_n} \equiv 1$. For each $J = (j_n) \in \mathbb{Z}^{\infty *}\backslash\{0\}$, define $n(J)$ to be the largest $n \in \mathbb{N}$ such that $j_n \ne 0$, and let $n(0) = 0$. With a slight abuse of notation, we let $\mathbb{Z}^n$ denote $\{J \in \mathbb{Z}^{\infty *} : n(J) \le n\}$ for each $n \ge 0$. Note that $\mathbb{Z}^{\infty *} = \cup_{n=1}^{\infty}\mathbb{Z}^n$.

Identifying $\mathbb{T}$ with the interval $[-\pi, \pi)$, define a sequence, $(s_n)_{n=1}^{\infty}$, of functions on $\mathbb{T}^{\mathbb{N}}$ by $s_1 \equiv 1$, and for $n \ge 2$, $s_n(\theta_1, \theta_2, \ldots) = \mathrm{sgn}(\theta_{n-1})$. Suppose (d_n) is a dyadic martingale difference sequence on $[0,1]$ with $d_1 = 0$ and $d_n = v_n(r_1, \ldots, r_{n-1})r_n$ where $v_n : \{-1,1\}^{n-1} \to X$ for all $n \ge 2$. Letting $d'_1 = 0$ and $d'_n = v_n(s_1, \ldots, s_{n-1})s_n$ for $n \ge 2$ we obtain a martingale difference sequence on $\mathbb{T}^{\mathbb{N}}$ such that the sequences (d_n) and (d'_n) are identically distributed. Call such a sequence a dyadic martingale difference sequence on $\mathbb{T}^{\mathbb{N}}$. Therefore, to prove that $X \in \mathrm{UMD}$, it suffices to show that there exists a constant satisfying (1.1) for dyadic martingale difference sequences on $\mathbb{T}^{\mathbb{N}}$. Since we may approximate each d'_n by a function with finite spectrum in $\mathbb{Z}^n \backslash \mathbb{Z}^{n-1}$, to show that X has the UMD property, we see that it is sufficient to show the following. For $p \in (1, \infty)$, there exists a constant $C_p(X)$ such that if K_j is a finite subset of $\mathbb{Z}^j \backslash \mathbb{Z}^{j-1}$, and $a_J \in X$ for all $J \in K_j$, and $(\epsilon_n) \in \{-1, 1\}^{\mathbb{N}}$, then

$$\left\| \sum_{k=1}^{n} \epsilon_k \left(\sum_{J \in K_j} a_J \chi_J \right) \right\|_{L^p(\mathbb{T}^{\mathbb{N}}, X)} \leq C_p(X) \left\| \sum_{j=1}^{n} \sum_{J \in K_j} a_J \chi_J \right\|_{L^p(\mathbb{T}^{\mathbb{N}}, X)}. \tag{3.1}$$

This reduction appears in Bourgain [6]. We will show that if X is an HT space, (3.1) follows from Theorem 2.1 with specific choices of the order P on $\mathbb{Z}^{\infty *}$. For this purpose, define a reversed lexicographic order on $\mathbb{Z}^{\infty *}$ as follows: $P = \{J = (j_n) \in \mathbb{Z}^{\infty *} : j_{n(J)} > 0\} \cup \{0\}$ where as previously, if $J = (j_n)$, then $j_{n(J)}$ is the last non-zero coordinate of J. Thus, $\mathrm{sgn}_P(\chi_J) = \mathrm{sgn}(j_{n(J)})$. Observe that if $\epsilon = (\epsilon_n) \in \{-1, 1\}^{\mathbb{N}}$, then the set $P(\epsilon) = \{J = (j_n) \in \mathbb{Z}^{\infty *} : \epsilon_{n(J)} j_{n(J)} > 0\} \cup \{0\}$ is also an order on $\mathbb{Z}^{\infty *}$. In this case, $\mathrm{sgn}_{P(\epsilon)}(\chi_J) = \epsilon_{n(J)} \mathrm{sgn}(j_{n(J)})$. We now state a simple identity that links the unconditionality of martingale difference sequences to harmonic conjugation with respect to orders: for every $n \geq 1$, and all $J = (j_n) \in \mathbb{Z}^n \backslash \mathbb{Z}^{n-1}$, we have

$$\epsilon_n = \mathrm{sgn}_{P(\epsilon)}(\chi_J) \mathrm{sgn}_P(\chi_J). \tag{3.2}$$

To verify (3.2), simply note that for each $n \in \mathbb{N}$, if $J \in \mathbb{Z}^n \backslash \mathbb{Z}^{n-1}$, then $n(J) = n$.

From (3.2), one immediately obtains that

$$T_P \circ T_{P(\epsilon)} \left(\sum_{k=1}^{n} \sum_{J \in K_k} a_J \chi_J \right) = \sum_{k=1}^{n} \epsilon_k \left(\sum_{J \in K_k} a_J \chi_J \right), \tag{3.3}$$

which expresses the martingale transform on the right side as a composition of two conjugate function operators. Applying Theorem 2.1 twice yields (3.1) and implies our next and last result.

Theorem 3.1. *Suppose X is a Banach space, and let $1 < p < \infty$. If X has ACF, then X is a UMD space and (1.1) holds with $C_p(X) \leq (A_p(X))^2$.*

Remarks. (a) Combining the implications above, we see that for a Banach space X, the properties UMD, ACF and HT are equivalent.

(b) Burkholder also proved in [8] that when X is a UMD space, the Hilbert transform is weak-type bounded on $L^1(\mathbb{R}, X)$. This weak-type estimate also transfers to the conjugate function operator on $L^1(G, X)$ defined with respect to an arbitrary order on Γ. For a proof see [13]. Thus, when $X \in$ UMD, we have vector-valued analogues of the classical results of M. Riesz and Kolmogorov.

Transference of operators on vector-valued L^p-spaces can be carried out in much greater generality than that shown here. A representation satisfying the vector-valued version of the distributional control condition introduced in [1] will transfer strong-type and weak-type bounds for maximal operators. This work will appear in [13]. The contents of this article are part of the second author's dissertation.

REFERENCES

1. N. Asmar, E. Berkson and T. A. Gillespie, Distributional control and generalized analyticity, *Integral Equations and Operator Theory* **14** (1991), 311–341.

2. N. Asmar, E. Berkson and T. A. Gillespie, Representations of groups with ordered duals and generalized analyticity, *J. Funct. Anal.* **90** (1990), 206–235.

3. N. Asmar and E. Hewitt, Marcel Riesz's theorem on conjugate Fourier series and its descendants, in *Proceedings, Analysis Conference, Singapore, 1986* (S. T. L. Choy *et al.*, eds., Elsevier Science, New York, 1988), 1–56.

4. N. Asmar and S. Montgomery-Smith, Hahn's Embedding Theorem for orders and harmonic analysis on groups with ordered duals, *Colloq. Math.* **70** (1996), 235–252.

5. E. Berkson, T. A. Gillespie and P. S. Muhly, Generalized analyticity in UMD spaces, *Ark. Math.* **27** (1989), 1–14.

6. J. Bourgain, Some remarks on Banach spaces in which martingale difference sequences are unconditional, *Ark. Math.* **21** (1983), 163–168.

7. D. L. Burkholder, A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional, *Ann. Probab.* **9** (1981), 997–1011.

8. D. L. Burkholder, A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions, in *Proceedings, Conference on Harmonic Analysis in Honor of A. Zygmund, Chicago, 1981* (W. Becker *et al.*, eds., Wadsworth, Belmont, CA, 1983), 270–286.

9. D. L. Burkholder, Martingales and Fourier analysis in Banach spaces, in *C.I.M.E. Lectures, Varenna, Italy, 1985* (Lecture Notes in Mathematics, **1206**, 1986), 61–108.

10. A. P. Calderón, Ergodic theory and translation-invariant operators, *Proc. Nat. Acad. Sci., U.S.A.* **59** (1968), 349–353.

11. R. R. Coifman and G. Weiss, *Transference methods in analysis* (Regional Conference Series in Math., **31**, *Amer. Math. Soc.*, Providence, R. I., 1977).

12. L. Fuchs, *Partially ordered algebraic systems* (Pergamon Press, Oxford, New York, 1960).

13. B. P. Kelly, *Distributional controlled representations acting on vector-valued functions spaces*, Doctoral Dissertation, University of Missouri, 1994.

14. B. Maurey, *Système de Haar* (Séminaire Maurey-Schwartz, 1974–1975, École Polytechnique, Paris, 1975).

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