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On two congruence conjectures of Z.-W. Sun involving Franel numbers

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In this paper, we mainly prove the following conjectures of Z.-W. Sun (*J. Number Theory* **133** (2013), 2914–2928): let $p>2$ be a prime. If $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1 \pmod 3$, then

$$x\equiv\frac{1}{4}\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k}\equiv\frac{1}{2}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} \pmod{p^2},$$

and if $p\equiv 1 \pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} \pmod{p^3},$$

where $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$ stands for the nth Franel number.

Keywords: Congruences; Franel numbers; p-adic gamma function; gamma function

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1. Introduction

In 1894, Franel [**2**] found that the numbers

$$f_n=\sum_{k=0}^{n}\binom{n}{k}^3 \quad (n=0,1,2,\ldots)$$

satisfy the recurrence relation (cf. [**14**, A000172]):

$$(n+1)^2f_{n+1}=(7n^2+7n+2)f_n+8n^2f_{n-1} \quad (n=1,2,3,\ldots).$$

These numbers are now called Franel numbers. Callan [**1**] found a combinatorial interpretation of the Franel numbers. The Franel numbers play important roles in combinatorics and number theory. The sequence $\{f_n\}_{n\geqslant 0}$ is one of the five sporadic sequences (cf. [**23**, § 4]) which are integral solutions of certain Apéry-like recurrence equations and closely related to the theory of modular forms. In 2013, Sun

[19] revealed some unexpected connections between the numbers f_n and representations of primes $p \equiv 1(\mathrm{mod} 3)$ in the form x^2+3y^2 with $x, y \in \mathbb{Z}$, for example, Sun [19, (1.2)] showed that

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}, \tag{1.1}$$

and in the same paper, Sun proposed some conjectures involving Franel numbers, one of which is

CONJECTURE 1.1. *Let $p > 2$ be a prime. If $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, then*

$$x \equiv \frac{1}{4} \sum_{k=0}^{p-1} (3k+4) \frac{f_k}{2^k} \equiv \frac{1}{2} \sum_{k=0}^{p-1} (3k+2) \frac{f_k}{(-4)^k} \pmod{p^2}.$$

For more details on Franel numbers, we refer the readers to [3, 4, 6, 8, 9, 18, 20] and so on.

In this paper, our first goal is to prove the above conjecture.

THEOREM 1.1. *Conjecture 1.1 is true.*

Combining (1.1) and theorem 1.1, we immediately obtain the following result.

COROLLARY 1.1. *For any prime $p \equiv 1 \pmod 3$, we have*

$$\sum_{k=0}^{p-1} \frac{k f_k}{2^k} \equiv 2 \sum_{k=0}^{p-1} \frac{k f_k}{(-4)^k} \pmod{p^2}.$$

Sun [19] also gave the following conjecture.

CONJECTURE 1.2. *Let $p > 2$ be a prime. If $p \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \pmod{p^3}.$$

Our last goal is to prove this conjecture.

THEOREM 1.2. *Conjecture 1.2 is true.*

We are going to prove theorem 1.1 in §2. Section 3 is devoted to proving theorem 1.2. Our proofs make use of some combinatorial identities which were found by the package Sigma [13] via software Mathematica and the p-adic gamma function. The proof of theorem 1.2 is somewhat difficult and complex because it is rather convoluted. Throughout this paper, prime p always $\equiv 1 \pmod 3$, so in the following lemmas $p > 5$ or $p > 3$ or $p > 2$ is the same, we mention it here first.

2. Proof of theorem 1.1

For a prime p, let $\mathbb{Z}_p$ denote the ring of all p-adic integers and let $\mathbb{Z}_p^\times := \{a \in \mathbb{Z}_p : a \text{ is prime to } p\}$. For each $\alpha \in \mathbb{Z}_p$, define the p-adic order $\nu_p(\alpha) := \max\{n \in \mathbb{N} : p^n \mid \alpha\}$ and the p-adic norm $|\alpha|_p := p^{-\nu_p(\alpha)}$. Define the p-adic gamma function $\Gamma_p(\cdot)$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1\leqslant k<n\\(k,p)=1}} k, \quad n = 1, 2, 3, \ldots,$$

and

$$\Gamma_p(\alpha) = \lim_{\substack{|\alpha-n|_p\to 0\\ n\in\mathbb{N}}} \Gamma_p(n), \quad \alpha \in \mathbb{Z}_p.$$

In particular, we set $\Gamma_p(0) = 1$. In the following, we need to use the most basic properties of Γ_p, and all of them can be found in [11, 12]. For example, we know that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } |x|_p = 1,\\ -1, & \text{if } |x|_p < 1.\end{cases} \tag{2.1}$$

$$\Gamma_p(1-x)\Gamma_p(x) = (-1)^{a_0(x)}, \tag{2.2}$$

where $a_0(x) \in \{1, 2, \ldots, p\}$ such that $x \equiv a_0(x) \pmod p$. And a property we need here is the fact that for any positive integer n,

$$z_1 \equiv z_2 \pmod{p^n} \quad \text{implies} \quad \Gamma_p(z_1) \equiv \Gamma_p(z_2) \pmod{p^n}. \tag{2.3}$$

LEMMA 2.1. ([19, lemma 2.2]) *For any $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \binom{n}{k}^3 z^k = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+k}{3k}\binom{2k}{k}\binom{3k}{k} z^k(1+z)^{n-2k} \tag{2.4}$$

and

$$f_n = \sum_{k=0}^{n} \binom{n+2k}{3k}\binom{2k}{k}\binom{3k}{k}(-4)^{n-k}. \tag{2.5}$$

For $n, m \in \{1, 2, 3, \ldots\}$, define

$$H_n^{(m)} := \sum_{1\leqslant k\leqslant n} \frac{1}{k^m}, \quad H_0^{(m)} := 0,$$

these numbers with $m = 1$ are often called the classic harmonic numbers. Recall that the Bernoulli polynomials are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots).$$

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Lemma 2.2. ([**15**, **16**]) *Let $p>5$ be a prime. Then*

$$H_{p-1}^{(2)}\equiv 0 \pmod p,\quad H_{(p-1)/2}^{(2)}\equiv 0 \pmod p,\quad H_{p-1}\equiv 0 \pmod{p^2},$$

$$\frac{1}{5}H_{\lfloor p/6\rfloor}^{(2)}\equiv H_{\lfloor p/3\rfloor}^{(2)}\equiv \frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod p,$$

$$H_{\lfloor p/3\rfloor}\equiv -\frac{3}{2}q_p(3)+\frac{3p}{4}q_p^2(3)-\frac{p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2},$$

$$H_{(p-1)/2}\equiv -2q_p(2)+pq_p^2(2) \pmod{p^2},\ H_{\lfloor p/4\rfloor}^{(2)}\equiv (-1)^{(p-1)/2}4E_{p-3} \pmod p,$$

$$H_{\lfloor p/6\rfloor}\equiv H_{\lfloor p/3\rfloor}+H_{(p-1)/2}-\frac{p}{4}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2},$$

$$H_{\lfloor 2p/3\rfloor}\equiv -\frac{3}{2}q_p(3)+\frac{3p}{4}q_p^2(3)+\frac{p}{3}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2},$$

where $q_p(a)=(a^{p-1}-1)/p$ stands for the Fermat quotient.

Lemma 2.3. *Let $p>5$ be a prime. If $0\leqslant j\leqslant (p-1)/2$, then we have*

$$\binom{3j}{j}\binom{p+j}{3j+1}\equiv \frac{p}{3j+1}(1-pH_{2j}+pH_j) \pmod{p^3}.$$

Proof. If $0\leqslant j\leqslant (p-1)/2$ and $j\neq (p-1)/3$, then we have

$$\begin{aligned}\binom{3j}{j}\binom{p+j}{3j+1}&=\frac{(p+j)\cdots(p+1)p(p-1)\cdots(p-2j)}{j!(2j)!(3j+1)}\\&\equiv \frac{pj!(1+pH_j)(-1)^{2j}(2j)!(1-pH_{2j})}{j!(2j)!(3j+1)}\\&\equiv \frac{p}{3j+1}(1-pH_{2j}+pH_j) \pmod{p^3}.\end{aligned}$$

If $j=(p-1)/3$, then by lemma 2.2, we have

$$\begin{aligned}&\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{p-1}{3}}{\frac{p-1}{3}}\\&\equiv \left(1-pH_{(p-1)/3}+\frac{p^2}{2}(H_{(p-1)/3}^2-H_{(p-1)/3}^{(2)})\right)\\&\quad\left(1+pH_{(p-1)/3}+\frac{p^2}{2}(H_{(p-1)/3}^2-H_{(p-1)/3}^{(2)})\right)\\&\equiv 1-p^2H_{(p-1)/3}^{(2)}\equiv 1-\frac{p^2}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}\end{aligned}$$

and

$$1-pH_{(2p-2)/3}+pH_{(p-1)/3}\equiv 1-\frac{p^2}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

Now the proof of lemma 2.3 is complete. □

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Proof of theorem 1.1. With the help of (2.4), we have

$$\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k}=\sum_{k=0}^{p-1}\frac{3k+4}{2^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k+j}{3j}\binom{2j}{j}\binom{3j}{j}2^{k-2j}$$
$$=\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{3j}{j}}{4^j}\sum_{k=2j}^{p-1}(3k+4)\binom{k+j}{3j}. \tag{2.6}$$

By loading the package Sigma in software Mathematica, we find the following identity:

$$\sum_{k=2j}^{n-1}(3k+4)\binom{k+j}{3j}=\frac{9nj+3n+9j+5}{3j+2}\binom{n+j}{3j+1}.$$

Thus, replacing n by p in the above identity and then substitute it into (2.6), we have

$$\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k}=\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{3j}{j}}{4^j}\frac{9pj+3p+9j+5}{3j+2}\binom{p+j}{3j+1}.$$

Hence, we immediately obtain the following result by lemma 2.3,

$$\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k}\equiv p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\frac{9j+5}{(3j+1)(3j+2)}\pmod{p^2}. \tag{2.7}$$

It is easy to verify that

$$\begin{aligned}
&p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\frac{9j+5}{(3j+1)(3j+2)}\\
&=p\sum_{\substack{j=0\\ j\neq(p-1)/3}}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\frac{9j+5}{(3j+1)(3j+2)}+\frac{3p+2}{p+1}\binom{(2p-2)/3}{(p-1)/3}4^{(1-p)/3}\\
&\equiv p\sum_{\substack{j=0\\ j\neq(p-1)/3}}^{(p-1)/2}\binom{\frac{p-1}{2}}{j}(-1)^j\frac{9j+5}{(3j+1)(3j+2)}+\frac{3p+2}{p+1}\binom{(2p-2)/3}{(p-1)/3}4^{(1-p)/3}\\
&\equiv S_1+S_2\pmod{p^2},
\end{aligned} \tag{2.8}$$

where

$$S_1=p\sum_{j=0}^{(p-1)/2}\binom{(p-1)/2}{j}(-1)^j\left(\frac{2}{3j+1}+\frac{1}{3j+2}\right) \tag{2.9}$$

and

$$S_2=\frac{3p+2}{p+1}\left(\binom{(2p-2)/3}{(p-1)/3}4^{(1-p)/3}-\binom{(p-1)/2}{(p-1)/3}\right).$$

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Applying the famous partial fraction identity

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+x}=\frac{n!}{x(x+1)\cdots(x+n)} \tag{2.10}$$

with $x=1/3, n=(p-1)/2$ and $x=2/3, n=(p-1)/2$, we may simplify (2.9) as

$$S_1=\frac{4p}{3p-1}\frac{(1)_{(p-1)/2}}{(1/3)_{(p-1)/2}}+\frac{2p}{3p+1}\frac{(1)_{(p-1)/2}}{(2/3)_{(p-1)/2}},$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ is the rising factorial or the Pochhammer symbol.

In view of (2.2), we have

$$\begin{aligned}\frac{4p}{3p-1}\frac{(1)_{(p-1)/2}}{(\frac{1}{3})_{(p-1)/2}}&=\frac{4p}{3p-1}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3}+\frac{p-1}{2})}=\frac{4p}{3p-1}\frac{(-1)^{(p+1)/2}\Gamma_p(\frac{p+1}{2})\Gamma_p(\frac{1}{3})}{(-1)^{(p-1)/2}\frac{p}{3}\Gamma_p(\frac{1}{3}+\frac{p-1}{2})}\\&=\frac{12}{1-3p}\frac{\Gamma_p(\frac{p+1}{2})\Gamma_p(\frac{1}{3})}{\Gamma_p(\frac{p}{2}-\frac{1}{6})}=\frac{12(-1)^{(p-1)/6}}{1-3p}\Gamma_p\left(\frac{p+1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{7}{6}-\frac{p}{2}\right),\end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. In view of [**7**, theorem 14] and [**5**, (2.4)] (or [**10**, (3.2)]), for $\alpha, s\in\mathbb{Z}_p$, we have

$$\Gamma_p(\alpha+ps)\equiv\Gamma_p(\alpha)+ps\Gamma_p'(\alpha)\ (\mathrm{mod}\ p^2) \tag{2.11}$$

and

$$\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)}\equiv 1+H_{p-\langle-\alpha\rangle_p-1}\ (\mathrm{mod}\ p), \tag{2.12}$$

where $\Gamma_p'(x)$ denotes the p-adic derivative of $\Gamma_p(x)$, $\langle\alpha\rangle_n$ denotes the least non-negative residue of α modulo n, i.e. the integer lying in $\{0,1,\ldots,n-1\}$ such that $\langle\alpha\rangle_n\equiv\alpha\ (\mathrm{mod}\ n)$.

Therefore,

$$\begin{aligned}&\frac{4p}{3p-1}\frac{(1)_{(p-1)/2}}{(\frac{1}{3})_{(p-1)/2}}\\&\quad\equiv\frac{12(-1)^{(p-1)/6}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{7}{6}\right)}{1-3p}\left(1+\frac{p}{2}(H_{(p-1)/2}-H_{(p-7)/6})\right)\ (\mathrm{mod}\ p^2).\end{aligned}$$

In view of (2.1) and (2.2), we have

$$\begin{aligned}&\frac{4p}{3p-1}\frac{(1)_{(p-1)/2}}{(\frac{1}{3})_{(p-1)/2}}\\&\quad\equiv\frac{2(1+3p)\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)}{\Gamma_p\left(\frac{5}{6}\right)}\left(1+\frac{p}{2}(H_{(p-1)/2}-H_{(p-7)/6})\right)\ (\mathrm{mod}\ p^2).\end{aligned}$$

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In view of [22, proposition 4.1], we have

$$\frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)}{\Gamma_p\left(\frac{5}{6}\right)} \equiv \frac{\binom{(5p-5)/6}{(p-1)/3}}{\left(1+\frac{p}{6}(5H_{(5p-5)/6}-3H_{(p-1)/2}-2H_{(p-1)/3})\right)} \pmod{p^2}.$$

Then with the help of [22, theorem 4.12] and lemma 2.2, we have

$$\frac{4p}{3p-1}\frac{(1)_{(p-1)/2}}{(1/3)_{(p-1)/2}} \equiv 4x+3pxq_p(3)-\frac{p}{x} \pmod{p^2} \tag{2.13}$$

and

$$\frac{2p}{3p+1}\frac{(1)_{(p-1)/2}}{(2/3)_{(p-1)/2}} \equiv \frac{p}{x} \pmod{p^2}. \tag{2.14}$$

Hence,

$$S_1 \equiv 4x+3pxq_p(3) \pmod{p^2}. \tag{2.15}$$

□

LEMMA 2.4. *Let $p>3$ be a prime. For any p-adic integer t, we have*

$$\binom{\frac{p-1}{2}+pt}{\frac{p-1}{3}} \equiv \binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(1+pt\left(H_{(p-1)/2}-H_{(p-1)/6}\right)\right) \pmod{p^2}. \tag{2.16}$$

Proof. Set $m=(p-1)/2$. It is easy to check that

$$\begin{aligned}\binom{m+pt}{(p-1)/3} &= \frac{(m+pt)\cdots(m+pt-(p-1)/3+1)}{((p-1)/3)!}\\ &\equiv \frac{m\cdots(m-(p-1)/3+1)}{((p-1)/3)!}(1+pt(H_m-H_{m-(p-1)/3}))\\ &= \binom{m}{(p-1)/3}(1+pt(H_m-H_{m-(p-1)/3})) \pmod{p^2}.\end{aligned}$$

So lemma 2.4 is finished. □

Now we evaluate S_2 modulo p^2. It is easy to obtain that

$$\begin{aligned}S_2 &\equiv 2\left(\binom{-\frac{1}{2}}{\frac{p-1}{3}}-\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\right) \equiv -p\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(H_{(p-1)/2}-H_{(p-1)/6}\right)\\ &\equiv -3pxq_p(3) \pmod{p^2}\end{aligned} \tag{2.17}$$

with the help of lemmas 2.2, 2.4 and [22, theorem 4.12].

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Therefore, in view of (2.7), (2.8), (2.15) and (2.17), we immediately get the desired result

$$\frac{1}{4}\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k} \equiv x \pmod{p^2}.$$

On the contrary, we use equation (2.5) to obtain

$$\begin{aligned}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} &= \sum_{k=0}^{p-1}\frac{3k+2}{(-4)^k}\sum_{j=0}^{k}\binom{k+2j}{3j}\binom{2j}{j}\binom{3j}{j}(-4)^{k-j} \\ &= \sum_{j=0}^{p-1}\frac{\binom{2j}{j}\binom{3j}{j}}{(-4)^j}\sum_{k=j}^{p-1}(3k+2)\binom{k+2j}{3j}.\end{aligned}$$

By using the package Sigma again, we find the following identity:

$$\sum_{k=j}^{n-1}(3k+2)\binom{k+2j}{3j} = \frac{9nj+3n+1}{3j+2}\binom{n+2j}{3j+1}.$$

Thus,

$$\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} = \sum_{j=0}^{p-1}\frac{\binom{2j}{j}\binom{3j}{j}\binom{p+2j}{3j+1}}{(-4)^j}\frac{9pj+3p+1}{3j+2}. \tag{2.18}$$

Lemma 2.5. *Let $p > 5$ be a prime. If $0 \leqslant j \leqslant (p-1)/2$ and $j \neq (p-1)/3$, then*

$$\binom{3j}{j}\binom{p+2j}{3j+1} \equiv \frac{p(-1)^j}{3j+1}(1+pH_{2j}-pH_j) \pmod{p^3}.$$

If $(p+1)/2 \leqslant j \leqslant p-1$, then

$$\binom{3j}{j}\binom{p+2j}{3j+1} \equiv \frac{2p(-1)^j}{3j+1} \pmod{p^2}.$$

Proof. If $0 \leqslant j \leqslant (p-1)/2$ and $j \neq (p-1)/3$, then we have

$$\begin{aligned}\binom{3j}{j}\binom{p+2j}{3j+1} &= \frac{(p+2j)\cdots(p+1)p(p-1)\cdots(p-j)}{j!(2j)!(3j+1)} \\ &\equiv \frac{p(2j)!(1+pH_{2j})(-1)^j(j)!(1-pH_j)}{j!(2j)!(3j+1)} \\ &\equiv \frac{p(-1)^j}{3j+1}(1+pH_{2j}-pH_j) \pmod{p^3}.\end{aligned}$$

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If $(p+1)/2 \leqslant j \leqslant p-1$, then

$$\begin{aligned}\binom{3j}{j}\binom{p+2j}{3j+1} &= \frac{(p+2j)\cdots(2p+1)(2p)(2p-1)\cdots(p+1)p(p-1)\cdots(p-j)}{j!(2j)!(3j+1)} \\ &\equiv \frac{2p^2(2j)\cdots(p+1)(p-1)!(-1)^j(j)!}{j!(2j)!(3j+1)} = \frac{2p(-1)^j}{3j+1} \pmod{p^2}.\end{aligned}$$

Now the proof of lemma 2.5 is complete. □

It is known that $\binom{2k}{k} \equiv 0 \pmod p$ for each $(p+1)/2 \leqslant k \leqslant p-1$, and it is easy to check that for each $0 \leqslant j \leqslant (p-1)/2$:

$$\binom{3j}{j}\binom{p+2j}{3j+1} \equiv \frac{p(-1)^j}{3j+1} \pmod{p^2}.$$

These, with (2.18) yield

$$\begin{aligned}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} &\equiv \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{(-4)^j}\frac{p(-1)^j}{3j+1}\frac{9pj+3p+1}{3j+2} \\ &+ \sum_{j=(p+1)/2}^{p-1}\frac{\binom{2j}{j}}{(-4)^j}\frac{2p(-1)^j}{3j+1}\frac{1}{3j+2} \equiv \sum_{j=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{j}\frac{p(-1)^j}{3j+1}\frac{1}{3j+2} + S_3 \\ &= p\sum_{j=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{j}(-1)^j\left(\frac{1}{3j+1}-\frac{1}{3j+2}\right) + S_3 \pmod{p^2}, \qquad (2.19)\end{aligned}$$

where

$$\begin{aligned}S_3 &= \binom{\frac{2p-2}{3}}{\frac{p-1}{3}}\frac{1}{(p+1)4^{(p-1)/3}} - \binom{\frac{p-1}{2}}{\frac{p-1}{3}}\frac{1}{p+1} - \binom{\frac{4p-4}{3}}{\frac{2p-2}{3}}\frac{1}{4^{(2p-2)/3}} \\ &= \frac{1}{p+1}\left(\binom{-1/2}{(p-1)/3} - \binom{\frac{p-1}{2}}{\frac{p-1}{3}}\right) - \binom{-1/2}{(2p-2)/3}.\end{aligned}$$

As above, with (2.10), (2.13), (2.14), lemma 2.2 and [**22**, theorem 4.12], we have the following congruence modulo p^2:

$$p\sum_{j=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{j}(-1)^j\left(\frac{1}{3j+1}-\frac{1}{3j+2}\right) \equiv 2x + \frac{3px}{2}q_p(3) - \frac{3p}{2x}. \qquad (2.20)$$

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Now we evaluate S_3. It is easy to see that

$$\begin{aligned}\binom{-1/2}{(2p-2)/3} &= \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-\frac{2p-2}{3}+1)}{(\frac{2p-2}{3})!}\\ &= \frac{(\frac{1}{2})(\frac{3}{2})\cdots(\frac{p}{2}-1)\frac{p}{2}(\frac{p}{2}+1)\cdots(\frac{p}{2}+\frac{p-7}{6})}{(\frac{2p-2}{3})!}\\ &= \frac{(\frac{p}{2}-\frac{p-1}{2})\cdots(\frac{p}{2}-1)\frac{p}{2}(\frac{p}{2}+1)\cdots(\frac{p}{2}+\frac{p-7}{6})}{(\frac{2p-2}{3})!}\\ &\equiv \frac{(-1)^{(p-1)/2}\frac{p}{2}(\frac{p-1}{2})!(\frac{p-7}{6})!}{(\frac{2p-2}{3})!} = \frac{(-1)^{(p-1)/2}3p}{p-1}\frac{1}{\binom{\frac{2p-2}{3}}{\frac{p-1}{2}}}\\ &\equiv \frac{-3p(-1)^{(p-1)/2}}{\binom{\frac{2p-2}{3}}{\frac{p-1}{2}}} \pmod{p^2}.\end{aligned}$$

In view of (2.17) and [**22**, theorem 4.12], we immediately obtain

$$S_3 \equiv -\frac{3px}{2}q_p(3) + \frac{3p}{2x} \pmod{p^2}.$$

This, with (2.19) and (2.20) yields

$$\frac{1}{2}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} \equiv x \pmod{p^2}$$

Now the proof of theorem 1.1 is complete.

3. Proof of theorem 1.2

Proof of theorem 1.2. With the help of (2.4), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{f_k}{2^k} &= \sum_{k=0}^{p-1}\frac{1}{2^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k+j}{3j}\binom{2j}{j}\binom{3j}{j}2^{k-2j}\\ &= \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{3j}{j}}{4^j}\sum_{k=2j}^{p-1}\binom{k+j}{3j}. \qquad (3.1)\end{aligned}$$

By loading the package Sigma in software Mathematica, we have the following identity:

$$\sum_{k=2j}^{n-1}\binom{k+j}{3j} = \binom{n+j}{3j+1}.$$

Thus, replace n by p in the above identity and then substitute it into (3.1), we have

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k} = \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{3j}{j}}{4^j}\binom{p+j}{3j+1}.$$

Hence, we immediately obtain the following result by lemma 2.3:

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv p\sum_{j=0,j\neq(p-1)/3}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\frac{1-pH_{2j}+pH_j}{(3j+1)}+S_1 \pmod{p^3}, \tag{3.2}$$

where

$$S_1=\frac{\binom{\frac{2p-2}{3}}{\frac{p-1}{3}}\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{p-1}{3}}{p}}{4^{(p-1)/3}}=\binom{-\frac{1}{2}}{\frac{p-1}{3}}\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{p-1}{3}}{p}.$$

It is easy to verify that

$$\begin{aligned}
&p\sum_{j=0,j\neq(p-1)/3}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\frac{1-pH_{2j}+pH_j}{(3j+1)}\\
&\equiv p\sum_{j=0,j\neq(p-1)/3}^{(p-1)/2}\frac{\binom{\frac{p-1}{2}}{j}(-1)^j(1-pH_{2j}+pH_j)}{(3j+1)\left(1-p\sum_{r=1}^{j}\frac{1}{2r-1}\right)}\\
&\equiv p\sum_{j=0}^{(p-1)/2}\frac{\binom{\frac{p-1}{2}}{j}(-1)^j\left(1+\frac{p}{2}H_j\right)}{(3j+1)}-S_2 \pmod{p^3},
\end{aligned}$$

where

$$S_2=\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(1+\frac{p}{2}H_{(p-1)/3}\right).$$

So,

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv p\sum_{j=0}^{(p-1)/2}\frac{\binom{\frac{p-1}{2}}{j}(-1)^j\left(1+\frac{p}{2}H_j\right)}{(3j+1)}+S_1-S_2 \pmod{p^3}. \tag{3.3}$$

It is easy to see that

$$\frac{2p}{3p-1}\frac{(1)_{(p-1)/2}}{(\frac{1}{3})_{(p-1)/2}}=\frac{(\frac{p-1}{2})!}{\frac{1}{3}\cdots(\frac{p}{3}-1)(\frac{p}{3}+1)\cdots(\frac{p}{3}+\frac{p-1}{6})}\equiv\binom{\frac{p-1}{2}}{\frac{p-1}{3}} \pmod{p}. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k}&=\sum_{k=0}^{p-1}\frac{1}{(-4)^k}\sum_{j=0}^{k}\binom{k+2j}{3j}\binom{2j}{j}\binom{3j}{j}(-4)^{k-j}\\
&=\sum_{j=0}^{p-1}\frac{\binom{2j}{j}\binom{3j}{j}}{(-4)^j}\sum_{k=j}^{p-1}\binom{k+2j}{3j}=\sum_{j=0}^{p-1}\frac{\binom{2j}{j}\binom{3j}{j}}{(-4)^j}\binom{p+2j}{3j+1}.
\end{aligned}$$

So by lemma 2.5 and the fact that for each $0\leqslant k\leqslant(p-1)/2$,

$$\frac{\binom{2k}{k}}{(-4)^k}\equiv\frac{\binom{\frac{p-1}{2}}{k}}{(1-p\sum_{j=1}^{k}\frac{1}{2j-1})} \pmod{p^2},$$

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and for each $(p+1)/2 \leqslant j \leqslant p-1$,

$$j\binom{2j}{j}\binom{2p-2j}{p-j} \equiv 2p \pmod{p^2},$$

we have the following congruence modulo p^3:

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} - S_3 &\equiv p\sum_{\substack{j=0\\ j\neq(p-1)/3}}^{(p-1)/2}\frac{\binom{2j}{j}(1+pH_{2j}-pH_j)}{(3j+1)4^j} + 2p\sum_{j=(p+1)/2}^{p-1}\frac{\binom{2j}{j}}{(3j+1)4^j}\\
&\equiv \sum_{\substack{j=0\\ j\neq(p-1)/3}}^{(p-1)/2}\frac{p(-1)^j\binom{\frac{p-1}{2}}{j}\left(1+2pH_{2j}-\frac{3}{2}pH_j\right)}{3j+1} + \sum_{j=(p+1)/2}^{p-1}\frac{4p^2}{4^j(3j+1)j\binom{2p-2j}{p-j}}\\
&\equiv \sum_{j=0}^{(p-1)/2}\frac{p(-1)^j\binom{\frac{p-1}{2}}{j}\left(1+2pH_{2j}-\frac{3}{2}pH_j\right)}{3j+1} + \sum_{j=1}^{(p-1)/2}\frac{p^2 4^j}{(3j-1)j\binom{2j}{j}} - S_4,
\end{aligned}$$

where

$$\begin{aligned}
S_3 &= \frac{\binom{\frac{2p-2}{3}}{\frac{p-1}{3}}\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{2p-2}{3}}{p}}{(-4)^{\frac{p-1}{3}}} = \binom{-\frac{1}{2}}{\frac{p-1}{3}}\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{2p-2}{3}}{p},\\
S_4 &= \binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(1+2pH_{(2p-2)/3}-\frac{3}{2}pH_{(p-1)/3}\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} - \sum_{k=0}^{p-1}\frac{f_k}{2^k}\\
&\quad\equiv 2p^2\sum_{j=0}^{(p-1)/2}\frac{\binom{\frac{p-1}{2}}{j}(-1)^j(H_{2j}-H_j)}{3j+1} + S_5 + \sum_{j=1}^{(p-1)/2}\frac{p^2 4^j}{(3j-1)j\binom{2j}{j}} \pmod{p^3},
\end{aligned} \tag{3.5}$$

where

$$S_5 = S_3 - S_4 + S_2 - S_1.$$

By Sigma, we can find and prove the following identity:

$$\begin{aligned}
&\sum_{j=0}^{n}\frac{2\binom{n}{j}(-1)^j(H_{2j}-H_j)}{3j+1}\\
&\quad = \frac{1}{3n+1}\prod_{k=1}^{n}\frac{3k}{3k-2}\left(\sum_{k=1}^{n}\frac{1}{k}\prod_{j=1}^{k}\frac{3j-2}{3j} - \sum_{k=1}^{n}\frac{1}{k}\prod_{j=1}^{k}\frac{2(3j-2)}{3(2j-1)}\right)\\
&\quad = \frac{(1)_n}{(3n+1)\left(\frac{1}{3}\right)_n}\left(\sum_{k=1}^{n}\frac{\left(\frac{1}{3}\right)_k}{k(1)_k} - \sum_{k=1}^{n}\frac{\left(\frac{1}{3}\right)_k}{k\left(\frac{1}{2}\right)_k}\right).
\end{aligned} \tag{3.6}$$

In view of [17, lemma 3.1] and lemma 2.2, we have

$$\sum_{k=1}^{(p-1)/2}\frac{\left(\frac{1}{3}\right)_k}{k(1)_k}=\sum_{k=1}^{(p-1)/2}\frac{\binom{-1/3}{k}}{k\binom{-1}{k}}\equiv\frac{3}{2}q_p(3)-\frac{3p}{4}q_p^2(3)-\frac{p}{3}\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}\pmod{p^2}. \tag{3.7}$$

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{\left(\frac{1}{3}\right)_k}{k\left(\frac{1}{2}\right)_k}=\sum_{k=1}^{(p-1)/2}\frac{\binom{-1/3}{k}}{k\binom{-1/2}{k}}&\equiv\frac{4p}{3}(-1)^{(p-1)/2}E_{p-3}+\frac{3}{2}q_p(3)-\frac{3p}{4}q_p^2(3)\\&-\frac{2p}{3}(-1)^{(p-1)/2}\sum_{k=1}^{(p-1)/3}\frac{4^k}{(2k-1)k\binom{2k}{k}}\pmod{p^2}.\end{aligned} \tag{3.8}$$

It is easy to check that

$$\sum_{k=1}^{(p-1)/3}\frac{4^k}{(2k-1)k\binom{2k}{k}}=2\sum_{k=1}^{(p-1)/3}\frac{4^k}{(2k-1)\binom{2k}{k}}-\sum_{k=1}^{(p-1)/3}\frac{4^k}{k\binom{2k}{k}}. \tag{3.9}$$

And by [21, (6)], we have

$$\frac{1}{\binom{n+1+k}{k}}=(n+1)\sum_{r=0}^{n}\binom{n}{r}(-1)^r\frac{1}{k+r+1}. \tag{3.10}$$

$$\begin{aligned}2\sum_{k=1}^{(p-1)/3}\frac{4^k}{(2k-1)\binom{2k}{k}}&\equiv 2\sum_{k=1}^{(p-1)/3}\frac{(-1)^k}{(2k-1)\binom{\frac{p-1}{2}}{k}}\\&\equiv(-1)^{(p+1)/2}\sum_{k=(p-1)/6}^{(p-3)/2}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}}\\&=(-1)^{(p+1)/2}\left(\sum_{k=0}^{(p-3)/2}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}}-\sum_{k=0}^{(p-7)/6}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}}\right)\pmod{p}.\end{aligned} \tag{3.11}$$

By Sigma, we find the following identity which can be proved by induction on n:

$$\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)\binom{n}{k}}=\frac{2(-1)^n-1}{n+1}-(n+1)H_n^{(2)}-2(n+1)\sum_{k=1}^{n}\frac{(-1)^k}{k^2}.$$

So by setting $n=(p-1)/2$ in the above identity and with lemma 2.2, we have

$$\sum_{k=0}^{(p-3)/2}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}}\equiv 2\left((-1)^{(p-1)/2}-1\right)-(-1)^{(p-1)/2}2E_{p-3}\pmod{p}. \tag{3.12}$$

And by (3.10), we have

$$\begin{aligned}\sum_{k=0}^{(p-7)/6}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}} &\equiv \sum_{k=0}^{(p-7)/6}\frac{1}{(k+1)\binom{\frac{p-1}{2}+k}{k}}\\ &= \sum_{k=0}^{(p-7)/6}\frac{1}{k+1}\frac{p-1}{2}\sum_{r=0}^{(p-3)/2}\binom{\frac{p-3}{2}}{r}(-1)^r\frac{1}{k+r+1}\\ &\equiv -\frac{1}{2}\sum_{k=1}^{(p-1)/6}\frac{1}{k}\sum_{r=0}^{(p-3)/2}\binom{\frac{p-3}{2}}{r}(-1)^r\frac{1}{k+r}\\ &= -\frac{1}{2}H_{(p-1)/6}^{(2)}-\frac{1}{2}\sum_{r=1}^{(p-3)/2}\frac{(-1)^r}{r}\binom{\frac{p-3}{2}}{r}\sum_{k=1}^{(p-1)/6}\left(\frac{1}{k}-\frac{1}{k+r}\right) \pmod p.\end{aligned}$$

It is easy to check that

$$H_{(p-1)/6}-\sum_{k=1}^{(p-1)/6}\frac{1}{k+r}\equiv -\sum_{k=1}^{r}\frac{1}{k(6k-1)} \pmod p.$$

By Sigma again, we have

$$\sum_{r=1}^{n}\frac{(-1)^r}{r}\binom{n}{r}\sum_{k=1}^{r}\frac{1}{k(6k-1)}=H_n^{(2)}-\sum_{k=1}^{n}\frac{(1)_k}{k^2\left(\frac{5}{6}\right)_k}.$$

So in view of lemma 2.2 and [**22**], we have

$$\begin{aligned}&\sum_{k=0}^{(p-7)/6}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}}\\ &\quad\equiv \frac{(-1)^{(p-1)/2}}{x}-2-\frac{5}{4}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2\binom{-\frac{5}{6}}{k}} \pmod p.\end{aligned}$$

Thus, by (3.10), we have

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2\binom{-\frac{5}{6}}{k}} &= -\frac{6}{5}\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k\binom{-\frac{11}{6}}{k-1}}\equiv\frac{6}{5}\sum_{k=0}^{(p-3)/2}\frac{(-1)^k}{(k+1)\binom{\frac{5p-11}{6}}{k}}\\ &= \frac{6}{5}\sum_{k=0}^{(p-3)/2}\frac{1}{(k+1)\binom{\frac{p+5}{6}+k}{k}}=\frac{6}{5}\sum_{k=0}^{(p-3)/2}\frac{1}{k+1}\frac{p+5}{6}\sum_{r=0}^{(p-1)/6}(-1)^r\binom{\frac{p-1}{6}}{r}\frac{1}{k+1+r}\\ &\equiv \sum_{k=1}^{(p-1)/2}\frac{1}{k}\sum_{r=0}^{(p-1)/6}(-1)^r\binom{\frac{p-1}{6}}{r}\frac{1}{k+r}\\ &= H_{(p-1)/2}^{(2)}+\sum_{r=1}^{(p-1)/6}\frac{(-1)^r}{r}\binom{\frac{p-1}{6}}{r}\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k}-\frac{1}{k+r}\right) \pmod p.\end{aligned}$$

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Also it is easy to see that

$$H_{(p-1)/2}-\sum_{k=1}^{(p-1)/2}\frac{1}{k+r}\equiv-\sum_{k=1}^{r}\frac{1}{k(2k-1)} \pmod p.$$

And by Sigma, we have

$$\sum_{r=1}^{n}\frac{(-1)^r}{r}\binom{n}{r}\sum_{k=1}^{r}\frac{1}{k(2k-1)}=H_n^{(2)}-\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}.$$

So in view of lemma 2.2, we have

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2\binom{-\frac{5}{6}}{k}}\equiv\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}-\frac{5}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod p.$$

Hence,

$$\sum_{k=0}^{(p-7)/6}\frac{(-1)^k}{(k+1)\binom{\frac{p-1}{2}}{k}}\equiv\frac{(-1)^{(p-1)/2}}{x}-2-\frac{1}{2}\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}} \pmod p.$$

This, with (3.11) and (3.12) yields

$$\begin{aligned}2\sum_{k=1}^{(p-1)/3}&\frac{4^k}{(2k-1)\binom{2k}{k}}\\&\equiv-2+\frac{1}{x}+2E_{p-3}-\frac{1}{2}(-1)^{(p-1)/2}\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}} \pmod p. \qquad (3.13)\end{aligned}$$

By Sigma, we find the following identity which can be proved by induction on n:

$$\sum_{k=1}^{n}\frac{4^k}{k\binom{2k}{k}}=-2+2\frac{4^n}{\binom{2n}{n}}. \qquad (3.14)$$

So in view of [22], we have

$$\sum_{k=1}^{(p-1)/3}\frac{4^k}{k\binom{2k}{k}}\equiv-2+\frac{2}{\binom{\frac{p-1}{2}}{\frac{p-1}{3}}}\equiv-2+\frac{1}{x} \pmod p.$$

This, with (3.9) and (3.13) yields

$$\sum_{k=1}^{(p-1)/3}\frac{4^k}{(2k-1)k\binom{2k}{k}}\equiv 2E_{p-3}-\frac{1}{2}(-1)^{(p-1)/2}\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}} \pmod p.$$

Thus, with (3.8) we have

$$\sum_{k=1}^{(p-1)/2}\frac{\left(\frac{1}{3}\right)_k}{k\left(\frac{1}{2}\right)_k}\equiv\frac{3}{2}q_p(3)-\frac{3p}{4}q_p^2(3)+\frac{p}{3}\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}} \pmod{p^2}. \qquad (3.15)$$

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So by (3.7), we have

$$\sum_{k=1}^{(p-1)/2}\frac{\left(\frac{1}{3}\right)_k}{k(1)_k}-\sum_{k=1}^{(p-1)/2}\frac{\left(\frac{1}{3}\right)_k}{k\left(\frac{1}{2}\right)_k}\equiv-\frac{p}{3}\left(\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}+\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}\right)\pmod{p^2}.$$

Therefore, by (3.6) and (3.4), we deduce

$$\begin{aligned}&2p^2\sum_{j=0}^{(p-1)/2}\frac{\binom{\frac{p-1}{2}}{j}(-1)^j(H_{2j}-H_j)}{3j+1}\\&\quad\equiv-\frac{p^2}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}+\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}\right)\pmod{p^3}.\end{aligned}\tag{3.16}$$

Now, we evaluate the second sum on the right-hand side of (3.5). It is easy to see

$$\sum_{j=1}^{(p-1)/2}\frac{4^j}{(3j-1)j\binom{2j}{j}}=3\sum_{j=1}^{(p-1)/2}\frac{4^j}{(3j-1)\binom{2j}{j}}-\sum_{j=1}^{(p-1)/2}\frac{4^j}{j\binom{2j}{j}}.\tag{3.17}$$

By (3.14), we have

$$\sum_{j=1}^{(p-1)/2}\frac{4^j}{j\binom{2j}{j}}\equiv-2+2(-1)^{(p-1)/2}\pmod{p}.\tag{3.18}$$

Now we consider the first sum of the right-hand side in (3.17):

$$\sum_{j=1}^{(p-1)/2}\frac{4^j}{(3j-1)\binom{2j}{j}}=\sum_{j=1}^{(p-1)/3}\frac{4^j}{(3j-1)\binom{2j}{j}}+\sum_{j=(p+2)/3}^{(p-1)/2}\frac{4^j}{(3j-1)\binom{2j}{j}}.$$

The following identity is very important to us:

$$\sum_{k=1}^{n}\frac{4^k}{(k+n)\binom{2k}{k}}=-2+2\frac{4^n}{\binom{2n}{n}}-\frac{n\binom{2n}{n}}{4^n}\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}.\tag{3.19}$$

This, with [**22**] yields

$$\begin{aligned}3\sum_{j=1}^{(p-1)/3}\frac{4^j}{(3j-1)\binom{2j}{j}}&\equiv\sum_{j=1}^{(p-1)/3}\frac{4^j}{\left(j+\frac{p-1}{3}\right)\binom{2j}{j}}\\&\equiv-2+\frac{2}{\binom{-1/2}{\frac{p-1}{3}}}+\frac{1}{3}\binom{-1/2}{\frac{p-1}{3}}\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}\\&\equiv-2+\frac{1}{x}+\frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}\pmod{p}.\end{aligned}\tag{3.20}$$

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And by (3.19), we have

$$3\sum_{j=(p+2)/3}^{(p-1)/2}\frac{4^j}{(3j-1)\binom{2j}{j}}\equiv 3\sum_{j=0}^{(p-7)/6}\frac{(-1)^{(p-1)/2-j}}{(3(\frac{p-1}{2}-j)-1)\binom{\frac{p-1}{2}}{j}}$$
$$\equiv 6(-1)^{(p+1)/2}\sum_{j=0}^{(p-7)/6}\frac{4^j}{(6j+5)\binom{2j}{j}}\equiv(-1)^{(p+1)/2}\sum_{j=0}^{(p-7)/6}\frac{(-1)^j}{(j+\frac{p+5}{6})\binom{\frac{p-1}{2}}{j}}$$
$$\equiv\frac{6}{5}(-1)^{(p+1)/2}+(-1)^{(p+1)/2}\sum_{j=1}^{(p+5)/6}\frac{4^j}{(j+\frac{p+5}{6})\binom{2j}{j}}+\frac{3}{\binom{\frac{p-1}{2}}{\frac{p-1}{3}}}\pmod p. \quad (3.21)$$

In view of (3.19) and [22], we have

$$\sum_{j=1}^{(p+5)/6}\frac{4^j}{(j+\frac{p+5}{6})\binom{2j}{j}}\equiv-\frac{16}{5}+\frac{5(-1)^{(p-1)/6}}{2x}$$
$$-\frac{(-1)^{(p-1)/6}}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}\pmod p.$$

This, with (3.21) yields

$$3\sum_{j=(p+2)/3}^{(p-1)/2}\frac{4^j}{(3j-1)\binom{2j}{j}}\equiv 2(-1)^{(p-1)/2}-\frac{1}{x}+\frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}\pmod p.$$

Combining this with (3.20), we have

$$3\sum_{j=1}^{(p-1)/2}\frac{4^j}{(3j-1)\binom{2j}{j}}$$
$$\equiv -2+2(-1)^{(p-1)/2}+\frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}+\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}\right)\pmod p.$$

Thus, by (3.17) and (3.18), we have

$$\sum_{j=1}^{(p-1)/2}\frac{4^j}{(3j-1)j\binom{2j}{j}}\equiv\frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(\sum_{k=1}^{(p-1)/3}\frac{4^k}{k^2\binom{2k}{k}}+\sum_{k=1}^{(p-1)/6}\frac{4^k}{k^2\binom{2k}{k}}\right)\pmod p.$$

This, with (3.5) and (3.16) yields

$$\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k}-\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv S_5 \pmod{p^3}. \quad (3.22)$$

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While

$$S_5 = \binom{-\frac{1}{2}}{\frac{p-1}{3}}\binom{p-1}{\frac{p-1}{3}}\left(\binom{p+\frac{2p-2}{3}}{\frac{2p-2}{3}} - \binom{p+\frac{p-1}{3}}{\frac{p-1}{3}}\right) + 2p\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\left(H_{(p-1)/3} - H_{(2p-2)/3}\right).$$

It is easy to check that

$$\binom{p+\frac{2p-2}{3}}{\frac{2p-2}{3}} \equiv 1 + pH_{(2p-2)/3} + \frac{p^2}{2}\left(H^2_{(2p-2)/3} - H^{(2)}_{(2p-2)/3}\right) \pmod{p^3}$$

and

$$\binom{p+\frac{p-1}{3}}{\frac{p-1}{3}} \equiv 1 + pH_{(p-1)/3} + \frac{p^2}{2}\left(H^2_{(p-1)/3} - H^{(2)}_{(p-1)/3}\right) \pmod{p^3}.$$

So by lemma 2.2 and the fact that $H^{(2)}_{p-1-k} \equiv -H^{(2)}_k \pmod p$ for each $0 \leqslant k \leqslant p-1$, we have

$$\begin{aligned}\binom{p+\frac{2p-2}{3}}{\frac{2p-2}{3}} - \binom{p+\frac{p-1}{3}}{\frac{p-1}{3}} &\equiv p(H_{(2p-2)/3} - H_{(p-1)/3}) + \frac{p^2}{2}(H^{(2)}_{(p-1)/3} - H^{(2)}_{(2p-2)/3}) \\ &\equiv p^2\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}\end{aligned}$$

and

$$2p\left(H_{(p-1)/3} - H_{(2p-2)/3}\right) \equiv -p^2\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

So by $\binom{-\frac{1}{2}}{\frac{p-1}{3}} \equiv \binom{\frac{p-1}{2}}{\frac{p-1}{3}} \pmod p$ and $\binom{p-1}{\frac{p-1}{3}} \equiv (-1)^{\frac{p-1}{3}} = 1 \pmod p$, we can immediately obtain that

$$S_5 \equiv 0 \pmod{p^3}.$$

This, with (3.22) yields

$$\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} \equiv \sum_{k=0}^{p-1}\frac{f_k}{2^k} \pmod{p^3}.$$

Now the proof of theorem 1.2 is complete. □

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