# VIII

## Weak interactions of kaons

The kaon is the lightest hadron having a nonzero strangeness quantum number. It is unstable and decays weakly into states with zero strangeness, containing pions, photons, and/or leptons. We shall consider decays in the leptonic, semileptonic, and hadronic sectors to illustrate aspects of both weak and strong interactions.

### VIII–1 Leptonic and semileptonic processes

#### *Leptonic decay*

The simplest weak decay of the charged kaon, denoted by the symbol $K_{\ell 2}$, is into purely leptonic channels $K^+ \to \mu^+\nu_\mu$, $K^+ \to e^+\nu_e$. Such decays are characterized by the constant $F_K$,

$$\langle 0 \left| \bar{s}\gamma_\mu\gamma_5 u \right| K^+(\mathbf{k})\rangle = i\sqrt{2}\, F_K k_\mu. \tag{1.1}$$

As discussed previously, because of $SU(3)$ breaking $F_K$ is about 20% larger than the corresponding pion decay constant $F_\pi$. As with the pion, but even more so because of the larger kaon mass, helicity arguments require strong suppression of the electron mode relative to that of the muon. The ratio of $e^+\nu_e$ to $\mu^+\nu_\mu$ decay rates, as in pion decay, provides a test of lepton universality [RPP 12],

$$\left.\frac{\Gamma^*_{K^+\to e^+\nu_e}}{\Gamma_{K^+\to\mu^+\nu_\mu}}\right|_{\text{expt}} = (2.488 \pm 0.012) \times 10^{-5}, \tag{1.2}$$

in good agreement with the suppression predicted theoretically [CiR 07],

$$\frac{\Gamma'_{K^+\to e^+\nu_e}}{\Gamma_{K^+\to\mu^+\nu_\mu}} = \frac{m_e^2}{m_\mu^2}\left[\frac{1 - m_e^2/m_K^2}{1 - m_\mu^2/m_K^2}\right]^2 (1+\delta) = (2.477 \pm 0.001) \times 10^{-5}, \tag{1.3}$$



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where $\delta = -0.04$ is the electromagnetic radiative correction including the bremsstrahlung component.[1] The notation $\Gamma'$ indicates that the experimenters have subtracted off the large structure-dependent components of $K^+ \to \ell^+\nu_\ell\gamma$ but have included the small bremsstrahlung component.

### *Kaon beta decay and $V_{\mathrm{us}}$*

The kaon beta decay reactions $K^+ \to \pi^0\ell^+\nu_\ell$ and $K^0 \to \pi^-\ell^+\nu_\ell$, called $K^+_{\ell 3}$ and $K^0_{\ell 3}$ respectively, also are important in Standard Model physics. They are each parameterized by two form factors,

$$\langle\pi^-(\mathbf{p})\left|\bar{s}\gamma_\mu u\right| K^0(\mathbf{k})\rangle = f_+^{K^0\pi^-}(q^2)\,(k+p)_\mu + f_-^{K^0\pi^-}(q^2)\,(k-p)_\mu\,,$$
$$\langle\pi^0(\mathbf{p})\left|\bar{s}\gamma_\mu u\right| K^+(\mathbf{k})\rangle = \left[\frac{f_+^{K^+\pi^0}(q^2)}{\sqrt{2}}\,(k+p)_\mu + \frac{f_-^{K^+\pi^0}(q^2)}{\sqrt{2}}\,(k-p)_\mu\right]. \quad (1.4)$$

Isospin invariance implies $f_\pm^{K^0\pi^-} = f_\pm^{K^+\pi^0} \equiv f_\pm$. $SU(3)$ symmetry can be invoked to relate these matrix elements to the strangeness-conserving transition $\pi^+ \to \pi^0\ell^+\nu_\ell$, resulting in $f_+(0) = -1$ and $f_-(0) = 0$. The deviation of $f_+(0)$ from unity is predicted to be second order in $SU(3)$ symmetry breaking, i.e., of order $\left(m_s - \hat{m}\right)^2$. This result, the Ademollo–Gatto [AdG 64] theorem, is proved by considering the commutation of quark vector charges,

$$\left[Q^{\bar{u}s}, Q^{\bar{s}u}\right] = Q^{\bar{u}u-\bar{s}s}, \quad (1.5)$$

where

$$Q^{\bar{i}j} \equiv \int d^3x\, \bar{q}^i(x)\gamma_0 q^j(x). \quad (1.6)$$

Taking matrix elements and inserting a complete set of intermediate states gives

$$1 = \sum_n \left(\left|\langle n\left|Q^{\bar{s}u}\right| K^0\rangle\right|^2 - \left|\langle n\left|Q^{\bar{u}s}\right| K^0\rangle\right|^2\right). \quad (1.7)$$

Finally, we isolate the single $\pi^-$ state from the sum and note that in the $SU(3)$ limit the charge operator can only connect the kaon to another state within the *same* $SU(3)$ multiplet. This implies[2]

$$\langle n \neq \pi^-\left|Q^{\bar{u}s}\right| K^0\rangle = \mathcal{O}(\epsilon), \quad (1.8)$$

[1] The dominant term here is the simple contact contribution $-3(\alpha/\pi)\ln(m_\mu/m_e)$ discussed in Sect. VII–1.
[2] This is easiest to obtain in the limit $\mathbf{p}_K \to \infty$.

where $\epsilon$ is a measure of $SU(3)$ breaking, and we thus conclude that

$$1 - \left[ f_+^{K^0\pi^-}(0) \right]^2 = \mathcal{O}(\epsilon^2), \tag{1.9}$$

which is the result we were seeking.

It is interesting that the $SU(2)$ mass difference $m_u \neq m_d$ can modify $f_+^{K^+\pi^0}(0)$ in first order despite the Ademollo–Gatto theorem. This can be seen by considering a $K^+$ in the formulae of Eq. (1.7). Now there exist two intermediate states in the same octet as the kaon, i.e. $\pi^0$ and $\eta^0$, and it is their *sum* which obeys the Ademollo–Gatto theorem,

$$\frac{1}{4} \left| f_+^{K^+\pi^0}(0) \right|^2 + \frac{3}{4} \left| f_+^{K^+\eta^0}(0) \right|^2 = 1 + \mathcal{O}(\epsilon^2). \tag{1.10}$$

In the isospin limit, each term must separately obey the theorem because of the isospin relation $f_+^{K^+\pi^0} = f_+^{K^0\pi^-}$. However, when $m_u \neq m_d$ each form factor in Eq. (1.10) can separately deviate from unity to first order in $m_u - m_d$ as long as the first order effect cancels in the sum. Indeed this is what happens, yielding (cf. Prob. VIII.1)

$$\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1 + \frac{3}{4} \left( \frac{m_d - m_u}{m_s - \hat{m}} \right) + \ell_{K\pi} \simeq 1.021, \tag{1.11}$$

where $\ell_{K\pi} = 0.004$ arises from chiral corrections at $\mathcal{O}(E^4)$ [GaL 85*b* 85*b*]. This number can also be easily extracted from experiment by using the ratio of $K^+$ and $K^0$ beta decay rates, with the result [CiR 07],

$$\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1.027 \pm 0.004, \tag{1.12}$$

in agreement with the prediction.

The prime importance of the $K_{\ell 3}$ process is that it provides the best determination of the weak mixing element $V_{\rm us}$. Because of the Ademollo-Gatto theorem, the reaction is protected from large symmetry breaking corrections. In addition, the use of chiral perturbation theory allows a reliable treatment of the reaction. The above study of the form factors indicates that the theory is under control within the limits of experimental precision. The value [LeR 84]

$$V_{\rm us} = 0.2253 \pm 0.0013 \tag{1.13}$$

follows from an analysis of the $K^0$ and $K^+$ decay rates.

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## VIII–2 The nonleptonic weak interaction

For leptonic and semileptonic processes, at most one hadronic current is involved. There exist also *nonleptonic* interactions, in which two hadronic charged weak currents are coupled by the exchange of $W^{\pm}$ gauge bosons,

$$\mathcal{H}_{\rm nl} = \frac{g_2^2}{8} \int d^4x \, D_F^{\mu\nu}(x, M_W) T \left( J_\mu^{\dagger \rm had}(x/2) \, J_\nu^{\rm had}(-x/2) \right),$$

$$J_\mu^{\rm had} = \left( \bar{u} \; \bar{c} \; \bar{t} \right) \mathbf{V} \gamma_\mu (1 + \gamma_5) \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \qquad (2.1)$$

with $\mathbf{V}$ being the CKM matrix, given in Eq. (II–4.17). Such interactions are difficult to analyze theoretically because the product of two hadronic currents is a complicated operator. If one imagines inserting a complete set of intermediate states between the currents, all states from zero energy to $M_W$ are important, and the product is singular at short distances. Thus, one needs to have theoretical control over the physics of low-, intermediate-, and high-energy scales in order to make reliable predictions. Because this is not the case at present, our predictive power is substantially limited.

Let us first consider the particular case of $\Delta S = 1$ nonleptonic decays. These are governed by the products of currents

$$\bar{d}\Gamma^\mu u \; \bar{u}\Gamma_\mu s, \qquad \bar{d}\Gamma^\mu c \; \bar{c}\Gamma_\mu s, \qquad \bar{d}\Gamma^\mu t \; \bar{t}\Gamma_\mu s, \qquad (2.2)$$

where $\Gamma_\mu \equiv \gamma_\mu(1 + \gamma_5)$ and color labels are suppressed. The first of these would naively be expected to be the most important, because kaons and pions predominantly contain $u, d, s$ quarks. However, the others also contribute through virtual effects. Some properties of the $\Delta S = 1$ nonleptonic interactions can be read off from these currents. The first product contains two flavor-$SU(3)$ octet currents, one carrying $I = 1/2$ and one carrying $I = 1$,

$$SU(3): \qquad (\mathbf{8} \otimes \mathbf{8})_{\rm symm} = \mathbf{8} \oplus \mathbf{27}, \qquad (2.3a)$$

$$\text{isospin}: \qquad \mathbf{1} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}} \oplus \frac{\mathbf{3}}{\mathbf{2}}, \qquad (2.3b)$$

where the symmetric product is taken because the two currents are members of the same octet. The singlet $SU(3)$ representation is excluded from Eq. (2.3a) because a $\Delta S = 1$ interaction changes the $SU(3)$ quantum numbers and hence cannot be an $SU(3)$ singlet. The other two products in Eq. (2.2) are purely $SU(3)$ octet and isospin one-half operators. The currents are also purely left-handed. Thus, the nonleptonic hamiltonian transforms under separate left-handed and right-handed chiral rotations as $(8_L, 1_R)$ and $(27_L, 1_R)$. These symmetry properties, valid regardless of

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the dynamical difficulties occurring in nonleptonic decay, allow one to write down effective chiral lagrangians for the nonleptonic kaon decays. The hamiltonian is a Lorentz scalar, charge neutral, $\Delta S = 1$ operator, and has the above specified chiral properties.

At order $E^2$, there exist two possible effective lagrangians for the octet part, viz., $\mathcal{L}_{\text{octet}} = \mathcal{L}_8 + \overline{\mathcal{L}}_8$, where in the notation of Sect. IV–6,

$$\mathcal{L}_8 = g_8 \operatorname{Tr}\left(\lambda_6 D_\mu U D^\mu U^\dagger\right), \qquad \overline{\mathcal{L}}_8 = \bar{g}_8 \operatorname{Tr}\left(\lambda_6 \chi U^\dagger\right) + \text{h.c.} \tag{2.4}$$

It can easily be checked that both $\mathcal{L}_8$ and $\overline{\mathcal{L}}_8$ are singlets under right-handed transformations, but transform as members of an octet for the left-handed transformations. The barred lagrangian in Eq. (2.4) can in fact be removed, so that it does not contribute to physical processes. This is seen in two ways. At the simplest level, direct calculation of $K \to 2\pi$ and $K \to 3\pi$ amplitudes using $\overline{\mathcal{L}}_8$, including all diagrams, yields a vanishing contribution. Alternatively, this can be understood by noting that in *QCD* the quantity $\chi$ appearing in Eq. (2.4) is proportional to the quark mass matrix, $\chi = 2B_0 m_q$. Thus, the effect of $\overline{\mathcal{L}}_8$ is equivalent to a modification of the mass matrix,

$$m_q \to m'_q = m_q + \bar{g}_8 \lambda_6 m_q. \tag{2.5}$$

This new mass matrix can be diagonalized by a chiral rotation

$$\operatorname{Tr}\left(m'_q U\right) \to \operatorname{Tr}\left(R m'_q L U\right) \equiv \operatorname{Tr}\left(m_D U\right), \tag{2.6}$$

with $m_D$ diagonal. The transformed theory clearly has conserved quantum numbers, as it is flavor diagonal. This means that the original theory also has conserved quantum numbers, one of which can be called strangeness. When particles are mass eigenstates, even in the presence of $\overline{\mathcal{L}}_8$, the kaon state does not decay. Hence, this $\overline{\mathcal{L}}_8$ can be discarded from considerations, leaving only $\mathcal{L}_8$ as responsible for octet $K$ decays [Cr 67]. This octet operator is necessarily $\Delta I = 1/2$ in character. Another allowed operator, transforming as $(27_L, 1_R)$, contains both $\Delta I = 1/2$ and $\Delta I = 3/2$ portions,

$$\mathcal{L}_{27} = \mathcal{L}_{27}^{(1/2)} + \mathcal{L}_{27}^{(3/2)}, \tag{2.7}$$

where

$$\mathcal{L}_{27}^{(1/2)} = g_{27}^{(1/2)} C_{ab}^{1/2} \operatorname{Tr}\left(\lambda^a \partial_\mu U U^\dagger \lambda^b \partial_\mu U U^\dagger\right) + \text{h.c.}, \tag{2.8a}$$
$$\mathcal{L}_{27}^{(3/2)} = g_{27}^{(3/2)} C_{ab}^{3/2} \operatorname{Tr}\left(\lambda^a \partial_\mu U U^\dagger \lambda^b \partial_\mu U U^\dagger\right) + \text{h.c.} \tag{2.8b}$$

The coefficients are given by

$$C^{1/2}_{6+i7/2,\,3} = 1, \qquad C^{1/2}_{4+i5/2,\,1-i2/2} = -\sqrt{2}, \qquad C^{1/2}_{6+i7/2,\,8} = -\frac{3\sqrt{3}}{2},$$
$$C^{3/2}_{6+i7/2,\,3} = 1, \qquad C^{3/2}_{4+i5/2,\,1-i2/2} = \frac{1}{\sqrt{2}}. \tag{2.9}$$

The complete classification at order $E^4$ is difficult, but has been obtained [KaMW 90]. We shall apply these lagrangians to the data in Sects. VIII–4, XII–6. There we shall see that $g_8 \gg g^i_{27}$, whereas naive expectations would have octet and 27-plet amplitudes being of comparable strength. This is part of the puzzle of the $\Delta I = 1/2$ rule. The reliable theoretical calculation of the nonleptonic decay amplitudes, which is tantamount to predicting the quantities $g_8$, $g^{(1/2)}_{27}$ and $g^{(3/2)}_{27}$, is one of the difficult problems mentioned earlier. It has not yet been convincingly accomplished. The best we can do is to describe the theoretical framework of the short distance expansion, to which we now turn.

## VIII–3 Matching to *QCD* at short distance

At short distances, the asymptotic freedom property of *QCD* allows a perturbative treatment of the product of currents. The philosophy is to use perturbative *QCD* to treat the strong interactions for energies $M_W \geq E \geq \mu$. The result is an effective lagrangian which depends on the scale $\mu$. Ultimately matrix elements must be taken which include the strong interaction below energy scale $\mu$ and the final result should be independent of $\mu$. The subject provides a classic example of the techniques of perturbative matching to effective lagrangians and the use of the renormalization group.

### *Short-distance operator basis*

As introduced in Sect. IV–7, the outcome of the short-distance calculation can be expressed as an effective nonleptonic hamiltonian expanded in a set of local operators with scale-dependent coefficients (*Wilson coefficients*) [Wi 69],

$$\mathcal{H}^{\Delta S=1}_{\rm nl} = \frac{G_F}{2\sqrt{2}} V^*_{\rm ud} V_{\rm us} \sum_i \mathcal{C}_i(\mu)\mathcal{O}_i. \tag{3.1}$$

As in any effective lagrangian, those operators of lowest dimension should be dominant. If the operator $O_i$ has dimension $d$, its Wilson coefficient obeys the scaling property $\mathcal{C}_i \sim M_W^{6-d}$. Let us first see how this hamiltonian is generated in perturbation theory. We can later use the renormalization group to sum the

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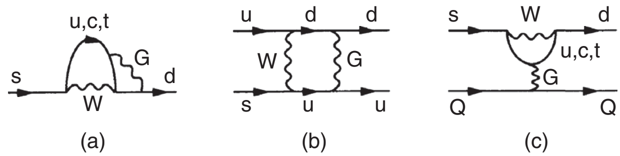


Fig. VIII–1 *QCD* Radiative corrections to the $\Delta S = 1$ nonleptonic hamiltonian.

leading logarithmic contributions. The lowest-order diagrams renormalizing the current product appear in Fig. VIII–1.

The process in Fig. VIII–1(a) corresponds to a left-handed, gauge-invariant operator of dimension 4,

$$O^{(d=4)} = \bar{d}\not{D}(1+\gamma_5)s. \tag{3.2}$$

This operator can be removed from consideration by a redefinition of the quark fields (cf. Prob. IV–1). The remaining operators are of dimension 6. Simple $W$ exchange with no gluonic corrections gives rise in the short-distance expansion to the local operator

$$O_A \equiv \bar{d}\gamma_\mu\,(1+\gamma_5)\,u\bar{u}\gamma^\mu\,(1+\gamma_5)\,s, \tag{3.3}$$

with a coefficient $\mathcal{C}_A = 2$ in the normalization of Eq. (3.1). The gluonic correction of Fig. VIII–1(b) generates an operator of the form

$$\bar{d}\gamma_\mu\,(1+\gamma_5)\,\lambda^a u\;\bar{u}\gamma^\mu\,(1+\gamma_5)\,\lambda^a s, \tag{3.4}$$

where the $\{\lambda^a\}$ are color $SU(3)$ matrices. However, use of the Fierz rearrangement property (see App. C) and the completeness property Eq. (II–2.8) of $SU(3)$ matrices allow this to be rewritten in color-singlet form

$$\bar{d}\gamma_\mu\,(1+\gamma_5)\,\lambda^a u\;\bar{u}\gamma^\mu\,(1+\gamma_5)\,\lambda^a s = -\frac{2}{3}O_A + 2O_B,$$

where

$$O_B \equiv \bar{u}\gamma_\mu\,(1+\gamma_5)\,u\bar{d}\gamma^\mu\,(1+\gamma_5)\,s. \tag{3.5}$$

The strong radiative correction is seen to generate a new operator $O_B$.

### *Perturbative analysis*

Consider now the one-loop renormalizations of the four-fermion interaction Fig. VIII–1(b). In calculating Feynman diagrams we typically encounter integrals such as (neglecting quark masses)

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$$I(\mu) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \frac{1}{k^2 - M_W^2} = -\frac{i}{16\pi^2 M_W^2} \ln \frac{\kappa^2}{\kappa^2 + M_W^2}\bigg|_{\mu}^{\infty}, \tag{3.6}$$

where we evaluate the integral at the lower end using a scale $\mu$. Clearly, $M_W$ presents a natural cut-off in the sense that

$$I(\mu) \simeq \frac{-i}{8\pi^2 M_W^2} \begin{cases} 0 & (\mu \sim M_W), \\ \ln \mu/M_W & (\mu \ll M_W). \end{cases} \tag{3.7}$$

The modification of the matrix element to first order in *QCD* is then

$$O_A \to O_A - \frac{g_3^2}{16\pi^2} \ln\left(\frac{M_W^2}{\mu^2}\right)(3O_B - O_A), \tag{3.8}$$

where $g_3$ is the quark–gluon coupling strength. The gluonic correction to $O_B$ must also be examined, and a similar analysis yields

$$O_B \to O_B - \frac{g_3^2}{16\pi^2} \ln\left(\frac{M_W^2}{\mu^2}\right)(3O_A - O_B). \tag{3.9}$$

We observe that the operators,

$$O_\pm = \frac{1}{2}(O_A \pm O_B), \tag{3.10}$$

are form-invariant, $O_\pm \to \mathcal{C}_\pm O_\pm$, with coefficients $\mathcal{C}_\pm$,

$$\mathcal{C}_\pm = 1 + d_\pm \frac{g_3^2}{16\pi^2} \ln \frac{M_W^2}{\mu^2}, \tag{3.11}$$

where $d_+ = -2$ and $d_- = +4$. The isospin content of the various operators can be determined in various ways. Perhaps the easiest method involves the use of raising and lowering operators [Ca 66],

$$I_+ d = u, \quad I_+ \bar{u} = -\bar{d}, \quad I_- u = d, \quad I_- \bar{d} = -\bar{u}, \tag{3.12}$$

to show that $I_+ O_- = 0$, implying that $O_-$ is the $I_z = 1/2$ member of an isospin doublet. With repeated use of raising and lowering operators, one can demonstrate that $O_-$ is purely $\Delta I = 1/2$ whereas $O_+$ is a combination of $\Delta I = 1/2$ and $\Delta I = 3/2$ operators.

From Eq. (3.11), we see that under one-loop corrections the operator $O_-$ is enhanced by the factor

$$\mathcal{C}_- = 1 + 4\frac{g_3^2}{16\pi^2} \ln \frac{M_W^2}{\mu^2} \simeq 2.1, \tag{3.13}$$

where we use $\alpha_s(\mu) \simeq 0.4$ ($\Lambda_{QCD} \simeq 0.2$ GeV) at $\mu \simeq 1$ GeV. Similarly $O_+$ is accompanied by the suppression factor

$$\mathcal{C}_+ = 1 - 2\frac{g_3^2}{16\pi^2}\ln\frac{M_W^2}{\mu^2} \simeq 0.4. \tag{3.14}$$

### *Renormalization-group analysis*

Choosing an even smaller value of $\mu$ would lead to an even larger correction. However, maintaining just the lowest-order perturbation in the $QCD$ interaction would then be unjustified. It is possible to do better than the lowest-order perturbative estimate by using the renormalization group to sum the logarithmic factors [GaL 74, AlM 74]. In a renormalizable theory physically measurable quantities can be written as functions of couplings which are renormalized at a renormalization scale $\mu_R$. Physical quantities calculated in the theory must be independent of $\mu_R$. Denoting an arbitrary physical quantity by $Q$, this may be written

$$Q = f\left(g_3(\mu_R), \mu_R\right), \tag{3.15}$$

where $f$ is some function of $\mu_R$ and $g_3$ is the strong coupling constant of $QCD$. Differentiating with respect to $\mu_R$, we have

$$\mu_R\frac{d}{d\mu_R}f\left(g_3(\mu_R), \mu_R\right) = 0, \tag{3.16}$$

which is the renormalization-group equation. It represents the feature that a change in the renormalization scale must be compensated by a modification of the coupling constants, leaving physical quantities invariant. In order to see how this program can be carried out for the effective weak hamiltonian, consider the following irreducible vertex function which represents a typical weak nonleptonic matrix element,

$$\begin{aligned}&\left\langle 0\left|T\left(J_\lambda^{\text{had}^\dagger}(x)J^\lambda_{\text{had}}(0)q_1(p_1)q_2(p_2)\bar{q}_3(p_3)\bar{q}_4(p_4)\right)\right|0\right\rangle^{\text{irr}}_{\text{ren}}\\ &\quad= \left(\sqrt{Z_2}\right)^4\left\langle 0\left|T\left(J_\lambda^{\text{had}^\dagger}(x)J^\lambda_{\text{had}}(0)q_1(p_1)q_2(p_2)\bar{q}_3(p_3)\bar{q}_4(p_4)\right)\right|0\right\rangle^{\text{irr}}_{\text{unren}},\end{aligned} \tag{3.17}$$

where the $\{q_i\}$ are quark fields carrying momenta $\{p_i\}$. $Z_2$ is the quark wavefunction renormalization constant for the fermion field, and subscripts 'ren', 'unren' denote renormalized and unrenormalized quantities.

Choosing the subtraction point $p_i^2 = -\mu_R^2$, we require that unrenormalized quantities be independent of $\mu_R$,

$$\mu_R\frac{d}{d\mu_R}\left\langle 0\left|T\left(J_\lambda^\dagger J^\lambda q_1q_2\bar{q}_3\bar{q}_4\right)\right|0\right\rangle^{\text{irr}}_{\text{unren}} = 0. \tag{3.18}$$

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This implies

$$\left(\mu_R\frac{\partial}{\partial\mu_R}+\beta_{QCD}\frac{\partial}{\partial g_{3r}}-4\gamma_F\right)\left\langle 0\left|T\left(J_\lambda^\dagger J^\lambda q_1q_2\bar{q}_3\bar{q}_4\right)\right|0\right\rangle_{\rm ren}^{\rm irr}=0,$$

where $g_{3r}$ is the renormalized strong coupling constant, $\beta_{QCD}$ is the $QCD$ beta function of Eq. (II–2.57(b)) and $\gamma_F$ is the quark field anomalous dimension of Eq. (II–2.69). As we have seen, $QCD$ radiative corrections generally mix the local operators appearing in the short-distance expansion,

$$\langle 0\,|T\,(O_nq_1q_2\bar{q}_3\bar{q}_4)|\,0\rangle_{\rm ren}^{\rm irr}=\sum_{n'}X_{nn'}\langle 0\,|T\,(O_{n'}q_1q_2\bar{q}_3\bar{q}_4)|\,0\rangle_{\rm unren}^{\rm irr}, \tag{3.19}$$

and the mixing matrix can be diagonalized to obtain a set of multiplicatively renormalized operators

$$\langle 0\,|T\,(O_kq_1q_2\bar{q}_3\bar{q}_4)|\,0\rangle_{\rm ren}^{\rm irr}=Z_k\langle 0\,|T\,(O_kq_1q_2\bar{q}_3\bar{q}_4)|\,0\rangle_{\rm unren}^{\rm irr}. \tag{3.20}$$

If operator $O_k$ has anomalous dimension $\gamma_k$, we can write

$$Z_k\sim 1+\gamma_k\ln\mu_R+\cdots, \tag{3.21}$$

and so the coefficient functions $\mathcal{C}_k(\mu_Rx)$ satisfy

$$\left(\mu_R\frac{\partial}{\partial\mu_R}+\beta_{QCD}\frac{\partial}{\partial g_{3r}}+\gamma_k-4\gamma_F\right)\mathcal{C}_k(\mu_Rx)=0. \tag{3.22}$$

From the above, we have for the operators $O_\pm$

$$\gamma_\pm-4\gamma_F\to\frac{g_3^2}{16\pi^2}d_\pm. \tag{3.23}$$

We can solve Eq. (3.22) with methods analogous to those employed in Sect. II–2. That is, because $QCD$ is asymptotically free and we are working at large momentum scales, we can use the perturbative result (cf. Eq. (II–2.57(b))),

$$\beta_{QCD}(g_{3r})=\mu_R\frac{\partial g_{3r}}{\partial\mu_R}=-\frac{g_{3r}^3}{16\pi^2}b+\cdots, \tag{3.24}$$

where $b=11-\frac{2}{3}n_f$, $n_f$ being the number of quark flavors. Upon inserting the leading term in the perturbative expression for $\alpha_s$ (cf. Eq. (II–2.74)),

$$\alpha_s(\mu_R)=\frac{12\pi}{(33-2n_f)\ln\mu_R^2/\Lambda^2}, \tag{3.25}$$

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one can verify that the solution to Eq. (3.22) is given by

$$\frac{\mathcal{C}_\pm(\mu_R)}{\mathcal{C}_\pm(M_W)} = \left(1 + \frac{g_3^2}{16\pi^2} b \ln \frac{M_W^2}{\mu_R^2}\right)^{d_\pm/b} . \tag{3.26}$$

Note that in the perturbative regime where $\alpha_s \ll 1$, we have

$$\frac{\mathcal{C}_\pm(\mu_R)}{\mathcal{C}_\pm(M_W)} = 1 + d_\pm \frac{g_3^2}{16\pi^2} \ln \frac{M_W^2}{\mu_R^2}, \tag{3.27}$$

which agrees with our previous result, Eq. (3.11). It is the renormalization group which has allowed us to sum all the 'leading logs'. Of course, at scale $M_W$ one must be able to reproduce the original weak hamiltonian, implying $\mathcal{C}_+(M_W) = \mathcal{C}_-(M_W) = 1$. Taking $\mu_\mathrm{R} \simeq 1$ GeV and $\alpha_s = 0.4$ as before, we find

$$\mathcal{H}_{\mathrm{nl}}^{\Delta S=1}(\mu_R) \;\propto\; \mathcal{C}_+(\mu_R) O_+ + \mathcal{C}_-(\mu_R) O_-, \tag{3.28}$$

with

$$\mathcal{C}_-(\mu_R) \simeq 1.5, \qquad \mathcal{C}_+(\mu_R) \simeq 0.8. \tag{3.29}$$

We observe then a $\Delta I = 1/2$ enhancement of a factor of two or so, which is encouraging but still considerably smaller than the experimental value of $A_0/A_2 \sim 22$ discussed in the next section.

Two additions to the above analysis must now be addressed. One is the proper treatment of heavy-quark thresholds. In reducing the energy scale from $M_W$ down to $\mu_R$, one passes through regions where there are successively six, five, four, or three light quarks, the word 'light' meaning relative to the energy scale $\mu_R$. The beta function changes slightly from region to region. A proper treatment must apply the renormalization group scheme in each sector separately. This is a straightforward generalization of the procedures described above.

The other addition is the inclusion of *penguin* diagrams of Fig. VIII–1(c) [ShVZ 77, ShVZ 79*b*, BiW 84], whimsically named because of a rough resemblance to this antarctic creature. The gluonic penguin is noteworthy because it is purely $\Delta I = 1/2$, thus helping to build a larger $\Delta I = 1/2$ amplitude, and because it is the main source of *CP* violation in the $\Delta S = 1$ hamiltonian. The electroweak penguin, wherein the gluon is replaced by a photon or a $Z^0$ boson, also enters the theory of *CP* violation. The *CP*-conserving portion of the penguin diagrams involves a GIM cancelation between the $c, u$ quarks and hence enters significantly at scales below the charmed quark mass. On the other hand, in the *CP* violating component, the GIM cancelation is between the $t, c$ quarks and thus this piece is short-distance dominated. At lowest order, before renormalization-group enhancement, one obtains the following Hamiltonians for the penguin and electroweak penguin interactions (cf. Fig. VIII–2),

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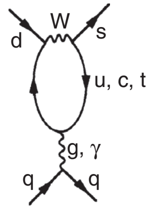


Fig. VIII–2 Penguin diagram.

$$\begin{aligned}\mathcal{H}_{\mathrm{w}}^{(\mathrm{peng})} &= -\frac{G_F \alpha_s}{12\pi\sqrt{2}}\left[V_{\mathrm{ud}}^* V_{\mathrm{us}} \ln\frac{m_c^2}{\mu_R^2} + V_{\mathrm{td}}^* V_{\mathrm{ts}} \ln\frac{m_t^2}{m_c^2}\right] \bar{d}\gamma_\mu(1+\gamma_5)\lambda^a s \bar{q}\gamma^\mu \lambda^a q, \\ \mathcal{H}_{\mathrm{w}}^{(\mathrm{ewp})} &= -\frac{2G_F \alpha}{9\pi\sqrt{2}}\left[V_{\mathrm{ud}}^* V_{\mathrm{us}} \ln\frac{m_c^2}{\mu_R^2} + V_{\mathrm{td}}^* V_{\mathrm{ts}} \ln\frac{m_t^2}{m_c^2}\right] \bar{d}\gamma_\mu(1+\gamma_5) d \bar{q}\gamma^\mu Q_q q.\end{aligned} \tag{3.30}$$

We have used a scale $\mu_R$ instead of the up quark mass and have quoted only the logarithmic $m_t$ dependence. The quarks $q = u, d, s$ are summed over and $Q_q$ is the charge of quark $q$. Note that since the vector current can be written as a sum of left-handed and right-handed currents, this is the only place where right-handed currents enter $\mathcal{H}_{\mathrm{w}}$. The gluonic penguin contains the right-handed current in an $SU(3)$ singlet, hence retaining the $(8_L, 1_R)$ property of $\mathcal{H}_{\mathrm{w}}$. However, the electroweak penguin introduces a small $(8_L, 8_R)$ component.

The full result can be described with the four-quark $\Delta S = 1$ operators,

$$\begin{aligned} &O_1 = H_A - H_B, && O_4 = H_A + H_B - H_C, \\ &O_2 = H_A + H_B + 2H_C + 2H_D, && O_5 = \bar{d}\gamma_\mu(1+\gamma_5)\lambda^a s\, \bar{q}\gamma_\mu(1-\gamma_5)\lambda^a q, \\ &O_3 = H_A + H_B + 2H_C - 3H_D, && O_6 = \bar{d}\gamma_\mu(1+\gamma_5) s\, \bar{q}\gamma_\mu(1-\gamma_5) q, \\ &O_7 = \frac{3}{2}\bar{s}\gamma_\mu(1+\gamma_5) d\, \bar{q}\gamma^\mu(1-\gamma_5) Q_q q, \\ &O_8 = -\frac{3}{2}\bar{s}_i\gamma_\mu(1+\gamma_5) d_j\, \bar{q}_j\gamma^\mu(1-\gamma_5) Q_q q_i, \end{aligned} \tag{3.31}$$

where $q = u, d, s$ are summed over in $O_{5,6,7,8}$, $i$ and $j$ are color labels, $Q_q$ is the charge of quark $q$ and

$$\begin{aligned} &H_A = \bar{d}\gamma_\mu(1+\gamma_5) u\, \bar{u}\gamma^\mu(1+\gamma_5) s, && H_C = \bar{d}\gamma_\mu(1+\gamma_5) s\, \bar{d}\gamma^\mu(1+\gamma_5) d, \\ &H_B = \bar{d}\gamma_\mu(1+\gamma_5) s\, \bar{u}\gamma^\mu(1+\gamma_5) u, && H_D = \bar{d}\gamma_\mu(1+\gamma_5) s\, \bar{s}\gamma^\mu(1+\gamma_5) s. \end{aligned} \tag{3.32}$$

The operators are arranged such that $O_{1,2,5,6}$ have octet and $\Delta I = 1/2$ quantum numbers, $O_3(O_4)$ are in the 27-plet with $\Delta I = 1/2(\Delta I = 3/2)$, while $O_{7,8}$ arise only from the electroweak penguin diagram. The full hamiltonian is

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$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = \frac{G_F}{2\sqrt{2}} V_{\text{ud}}^* V_{\text{us}} \sum_{i=1}^{8} \mathcal{C}_i O_i . \tag{3.33}$$

A renormalization-group analysis of the coefficients [BuBH 90] yields

$$\begin{aligned}
\mathcal{C}_1 &= 1.90 - 0.62\tau, & \mathcal{C}_5 &= -0.011 - 0.079\tau, \\
\mathcal{C}_2 &= 0.14 - 0.020\tau, & \mathcal{C}_6 &= -0.001 - 0.029\tau, \\
\mathcal{C}_3 &= \mathcal{C}_4/5, & \mathcal{C}_7 &= -0.009 - (0.010 - 0.004\tau)\alpha, \\
\mathcal{C}_4 &= 0.49 - 0.005\tau, & \mathcal{C}_8 &= (0.002 + 0.160\tau)\alpha,
\end{aligned} \tag{3.34}$$

with $\Lambda \simeq 0.2$ GeV, $\mu_R \simeq 1$ GeV, $m_t = 150$ GeV, and $\tau = -V_{\text{td}}^* V_{\text{ts}} / V_{\text{ud}}^* V_{\text{us}}$. The number multiplying $\tau$ has a dependence on $m_t$ whereas (within the leading logarithm approximation) the remainder does not if $m_t > M_W$. The $\tau$ dependence in $\mathcal{C}_4$ arises only because of the electroweak penguin diagram. This hamiltonian summarizes the *QCD* short-distance analysis and is the basis for estimates of weak amplitudes. For a treatment of corrections beyond those of leading order, see [BuBL 96].

## VIII–4 The $\Delta I = 1/2$ rule

### *Phenomenology*

In the decays $K \to \pi\pi$, the $S$-wave two-pion final state has a total isospin of either 0 or 2 as a consequence of Bose symmetry. Thus, such decays can be parameterized (ignoring the tiny effect of *CP* violation) as

$$\begin{aligned}
A_{K^0 \to \pi^+\pi^-} &= A_0 \, e^{i\delta_0} + \frac{A_2}{\sqrt{2}} e^{i\delta_2}, \\
A_{K^0 \to \pi^0\pi^0} &= A_0 \, e^{i\delta_0} - \sqrt{2}\, A_2 e^{i\delta_2}, \\
A_{K^+ \to \pi^+\pi^0} &= \frac{3}{2} A_2' \, e^{i\delta_2},
\end{aligned} \tag{4.1}$$

where the subscripts 0, 2 denote the total $\pi\pi$ isospin and the strong interaction $S$-wave $\pi\pi$ phase shifts $\delta_I$ enter as prescribed by Watson's theorem (cf. Eq. (C–3.15)). There are, in principle, two distinct $I = 2$ amplitudes $A_2$ and $A_2'$. These are equal if there are no $\Delta I = 5/2$ components in the weak transition, as is the case in the Standard Model if electromagnetic corrections are neglected. Including electromagnetism leads to a small difference between $A_2$ and $A_2'$ [CiENPP 12], but we will neglect this possibility from now on, and employ just the two isospin amplitudes $A_0$ and $A_2$. The experimental decay rates themselves imply

$$\begin{aligned}|A_{K^0\to\pi^+\pi^-}| &= (2.772 \pm 0.0013) \times 10^{-7}\ \text{GeV},\\ |A_{K^0\to\pi^0\pi^0}| &= (2.592 \pm 0.0022) \times 10^{-7}\ \text{GeV},\\ |A_{K^+\to\pi^+\pi^0}| &= (0.1811 \pm 0.0004) \times 10^{-7}\ \text{GeV}.\end{aligned} \tag{4.2a}$$

The $\pi\pi$ phase difference $\delta_0 - \delta_2$ can be obtained via

$$\begin{aligned}\cos(\delta_0 - \delta_2) &= \frac{\sqrt{3}}{2\sqrt{2}} \cdot \frac{|\mathcal{A}_{+-}|^2 - |\mathcal{A}_{00}|^2 + 2|\mathcal{A}_{+0}|^2/3}{|\mathcal{A}_{+0}|\left[2|\mathcal{A}_{+-}|^2 + |\mathcal{A}_{00}|^2 - 4|\mathcal{A}_{+0}|^2/3\right]^{1/2}},\\ \delta_0 - \delta_2 &= (44.55 \pm 1.04)^0.\end{aligned} \tag{4.2b}$$

This is consistent with the phase difference which emerges from the analysis of $\pi\pi$ scattering. The magnitude of the isospin amplitudes can be found to be

$$\begin{aligned}|A_0| &= (2.711 \pm 0.0011) \times 10^{-7}\ \text{GeV},\\ |A_2| &= (1.207 \pm 0.0026) \times 10^{-8}\ \text{GeV}.\end{aligned} \tag{4.3}$$

The ratio of magnitudes,

$$|A_2/A_0| = 0.0445 \pm 0.0001 \ \simeq\ 1/22.47, \tag{4.4}$$

indicates a striking dominance of the $\Delta I = 1/2$ amplitude (which contributes to $A_0$) over the $\Delta I = 3/2$ amplitude (which contributes only to $A_2$). This enhancement of $A_0$ over $A_2$, together with related manifestations to be discussed later, is called the $\Delta I = 1/2$ rule. As we have seen in previous sections, a naive estimate (and even determinations which are less naive!) do not suggest this much of an enhancement. However, the factor 22.5 dominance of $\Delta I = 1/2$ effects over those with $\Delta I = 3/2$ is common to both kaon and hyperon decay.[3]

A similar enhancement of $\Delta I = 1/2$ is found in the $K \to \pi\pi\pi$ channel. In this case, it is customary to expand the transition amplitude about the center of the Dalitz plot. For the decay amplitude $K(k) \to \pi(p_1)\ \pi(p_2)\ \pi(p_3)$, the relevant variables are

$$\begin{aligned}s_i &= (k - p_i)^2\Big|_{i=1,2,3}, \qquad s_0 = \frac{1}{3}(s_1 + s_2 + s_3) = \frac{m_K^2}{3} + m_\pi^2,\\ \bar{x} &= \frac{s_1 - s_2}{s_0}, \qquad\qquad \bar{y} = \frac{s_3 - s_0}{s_0},\end{aligned} \tag{4.5}$$

where $s_3$ labels the 'odd' pion, i.e. the third pion in each of the final states $\pi^+\pi^-\pi^0$, $\pi^0\pi^0\pi^+$, $\pi^+\pi^+\pi^-$. The large $\Delta I = 1/2$ amplitudes are considered up to quadratic

[3] Although our discussion stresses the relative magnitudes of the $\Delta I = 1/2, 3/2$ amplitudes, the relative *phases* of these amplitudes turns out to place important restrictions on the structure of the $|\Delta S| = 1$ nonleptonic hamiltonian [GoH 75].

order in these variables while the $\Delta I = 3/2$ amplitudes contain only constant plus linear terms,

$$\begin{aligned}
\sqrt{2}\, A_{K^0\to\pi^+\pi^-\pi^0} &= a_1 - 2a_3 + (b_1 - 2b_3)\bar{y}e^{i\delta_{M1}} - \frac{2}{3}b_{23}\bar{x}e^{i\delta_{21}} \\
&\quad + c\left(\bar{y}^2 + \frac{1}{3}\bar{x}^2\right) + d\left(\bar{y}^2 - \frac{1}{3}\bar{x}^2\right)e^{i\delta_{M1}}, \\
\sqrt{2}\, A_{K^0\to\pi^0\pi^0\pi^0} &= 3(a_1 - 2a_3) + 3c\left(\bar{y}^2 + \frac{1}{3}\bar{x}^2\right), \\
A_{K^+\to\pi^+\pi^+\pi^-} &= 2(a_1 + a_3) - (b_1 + b_3)\bar{y}e^{i\delta_{M1}} + b_{23}\bar{y}e^{i\delta_{21}} \\
&\quad + 2c\left(\bar{y}^2 + \frac{1}{3}\bar{x}^2\right) + d\left(\bar{y}^2 - \frac{1}{3}\bar{x}^2\right)e^{i\delta_{M1}}, \\
A_{K^+\to\pi^0\pi^0\pi^+} &= a_1 + a_3 + (b_1 + b_3)\bar{y}e^{i\delta_{M1}} + b_{23}\bar{y}e^{i\delta_{21}} \\
&\quad + c\left(\bar{y}^2 + \frac{1}{3}\bar{x}^2\right) + d\left(\bar{y}^2 - \frac{1}{3}\bar{x}^2\right)e^{i\delta_{M1}}, \qquad (4.6)
\end{aligned}$$

where $a_1, b_1, c, d$ are $\Delta I = 1/2$ amplitudes, $a_3, b_3, b_{23}$ are $\Delta I = 3/2$ amplitudes, and the phases $\{\delta_I\}$ in $\delta_{M1}$ ($\equiv \delta_M - \delta_1$) and $\delta_{21}$ ($\equiv \delta_2 - \delta_1$) refer to final-state phase shifts in the $I = 1, 2$ and mixed symmetry $I = 1$ states respectively. Because of the relatively small $Q$ value for such decays ($Q_{\pi\pi\pi} = m_K - 3m_\pi \simeq 75\,\text{MeV}$), such phases are presumably small and are often omitted. Also, this representation in terms of simple energy-independent phase factors is clearly idealistic. Analysis of the available data [BiBD 03] yields (in units of $10^{-7}$)

$$\begin{aligned}
a_1 &= 9.32 \pm 0.04, \quad a_3 = 0.34 \pm 0.03, \\
b_1 &= 14.2 \pm 0.2, \quad b_3 = -0.6 \pm 0.1, \quad b_{23} = 2.7 \pm 0.3, \\
c &= -1.1 \pm 0.5, \quad d = -5.0 \pm 0.8.
\end{aligned} \qquad (4.7)$$

Dominance of the $\Delta I = 1/2$ signal is again clear in magnitude and in slope terms, e.g., we find at the center of the Dalitz plot,

$$|a_3/a_1| \simeq 1/27. \qquad (4.8)$$

In $SU(3)$ language, the dominance of $\Delta I = 1/2$ effects over $\Delta I = 3/2$ implies the dominance of octet transitions over those involving the 27-plet. This is a consequence, within the Standard Model, of the fact that the $\Delta I = 1/2$ 27-plet operator contributes relative to the $\Delta I = 3/2$ 27-plet operator with a fixed strength given by the scale-independent ratio of coefficients $\mathcal{C}_3/\mathcal{C}_4 \simeq 1/5$ (viz. Eq. (3.34)). The 27-plet operator then gives only a small contribution to the $\Delta I = 1/2$ amplitudes, with the major portion coming from the octet operators. We shall therefore ignore the $\Delta I = 1/2$ 27-plet contribution henceforth.

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### *Chiral lagrangian analysis*

The left-handed chiral property of the Standard Model may be directly tested by the use of chiral symmetry to relate the amplitudes in $K \to \pi\pi\pi$ to those in $K \to \pi\pi$. We have already constructed the effective lagrangians for $(8_L, 1_R)$ and $(27_L, 1_R)$ transitions. Dropping $g_{27}^{(1/2)}$, the nonleptonic decays are described by the two parameters $g_8$ and $g_{27}^{(3/2)}$ at $\mathcal{O}(E^2)$. Let us see how well this parameterization works, and afterwards add $\mathcal{O}(E^4)$ corrections. The two free parameters may be determined from $A_0$ and $A_2$ in $K \to \pi\pi$ decays. From the chiral lagrangians of Eqs. (2.4), (2.8b), we find

$$A_0 = \frac{\sqrt{2}\, g_8}{F_\pi^3}\left(m_K^2 - m_\pi^2\right), \qquad A_2 = \frac{2\, g_{27}^{(3/2)}}{F_\pi^3}\left(m_K^2 - m_\pi^2\right), \tag{4.9}$$

which yields upon comparison with Eq. (4.3),

$$g_8 \simeq 7.8 \times 10^{-8} F_\pi^2, \qquad g_{27}^{(3/2)} \simeq 0.25 \times 10^{-8} F_\pi^2. \tag{4.10}$$

The $K \to \pi\pi\pi$ amplitude may be predicted from these. Because there are only two factors of the energy, no quadratic terms are present in the predictions,

$$\begin{aligned}
A^{(1/2)}_{K_L^0 \to \pi^+\pi^-\pi^0} &= \frac{\sqrt{2} A_0 m_K^2}{6F_\pi (m_K^2 - m_\pi^2)}\left[1 + \frac{m_K^2 + 3m_\pi^2}{m_K^2}\,\bar{y}\right], \\
A^{(3/2)}_{K_L^0 \to \pi^+\pi^-\pi^0} &= -\frac{A_2 m_K^2}{3F_\pi (m_K^2 - m_\pi^2)}\left[1 - \frac{5}{4}\frac{m_K^2 + 3m_\pi^2}{m_K^2}\,\bar{y}\right], \\
A^{(3/2)}_{K^+ \to \pi^+\pi^+\pi^-} &= \frac{A_2 m_K^2}{3F_\pi (m_K^2 - m_\pi^2)}\left[1 + 4\frac{m_K^2 + 3m_\pi^2}{m_K^2}\,\bar{y}\right],
\end{aligned} \tag{4.11a}$$

which correspond to the numerical values (again in units of $10^{-7}$),

$$\begin{aligned}
A^{(1/2)}_{K_L^0 \to \pi^+\pi^-\pi^0} &= 7.5 + 9.1\,\bar{y}, \\
A^{(3/2)}_{K_L^0 \to \pi^+\pi^-\pi^0} &= -0.47 + 0.74\,\bar{y}, \\
A^{(3/2)}_{K^+ \to \pi^+\pi^+\pi^-} &= 0.47 + 2.3\,\bar{y}.
\end{aligned} \tag{4.11b}$$

These are to be compared to the experimental results,

$$\begin{aligned}
A^{(1/2)}_{K_L^0 \to \pi^+\pi^-\pi^0} &= 9.32 + 14.2\,\bar{y} - 6.1\,\bar{y}^2 + 1.3\,\bar{x}^2, \\
A^{(3/2)}_{K_L^0 \to \pi^+\pi^-\pi^0} &= -0.68 + 1.2\,\bar{y}, \\
A^{(3/2)}_{K^+ \to \pi^+\pi^+\pi^-} &= 0.68 + 3.3\,\bar{y}.
\end{aligned} \tag{4.12}$$

This comparison can be seen in Fig. VIII–3, where a slice across the Dalitz plot is given. Also shown are the extrapolations outside the physical region to the 'soft-pion point' where either $p_+^\mu \to 0$ or $p_0^\mu \to 0$. Predictions at these locations are obtained by using the soft-pion theorem (see Prob. VIII–2).

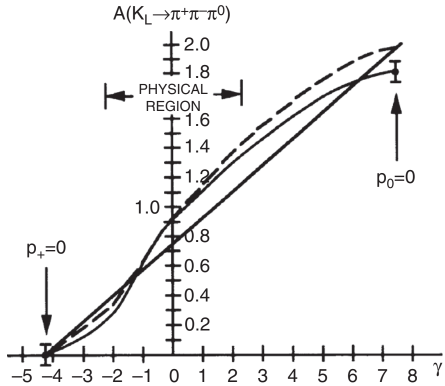


Fig. VIII–3 Dalitz plot.

The chiral relations clearly capture the main features of the amplitude and demonstrate that the $K \to 3\pi$ $\Delta I = 1/2$ enhancement is not independent of that observed in $K \to 2\pi$ decay. However, for the $\Delta I = 1/2$ transitions we may do somewhat better. The kinematic dependence of $\bar{x}^2$ or $\bar{y}^2$ can come only from a chiral lagrangian with four factors of the momentum, and only two combinations are possible:

$$\Lambda_{\text{quad}} = \gamma_1 k \cdot p_0 p_+ \cdot p_- + \gamma_2 \left(k \cdot k_+ p_0 \cdot p_- + k \cdot p_- p_0 \cdot p_+\right). \tag{4.13}$$

Such behavior can be generated from a variety of chiral lagrangians,

$$\begin{aligned}\mathcal{L}_{\text{quad}} = {} & g_8' \operatorname{Tr}\left(\lambda_6 \partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger\right) \\ & + g_8'' \operatorname{Tr}\left(\lambda_6 \partial_\mu U \partial_\nu U^\dagger \partial^\mu U \partial^\nu U^\dagger\right) + \cdots .\end{aligned} \tag{4.14}$$

However the predictions in terms of $\gamma_i$ are unique. Fitting the quadratic terms to determine $\gamma_1, \gamma_2$ yields the full amplitude,

$$\sqrt{2}\, A^{(1/2)}_{K^0 \to \pi^+\pi^-\pi^0} = (9.5 \pm 0.7) + (16.0 \pm 0.5)\, \bar{y} - 4.85\, \bar{y} + 0.88\, \bar{x}^2, \tag{4.15}$$

which provides an excellent representation of the data. Final-state interaction effects also provide an important contribution and must be included in a complete analysis [KaMW 90]. Note that in the process of determining the quadratic coefficients, the constant and linear terms have also become improved. This process cannot be repeated for $\Delta I = 3/2$ amplitudes due to a lack of data on quadratic terms.

### *Vacuum saturation*

Direct calculations of nonleptonic amplitudes have proven very difficult to perform. On the whole, no single analytical or numerical method for overcoming the

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general problem yet exists. In the following, we describe the simplest analytical approach, called *vacuum saturation*, which often serves as a convenient benchmark with which to compare the theory. For convenience we consider only $O_1$ (the largest $\Delta I = 1/2$ operator) and $O_4$ (the $\Delta I = 3/2$ operator),

$$\mathcal{H}_W \simeq \frac{G_F}{2\sqrt{2}} V_{\rm ud}^* V_{\rm us} (\mathcal{C}_1 O_1 + \mathcal{C}_4 O_4), \tag{4.16}$$

with $\mathcal{C}_1 \simeq 1.9$ and $\mathcal{C}_4 \simeq 0.5$. The vacuum saturation approximation consists of inserting the vacuum intermediate state between the two currents in all possible ways, e.g.,

$$\begin{aligned}
&\langle \pi^+(\mathbf{p}_+)\pi^-(\mathbf{p}_-) \left| \bar{d}\gamma^\mu (1+\gamma_5)\, u\bar{u}\gamma^\mu (1+\gamma_5)\, s \right| \bar{K}^0(\mathbf{k})\rangle \\
&\quad = \langle \pi^-(\mathbf{p}_-) \left| \bar{d}\gamma^\mu\gamma_5 u \right| 0\rangle \langle \pi^+(\mathbf{p}_+) \left| \bar{u}\gamma^\mu s \right| \bar{K}^0(\mathbf{k})\rangle \\
&\qquad + \langle \pi^+(\mathbf{p}_+)\pi^-(\mathbf{p}_-) \left| \bar{u}_\beta\gamma^\mu u_\alpha \right| 0\rangle \langle 0 \left| \bar{d}_\alpha\gamma^\mu\gamma_5 s_\beta \right| \bar{K}^0(\mathbf{k})\rangle \\
&\quad = -i\sqrt{2}\, F_\pi f_+ p_-^\mu (k + p_+)_\mu - \frac{i}{3}\sqrt{2}\, F_K f_+ k_\mu (p_- - p_+)^\mu .
\end{aligned} \tag{4.17}$$

In obtaining this result the Fierz rearrangement property

$$\bar{d}_\alpha\gamma^\mu (1+\gamma_5)\, u_\alpha \bar{u}_\beta\gamma^\mu (1+\gamma_5)\, s_\beta = \bar{d}_\alpha\gamma^\mu (1+\gamma_5)\, s_\beta \bar{u}_\beta\gamma^\mu (1+\gamma_5)\, u_\alpha$$

has been used, where $\alpha, \beta$ are color indices which are summed over. In addition, the color singlet property of currents is employed,

$$\langle 0 \left| \bar{d}_\alpha\gamma_\mu\gamma_5 s_\beta \right| \bar{K}^0(\mathbf{k})\rangle = i\sqrt{2}\, F_K k_\mu \frac{\delta_{\alpha\beta}}{3}. \tag{4.18}$$

Within the vacuum saturation approximation, we see that the amplitudes are expressed in terms of known semileptonic decay matrix elements. Putting in all of the constants, we find that

$$\begin{aligned}
A_0 &= \frac{G_F}{3} V_{\rm ud}^* V_{\rm us} F_\pi \left(m_K^2 - m_\pi^2\right) \mathcal{C}_1 \simeq 0.34 \times 10^{-7}\ \text{GeV}, \\
A_2 &= \frac{2\sqrt{2}\, G_F}{3} V_{\rm ud}^* V_{\rm us} F_\pi \left(m_K^2 - m_\pi^2\right) \mathcal{C}_4 \simeq 2.5 \times 10^{-8}\ \text{GeV}.
\end{aligned} \tag{4.19}$$

The above estimate of $A_2$ is seen to work reasonably well, but that of $A_0$ falls considerably short of the observed $\Delta I = 1/2$ amplitude. This demonstrates that vacuum saturation is not a realistic approximation. However, it does serve to indicate how much additional $\Delta I = 1/2$ enhancement is required to explain the data.

### *Nonleptonic lattice matrix elements*

While historically there have been many attempts to improve on the naive vacuum saturation method, the present state of the art is to rely on lattice calculations.

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However, nonleptonic matrix elements have been a particular challenge for lattice methods. The transition from a kaon to two pions requires three external sources to create the mesons involved, as well as a singular four-quark operator for the weak hamiltonian. In addition, there are diagrams where quarks in the hamiltonian form disconnected loops not connected to external states. Recent advances have allowed the extraction of the $A_2$ amplitude with reasonable precision [Bl *et al.* 12]

$$|A_2| = (1.381 \pm 0.046_{\text{stat}} \pm 0.258_{\text{syst}}) \times 10^{-8}\ \text{GeV} \tag{4.20}$$

consistent with the experimental result of Eq. (4.3). However, the isospin-zero final state in $A_0$ implies the existence of disconnected diagrams, which make the numerical evaluation difficult, and we do not yet have a reliable lattice calculation of $A_0$ [Bl *et al.* 11].

## VIII–5 Rare kaon decays

Thus far, we have discussed the dominant decay modes of the kaon. There are, however, many additional modes which, despite tiny branching ratios, have been the subject of intense experimental and theoretical activity. We can divide this activity into three main categories.

(1) Forbidden decays – These include tests of the flavor conservation laws of the Standard Model such as $K_L \to e^+\mu^-$. Positive signals would represent evidence for physics beyond our present theory.
(2) Rare decays within the Standard Model – These include decays which occur only at one-loop order. Such processes can be viewed as tests of chiral dynamics as developed in this and preceding chapters (e.g., radiative kaon decays) or as particularly sensitive to short-distance effects, which probe the particle content of the theory.
(3) *CP*-violation studies – There is now confirmation of *CP*-violating processes involving kaons and *B* mesons (and searches of the same for *D* mesons). Also, the observed baryon–antibaryon asymmetry of the Universe requires the existence of *CP* violation within the standard cosmological model. There remain, however, interesting opportunities for further studies of *CP* violation within the subfield of rare kaon decays.

Any of these have the potential to yield exciting physics. We shall content ourselves with discussing only a small sample of the many possibilities. Surveys of rare kaon modes appear in [CiENPP 12] and also in [RPP 12].

Consider first the rare decay $K^+ \to \pi^+\nu\bar{\nu}$, where the neutrino flavor $\nu = \nu_e, \nu_\mu, \nu_\tau$ is summed over. This mode is often called '$K^+$ to $\pi^+$ plus nothing', in reference to its unique experimental signature. This process can take place only

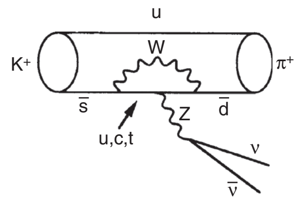


Fig. VIII–4 The decay $K^+ \to \pi^+ \nu_\ell \bar{\nu}_\ell$.

through loop diagrams, such as the ones in Fig. VIII–4. We content ourselves here to show just the effective hamiltonian for the dominant $t$-quark loop at leading order in *QCD*, [InL 81, HaL 89]

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \cdot \frac{\alpha}{2\pi \sin^2 \theta_{\text{w}}} V_{\text{ts}}^* V_{\text{td}} X_0(x_t) \sum_\ell \bar{\nu}_\ell \gamma^\mu (1+\gamma_5) \nu_\ell \, \bar{s} \gamma_\mu (1+\gamma_5) d, \quad (5.1)$$

where $x_t \equiv m_t^2/m_W^2$ and

$$X_0(x_t) = \frac{x_t}{8} \left( \frac{x_t + 2}{x_t - 1} + \frac{3x_t - 6}{(1 - x_t)^2} \right). \quad (5.2)$$

The overall factor of $m_t^2/m_W^2$ in $X_0(x_t)$ is associated with the GIM effect; the $c$-quark and $u$-quark amplitudes, were they included, would contain analogous mass factors such that in the limit $m_u = m_c = m_t$ the total amplitude would vanish via GIM cancelation. Although the calculation of many hadronic processes in this book are hindered by QCD uncertainties, such is not the case here. The quark matrix element is related by isospin to the known charged current amplitude

$$\langle \pi^+(\mathbf{p}) \left| \bar{s} \gamma_\mu d \right| K^+(\mathbf{k}) \rangle = \sqrt{2} \, \langle \pi^0(\mathbf{p}) \left| \bar{s} \gamma_\mu u \right| K^+(\mathbf{k}) \rangle = f_+(q^2) \, (k + p)_\mu \quad (5.3)$$

with $f_+(0) = -1$. This makes the $K^+ \to \pi^+ \nu_\ell \bar{\nu}_\ell$ example a theoretically 'clean' process and is responsible in large part for all the attention this transition has attracted. Our discussion is relatively brief, and a more careful analysis would include effects like *QCD* perturbative corrections, the $c$-quark loop contribution (roughly 30%), etc. A recent prediction, [BrGS 11], along with the current experimental result [RPP 12], reads

$$\text{Br}^{(\text{theo})}_{K^+ \to \pi^+ \nu_\ell \bar{\nu}_\ell} = (7.8 \pm 0.8) \times 10^{-11}, \quad \text{Br}^{(\text{expt})}_{K^+ \to \pi^+ \nu_\ell \bar{\nu}_\ell} = (1.7 \pm 1.1) \times 10^{-10}. \quad (5.4)$$

The experimental result, still consistent with zero, is reaching the sensitivity needed to probe the theory prediction and further advances are anticipated. The main source of uncertainty in the theory prediction is from CKM factors and quark mass values, which presumably can and will be improved upon.

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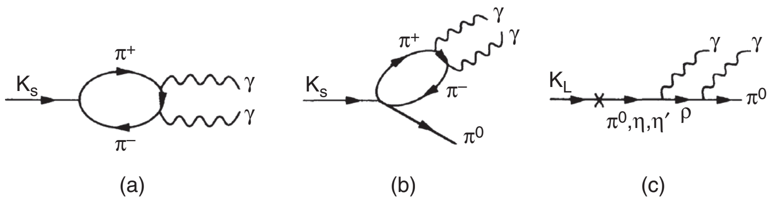


Fig. VIII–5 Long-distance contributions to radiative kaon decays.

A different class of rare decays consists of the radiative processes $K_S \to \gamma\gamma$ and $K_L \to \pi^0\gamma\gamma$. These transitions provide interesting tests of chiral perturbation theory at one-loop order. In this case, the long-distance process, Fig. VIII–5, is dominant. An important feature is that there is no tree-level contribution at order $E^2$ or $E^4$ from any of the strong or weak chiral lagrangians because all of the hadrons involved are neutral. Thus, the decays can only come from loop diagrams, or from lagrangians at $\mathcal{O}(E^6)$. There is also an interesting corollary of this result concerning the renormalization behavior of the loops. Since there are no tree-level counterterms at $\mathcal{O}(E^4)$ with which to absorb divergences from the loop diagrams, and recalling that we have proven all divergences can be handled in this fashion, it follows that the sum of the loop diagrams must be finite. This is in fact borne out by direct calculation.

For $K_S \to \gamma\gamma$, the prediction of chiral [D'AE 86, Go 86] loops is given in terms of known quantities such as

$$\begin{aligned}
\Gamma_{K_S\to\gamma\gamma} &= \frac{\alpha^2 m_K^2 g_8^2 F_\pi^2}{16\pi^3}\left(1-\frac{m_\pi^2}{m_K^2}\right)^2 \left|F\left(\frac{m_K^2}{m_\pi^2}\right)\right|^2, \\
F(z) &= 1 - z\left[\pi^2 - \ln^2 Q(z) - 2\pi i \ln Q(z)\right], \\
Q(z) &= \frac{1-\sqrt{1-4z}}{1+\sqrt{1-4z}},
\end{aligned} \tag{5.5}$$

where $g_8$ is the nonleptonic coupling defined previously in Eq. (2.4). Comparison of the theoretical one-loop branching ratio and the experimental result,

$$\mathrm{Br}^{(\mathrm{theo})}_{K_S\to\gamma\gamma} = 2.0\times 10^{-6}, \qquad \mathrm{Br}^{(\mathrm{expt})}_{K_S\to\gamma\gamma} = (2.63\pm 0.17)\times 10^{-6}, \tag{5.6}$$

shows reasonable agreement but implies the need to consider $\mathcal{O}(E^6)$ corrections. In particular, the 'unitarity correction' $K_S \to \pi^+\pi^- \to \gamma\gamma$ has been shown to provide an improved theoretical prediction [KaH 94].

The case of $K_L \to \pi^0\gamma\gamma$ is also instructive. Again, one-loop contributions are finite and unambiguous [EcPR 88]. Indeed, we know that $K_L \to \pi^0\gamma\gamma$ and $K_L \to \gamma\gamma$ are related by the soft-pion theorem in the limit $p_\pi^\mu \to 0$, yielding

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$$\frac{d\Gamma_{K_L\to\pi^0\gamma\gamma}}{dz} = \frac{\alpha^2 m_K^5}{(4\pi)^5} g_8^2 \left[\lambda\left(z, \frac{m_\pi^2}{m_K^2}\right)\right]^{1/2}$$
$$\times \left| \left(z - \frac{m_\pi^2}{m_K^2}\right) F(z\frac{m_K^2}{m_\pi^2}) + \left(1 - z + \frac{m_\pi^2}{m_K^2}\right) F(z) \right|^2, \qquad (5.7)$$

where $z = m_{\gamma\gamma}^2/m_K^2$ and

$$\lambda(a, b) \equiv 1 + a^2 + b^2 - 2(a + b + ab). \qquad (5.8)$$

If we compare the theoretical branching ratio based on the above description with the experimental value, we find

$$\mathrm{Br}^{(\mathrm{loop})}_{K_L\to\pi^0\gamma\gamma} = 0.68 \times 10^{-6}, \qquad \mathrm{Br}_{K_L^0\to\pi^0\gamma\gamma} = (1.273 \pm 0.033) \times 10^{-6}. \quad (5.9)$$

This indicates the need for an $\mathcal{O}(E^6)$ correction. Indeed, the most recent data input, from [Ab *et al*. (KTeV collab.) 08], provides evidence for a vector exchange contribution. It is easy to take this into account, viz. the diagram of Fig. VIII–5(c) shows the effect of $\rho$-exchange.

## Problems

(1) ***$K_{\ell 3}$* decay**

The ratio $f_+^{K^+\pi^0}(0)/f_+^{K^0\pi^-}(0)$ of semileptonic form factors is a measure of isospin violation. Part of this quantity arises from $\pi^0$-$\eta_8^0$ mixing.

(a) By diagonalizing the pseudoscalar meson mass matrix, show that $m_d \neq m_u$ induces the mixing $|\pi^0\rangle = \cos\epsilon\,|\varphi_3\rangle + \sin\epsilon\,|\varphi_8\rangle$ where $\epsilon \simeq \sqrt{3}(m_d - m_u)/[4(m_s - \hat{m})]$ and $\hat{m} \equiv (m_u + m_d)/2$.

(b) Demonstrate that this leads to the result (cf. Eq. (1.11))

$$\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = \frac{\cos\epsilon f_+^{K^+\varphi^3}(0) + \sin\epsilon f_+^{K^+\varphi^8}(0)}{f_+^{K^0\pi^-}(0)} \simeq 1 + \sqrt{3}\sin\epsilon.$$

(2) **Soft pions and $K \to 3\pi$ decay**

The results derived in Sect. VIII–4 with effective lagrangians can also be obtained by means of soft pion methods (see App. B–3).

(a) Using the soft pion theorem, show that the soft-pion limit of the $K \to 3\pi$ transition amplitude is given by

$$\lim_{q_a\to 0} \langle \pi^a_{q_a}\pi^b_{q_b}\pi^c_{q_c}|\mathcal{H}_{\mathrm{w}}(0)|K^n_k\rangle = \frac{-i}{F_\pi}\langle \pi^b_{q_b}\pi^c_{q_c}|[Q_5^a, \mathcal{H}_{\mathrm{w}}(0)]|K^n_k\rangle,$$

where $Q_5^a = \int d^3x\ A_0^a(\mathbf{x}, t)$ is the axial charge.

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(b) Demonstrate that this may be also written as

$$\lim_{q_a\to 0}\langle \pi^a_{q_a}\pi^b_{q_b}\pi^c_{q_c}|\mathcal{H}_\mathrm{w}(0)|K^n_k\rangle = \frac{-i}{F_\pi}\langle \pi^b_{q_b}\pi^c_{q_c}|[Q^a, \mathcal{H}_\mathrm{w}(0)]|K^n_k\rangle,$$

where $Q^a$ is an isotopic spin operator, and hence that

$$\lim_{q_0\to 0}\langle \pi^+_{q_+}\pi^-_{q_-}\pi^0_{q_0}|\mathcal{H}^I_\mathrm{w}(0)|K^0_k\rangle = \frac{-i}{2F_\pi}A^I_{K^0\to\pi^+\pi^-},$$

$$\lim_{q_+\to 0}\langle \pi^+_{q_+}\pi^-_{q_-}\pi^0_{q_0}|\mathcal{H}^I_\mathrm{w}(0)|K^0_k\rangle = \frac{-i}{F_\pi}(A^I_{K^0\to\pi^0\pi^0} - A^I_{K^0\to\pi^+\pi^-}),$$

$$\lim_{q_-\to 0}\langle \pi^+_{q_+}\pi^-_{q_-}\pi^0_{q_0}|\mathcal{H}^I_\mathrm{w}(0)|K^n_0\rangle = \frac{i}{F_\pi}(A^I_{K^0\to\pi^0\pi^0},$$
$$- A^I_{K^0\to\pi^+\pi^-} + \frac{1}{\sqrt{2}}A^I_{K^+\to\pi^+\pi^0}),$$

where $I = 1/2, 3/2$ signifies the isospin component of the quantities in question.

(c) Use a linear expansion of the $K \to 3\pi$ transition amplitude, (i.e. Eq. (4.6) with $c = d = 0$) to reproduce the results of Eq. (4.11), up to corrections of order $m_\pi^2$.

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