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Open index pairs, the fixed point index and rationality of zeta functions

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Abstract. We define open index pairs of an isolated invariant set, prove their existence and compute the fixed point index of an isolating neighbourhood in terms of the Lefschetz number of a certain map associated with the open index pair. We use this to establish rationality of zeta functions and Lefschetz zeta functions.

0. *Introduction*

The Conley index [**2**] of an isolated invariant set of a flow has become an important tool in the qualitative study of flows and differential equations. Recently Fried [**7**] observed that the Conley theory can be successfully applied to extend Manning's theorem [**9**] on the rationality of the zeta function of a basic set of an Axiom A diffeomorphism to the case of a C^1 diffeomorphism and its expansive, isolated invariant set with nondegenerate periodic points. Fried's proof is based on a lemma concerning the rationality of Lefschetz zeta functions. In order to show this lemma the diffeomorphism is suspended to obtain a C^1 flow and then the Conley–Easton theorem [**3**] on the existence of smooth isolating blocks for flows is used. This allows the fixed point (Lefschetz) indices to be expressed in terms of traces of certain matrices.

In this paper we prove the rationality of the Lefschetz zeta function for an isolated invariant set of maps of compact attraction on an arbitrary metric ANR, which generalizes Fried's lemma. As a consequence we also obtain a generalization of Manning's theorem. The proof goes along similar lines as in Fried's paper [**7**] with one important difference: we cannot use suspension because otherwise we could obtain a space which is not an ANR. Thus we have to use a substitute of the Conley–Easton theorem for discrete time dynamical systems. In the discrete case it seems natural to consider index pairs instead of isolating blocks. In particular, this approach was chosen in [**12**] and [**13**] to carry over the Conley index from the continuous to the discrete case. In order to be able to consider the fixed point index we have to ensure the existence of index pairs being ANRs. Since the existence of such index pairs seems to be settled only in case of smooth flows and diffeomorphisms, we introduce open index pairs and prove their existence. Then we extend the fixed point index formula established for the case of compact ANR index pairs in

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[**11**] (see also [**10**]) to the case of open index pairs. This formula, which is interesting in itself (see [**11**] for details) expresses the fixed point index of an isolated invariant set in terms of the (global) Lefschetz number of a certain map associated with the index pair. This is all that is needed to accomplish the proof of our generalization of Fried's lemma and Manning's theorem along Fried's lines.

1. *Preliminaries*

In this paper we assume the notation introduced in [**11**]. We briefly recall the main definitions.

Let X be a metric ANR and let $f: X \to X$ be a continuous map. The map f is said to be locally compact iff for every $x \in X$ there exists a neighbourhood U of x such that $\operatorname{cl} f(U)$ is compact. It is called a map of compact attraction if it is locally compact and there exists an attracting compact for f, i.e. a compact subset $A \subseteq X$ such that for every $x \in X$ the positive trajectory $\pi_f^+(x)$ has a cluster point in A or, equivalently, the positive limit set $\omega_f^+(x)$ intersects A.

Obviously, if X is compact, then every continuous map $f: X \to X$ is a map of compact attraction.

Recall (see [**4**, Lemma (2.1)] and also compare [**5**, I, Lemmas 5.1 and 5.II, the argument in Theorem 2.1]) that if A is an attracting compact for f then there exists an open neighbourhood U of A such that $f(U) \subseteq U$ and $\operatorname{cl} f(U)$ is compact. From this one can easily obtain the following.

Proposition 1. *If f is a map of compact attraction then for all $n \in \mathbb{N}$ f^n is also a map of compact attraction.* □

The map $f: (X, A) \to (X, A)$ will be called a map of compact attraction iff both f_X and f_A are maps of compact attraction, where $f_X: X \to X$ and $f_A: A \to A$ denote the restrictions of f to (X, ϕ) and (A, ϕ) respectively.

We recall that $\phi = \{\phi_i\}$, the endomorphism of a graded vector space E over the field of rational numbers, is called a Leray endomorphism (cf. [**8**]), iff $E' := E/N(\phi)$, where $N(\phi) := \bigcup \{\phi^{-n}(0) \mid n = 1, 2, \ldots\}$ is of finite type. If ϕ is a Leray endomorphism then we define its Lefschetz number by $\Lambda(\phi) := \Lambda(\phi')$, where $\phi': E' \to E'$ denotes the induced map. Note that if ϕ is a Leray endomorphism, so are any of its iterates.

The following proposition is a straightforward consequence of [**8**, Lemma 2.1].

Proposition 2. *Assume ϕ and ψ are two endomorphisms of graded vector spaces such that $\phi = gh$, $\psi = hg$ for some morphisms $h: E \to F$, $g: F \to E$. (This is in particular satisfied if ϕ and ψ are conjugated.) If one of them is a Leray endomorphism then so is the other and $\Lambda(\phi^k) = \Lambda(\psi^k)$ for all natural k.*

We recall that if $f: (X, A) \to (X, A)$ is a continuous map of a pair (X, A) of topological spaces into itself such that f_*, the map induced in singular homology, is a Leray endomorphism, then f is said to be a Lefschetz map and in such a case the Lefschetz number of f is given by $\Lambda(f) := \Lambda(f_*)$.

The following proposition follows directly from [**4**, Lemma (4.1)] and [5.1, Theorem 2.1].

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PROPOSITION 3. *Assume X and A are metric ANRs and* $f:(X, A)\to(X, A)$ *is a map of compact attraction. Then* f, f_X, f_A *are Lefschetz maps and*

$$\Lambda(f)=\Lambda(f_X)-\Lambda(f_A).$$

2. *Open index pairs*

The function $\sigma:\mathbb{Z}\to X$ is called a solution to f iff $f(\sigma(i))=\sigma(i+1)$ for all $i\in\mathbb{Z}$. For $N\subseteq X$ denote by Sltn (f, N) the set of all solutions σ to f such that $\sigma(\mathbb{Z})\subseteq N$. Sltn (f, X) will be shortened to Sltn (f).

We define Inv N, the invariant part of N, as the set of those $x\in N$ which admit a $\sigma\in$ Sltn (f, N) such that $\sigma(0)=x$. The set $S\subseteq X$ is invariant (with respect to f) iff $S=\operatorname{Inv} S$. Obviously, if f is a homeomorphism, then S is invariant iff $S=f(S)=f^{-1}(S)$.

A compact set S is an isolated invariant set iff it is the largest invariant set in some its neighbourhood N. In such a situation N is said to isolate S. If N is closed, it is called an isolating neighbourhood for S. We say that S is of Rybakowski type if it admits an isolating neighborhood N such that for every pair of sequences $\{x_n\}_{n=1,\infty}\subseteq N$ and $\{m_n\}_{n=1,\infty}\subseteq\mathbb{Z}^+$ such that $\{f^i(x_n)\mid i=0, 1, \ldots, m_n\}\subseteq N$ and $m_n\to\infty$, the sequence $\{f^{m_n}(x_n)\}_{n=1,\infty}$ is relatively compact.

Notice that if S is an isolated invariant set then the set of fixed points of f in S is isolated.

A straightforward consequence of Theorem 2 in [**11**] is the following

THEOREM 1. *For every isolated invariant set S of Rybakowski type there exists an open neighbourhood V of S and continuous functions* ϕ, γ: $V\to[0,\infty]$ *such that*

(1) $\phi(x)>0, f(x)\in V\Rightarrow\phi(f(x))<\phi(x)$,

(2) $\gamma(x)>0, f(x)\in V\Rightarrow\gamma(f(x))>\gamma(x)$,

(3) $S=\phi^{-1}(0)\cap\gamma^{-1}(0)$.

Definition 1. Assume (M, N) is a pair of subsets of X such that $N\subseteq M$. The pair (M, N) will be called an open index pair for S or briefly (in this paper only) an index pair iff M, N are open and the following conditions are satisfied

(4) $x\in N, f(x)\in M\Rightarrow f(x)\in N$,

(5) $x\in M, f(x)\notin M\Rightarrow x\in N$,

(6) $S=\operatorname{Inv}\operatorname{cl}(M\backslash N)\subseteq\operatorname{int}(M\backslash N)$.

We say that the index pair (M, N) is regular iff the following conditions are satisfied

(7) there exists an open set U such that $\operatorname{cl}_M N\subseteq U$ and $f(U\backslash N)\subseteq N$.

(8) $\operatorname{Cl}(f(N)\backslash M)\cap\operatorname{cl}(M\backslash N)=\emptyset$.

THEOREM 2. *Assume that S is an isolated invariant set of Rybakowski type. Then for every neighbourhood U of S there exists a regular open index pair* (M, N) *for S such that* $M\subseteq U$.

Proof. Choose V, ϕ, γ as in Theorem 1. Taking restrictions, if necessary, we can assume that cl V is an isolating neighbourhood for S. Put $W:=U\cap V\cap f^{-1}(V)\cap f^{-2}(V)$. Then W is an open neighbourhood of S and $W\cup f(W)\cup f^2(W)\subseteq V$. Find

$\varepsilon \in 0$ such that $\phi^{-1}([0, \varepsilon]) \cap \gamma^{-1}([0, \varepsilon]) \subseteq W$. Put

$$M := \phi^{-1}([0, \varepsilon)) \cap \gamma^{-1}([0, \varepsilon)), \quad N := \{x \in M \mid \gamma(f(x)) > \varepsilon/2\}.$$

We will show that (M, N) is an open regular index pair for S.

First observe that M and N are open in X. Assume $x \in N$ and $f(x) \in M$. Then $\gamma(f(x)) > \varepsilon/2$, hence we get from (2) that $\gamma(f^2(x)) > \gamma((x)) > \varepsilon/2$, i.e. $f(x) \in N$. Suppose $x \in M$, $f(x) \notin M$. Since $f(x) \in V$, we have $\phi(f(x)) < \phi(x) < \varepsilon$, thus it must be $\gamma(f(x)) \geq \varepsilon > \varepsilon/2$, which means $x \in N$ and (4), (5) are proved.

Since $S \subseteq \{x \in W \mid \phi(x) < \varepsilon,\ \gamma(f(x)) < \varepsilon/2\} \subseteq \operatorname{int}(M \backslash N)$, we have

$$S = \operatorname{Inv} S \subseteq \operatorname{Inv} \operatorname{int}(M \backslash N) \subseteq \operatorname{Inv} \operatorname{cl}(M \backslash N) \subseteq \operatorname{Inv} V = S$$

and (6) is proved. Put $U := (N \cup f^{-1}(N)) \cap M$. Then U is open and $f(U \backslash N) \subseteq N$. Take $y \in \operatorname{cl}_M N$. If $y \in N$ then $y \in U$. If $y \notin N$ then $\gamma(y) \leq \varepsilon$, $\phi(y) \leq \varepsilon$, $\gamma(f(y)) = \varepsilon/2$ and by (5) $f(y) \in M$. There is also $\gamma(f^2(y)) > \gamma(f(y)) = \varepsilon/2$, which shows that $f(y) \in N$. Thus $y \in f^{-1}(N)$ and we have proved (7).

In order to prove (8) assume it is not true. Then there exists $y \in \operatorname{cl}(M \backslash N) \cap \operatorname{cl}(f(N) \backslash M)$. In particular we have $\gamma(f(y)) \leq \varepsilon/2$ and $\gamma(f(y)) > \gamma(y) \geq \varepsilon/2$, a contradiction. □

From the above theorem and Theorem 1 in [**11**] we get the following

COROLLARY 1. *Every compact isolated invariant set S of a locally compact map (in particular of a map of compact attraction) admits regular, open index pairs arbitrarily close to S.*

The following proposition follows easily from (4), (5), (8) and the excision property of singular homology (see [**16**, Ch. 4, § 6, Corr. 5]).

PROPOSITION 4. *Assume (M, N) is a regular index pair for K. Then f maps the pair (M, N) into the pair $(M \cup f(N), N \cup f(N))$ and the inclusion $i_{M,N} : (M, N) \to (M \cup f(N), N \cup f(N))$ induces an isomorphism in homology.*

3. *Index maps and the fixed point index formula*

Let $f_{M,N}$ denote the mapping f considered as a mapping of the pair (M, N) into the pair $(M \cup f(N), N \cup f(N))$. Similarly as in the case of closed index pairs (see [**11**]), Proposition 4 enables us to define an endomorphism $I_{M,N} : H_*(M, N) \to H_*(M, N)$ by

$$I_{M,N} := (i_{M,N})_*^{-1} \circ (f_{M,N})_*.$$

We will call this map the index map of the index pair (M, N).

In the sequel Fix f will denote the set of fixed points of f and ind(f, V) will stand for the fixed point index as defined by Granas in [**8**, Theorem 12.1] with the normalization property in the form proved in [**4**, Proposition 3.5].

The main result of this paragraph is the following

THEOREM 3. *Assume X is a metric ANR and $f : X \to X$ is a map of compact attraction. If (M, N) is a regular open index pair, then $I_{M,N}$ is a Leray endomorphism and for every $n \in \mathbb{N}$*

$$\operatorname{ind}(f^n, \operatorname{int}(M \backslash N)) = \Lambda((I_{M,N})^n).$$

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A similar theorem for closed index pairs is proved in [**11**, Theorem 4]. Unfortunately the proof in the case of closed index pairs does not carry over directly to the case of open index pairs.

In course of the proof of Theorem 3, we will need the following

LEMMA 1. *Assume X, A are metric ANRs, $A\subseteq X$ and $f:(X, A)\to(X, A)$ is a map of compact attraction, such that* $\operatorname{Fix} f\cap \operatorname{bd} A=\emptyset$. *Then*

(9) $\operatorname{ind}(f, X\backslash \operatorname{cl} A)=\Lambda(f_X)-\Lambda(f_A)=\Lambda(f)$.

Proof. Put $U:=\operatorname{int} A$, $V:=X\backslash \operatorname{cl} A$. Using the argument of Fournier [**5**, I, Lemma 5.1; **5**, II, Theorem 2.1] we can find W open in A such that $\operatorname{Fix} f_A\subseteq W$, $\operatorname{cl} f(W)$ is a compact subset of W and

(10) $\Lambda(f_A)=\Lambda(f_W)$.

Let $W':=W\cap \operatorname{int} A$. The normalization, excision and commutativity properties of the fixed point index imply

(11) $\Lambda(f_W)=\operatorname{ind}(W, f_W, W)=\operatorname{ind}(W, f_{W'}, W')=\operatorname{ind}(X, f_{W'}, W')=\operatorname{ind}(X, f, U)$.

From the additivity property of the fixed point index we get

(12) $\Lambda(f_X)=\operatorname{ind}(X, f, X)=\operatorname{ind}(X, f, V)+\operatorname{ind}(X, f, U)$.

Hence (9) follows from (10), (11) and (12). □

Proof of Theorem 3. Let $I:=[0, 1]$, $J:=(-\frac{1}{4}, \frac{5}{4})$, $J_0:=(-\frac{1}{4}, \frac{1}{4})$, $j_1:=(\frac{3}{4}, \frac{5}{4})$. Put

$$Y:=M\times J_0\cup N\times J\cup X\times J_1.$$

Y is an ANR as an open subset of $X\times J$. Let U be open such that $N\subseteq U\subseteq M$, $\operatorname{cl}_M N\subseteq U$ and $f(U\backslash N)\subseteq N$. Let $W:=\operatorname{cl}_Y((M\backslash U)\times J_0)$ and $Z:=N\times J\cup X\times J_1$. Let $\alpha: Y\to I$ be a continuous function such that $\alpha|_W=0$, $\alpha|_{\operatorname{cl} Z}=1$. Define the map

$$g:(Y, Z)\ni(x, t)\to(f(x), \alpha(x, t))\in(Y, Z).$$

This map is well defined as one can see from the following properties based on the definition of the index pair

$$g(W)\subseteq M\times\{0\},$$

$$g(Z)\subseteq X\times\{1\},$$

$$g(Y\backslash(W\cup Z))\subseteq N\times I\subseteq Z.$$

We will prove that g is a map of compact attraction. Let $(x, t)\in Y$. There exists a neighborhood V of x and a compact $K\subseteq X$, such that $f(V)\subseteq K$. Then

$$g(V\times J)\subseteq f(V)\times\alpha(Y)\subseteq K\times I,$$

which shows that g_Y and g_Z are locally compact. Assume A is a compact set in X such that for every $x\in X$ $(\omega_f^+)(x)\cap A\neq\emptyset$.

Put $A':=A\times\{1\}$, $A'':=(\operatorname{Inv}^+\operatorname{cl}(M\backslash N)\cap A)\times\{0\}$. (We recall that for $C\subseteq X$ $\operatorname{Inv}^+ C:=\{x\in C\,|\,f^i(x)\in C$ for all $i\in\mathbb{N}\}$). Obviously A' and A'' are compact. We have also

$$A'\subseteq X\times\{1\}\subseteq Z\subseteq Y,\quad A''\subseteq M\times\{0\}\subseteq Y.$$

Take $(x, t)\in Y$. Then $g^n(x, t)$ has the form $(f^n(x), s_n(x))$ for some $s_n(x)\in I$. The following properties follow easily from the definition of g.

$$s_n(x)>0 \Rightarrow s_m(x)=1 \quad \text{for all } m>n,$$
$$s_n(x)=0 \quad \text{for all } n\in\mathbb{N} \Leftrightarrow (\pi_f^+)(x)\subseteq M\setminus U.$$

Let $y\in(\omega_f^+)(x)$. If for some $n\in\mathbb{N}$, $s_n(x)>0$, in particular if $(x,t)\in Z$, then $(y,1)\in(\omega_g^+)(x,t)\cap A'$. Otherwise

$$(\pi_f^+)(x)\subseteq M\setminus U, \quad \mathrm{cl}\,(\pi_f^+)(x)\subseteq \mathrm{cl}\,(M\setminus U)$$

and

$$y\in(\omega_f^+)(x)=\mathrm{Inv}^+\,(\omega_f^+)(x)\subseteq \mathrm{Inv}^+\,(\mathrm{cl}\,(\pi_f^+)(x))\subseteq \mathrm{Inv}^+\,(\mathrm{cl}\,(M\setminus N)),$$

i.e. $(y,0)\in A''\times(\omega_g^+)(x,t)$.

Thus A' is an attracting compact with respect to g_z and $A'\cup A''$ is an attracting compact with respect to g_Y. This means that g is a map of compact attraction.

Put $Q:=N\times J$, $P:=M\times J_0\cup Q$. P and Q are open subsets of Y, thus they are ANRs. An easy computation shows that (P,Q) is a regular index pair for g. Put

$$R:=\mathrm{int}\,(M\setminus U), \quad R':=R\times J_0=\mathrm{int}\,(P\setminus Q),$$
$$V:=f^{-n}(R)\cap R, \quad V':=V\times J_0,$$
$$\mu: R'\ni(x,t)\to x\in R, \quad \nu: R\ni x\to(x,0)\in R'.$$

Applying the commutativity property of the fixed point index to $\mu\circ g^n|_{R'}$ and ν we get from $\mu\circ g^n|_{R'}\circ\nu=f^n|_R$ and $\nu\circ\mu\circ g^n|_{R'}=g^n|_{V'}$ that

$$\mathrm{ind}\,(f^n,R)=\mathrm{ind}\,(g^n,V').$$

However

$$\mathrm{Fix}\,f^n\cap R=\mathrm{Fix}\,f^n\cap\mathrm{int}\,(M\setminus N),$$
$$\mathrm{Fix}\,g^n\cap V'=\mathrm{Fix}\,g^n\cap\mathrm{int}\,(P\setminus Q),$$

thus we get from the additivity and excision properties of the fixed point index that

(13) $\mathrm{ind}\,(f^n,\mathrm{int}\,(M\setminus N))=\mathrm{ind}\,(g^n,\mathrm{int}\,(P\setminus Q))$.

By Proposition 1 g^n is a map of compact attraction. It follows from Proposition 3 that g^n is a Lifschitz map and, since $P\setminus Q=Y\setminus Z$, we have by Lemma 1 applied to g^n

(14) $\mathrm{ind}\,(g^n,\mathrm{int}\,(P\setminus Q))=\mathrm{ind}\,(g^n,\mathrm{int}\,(Y\setminus Z))=\Lambda(g^n)=\Lambda((g_*)^n)$.

Consider now the following commutative diagram

$$\begin{array}{ccccc}
H_*(Y,Z) & \xrightarrow{\;g_*\;} & H_*(Y,Z) & \xleftarrow{\;id\;} & H_*(Y,Z) \\
\uparrow{\scriptstyle j_{1*}} & & \uparrow{\scriptstyle j_{2*}} & & \uparrow{\scriptstyle j_{1*}} \\
H_*(P,Q) & \xrightarrow{(g_{P,Q})_*} & H_*(P\cup g(Q),Q\cup g(Q)) & \xleftarrow{(i_{P,Q})_*} & H_*(P,Q) \\
\downarrow{\scriptstyle p_{1*}} & & \downarrow{\scriptstyle p_{2*}} & & \downarrow{\scriptstyle p_{1*}} \\
H_*(M,N) & \xrightarrow{(f_{M,N})_*} & H_*(M\cup f(N),N\cup f(N)) & \xleftarrow{(i_{M,N})_*} & H_*(M,N)
\end{array}$$

in which $j_1:(P,Q)\to(Y,Z)$ and $j_2:(P\cup g(Q),Q\cup g(Q))\to(Y,Z)$ are inclusions and

$$p_1:(P,Q)\ni(x,t)\to x\in(M,N),$$
$$p_2:(P\cup g(Q),Q\cup g(Q))\ni(x,t)\to x\in(M\cup f(N),N\cup f(N)),$$

are projections. Since $\text{cl}(Y\backslash P)\subseteq \text{cl}(X\times J_0)\subseteq \text{int } Z$, we see that j_1 is an excision, thus j_{1*} is an isomorphism. By Proposition 4 applied to (P, Q), $(i_{P,Q})_*$ is an isomorphism. This shows that g_* and $I_{P,Q}$ are conjugate and since g_* is a Lefschetz map, we see by Proposition 2 that $I_{P,Q}$ is also a Lefschetz map and

(15) $\Lambda((g_*)^n)=\Lambda((I_{P,Q})^n)$.

One can easily verify that the mapping $(M, N)\ni x\to(x, 0)\ni(P, Q)$ is the homotopy inverse of p_1, thus p_{1*} is an isomorphism. This shows that $I_{M,N}$ and $I_{P,Q}$ are conjugate and since $I_{P,Q}$ is a Lefschetz map, so is $I_{M,N}$ and

(16) $\Lambda((I_{P,Q})^n)=\Lambda((I_{M,N})^n)$.

The thesis follows now from (13), (14), (15) and (16). □

4. *Zeta functions*

Following Artin–Mazur [1] we define the zeta function of a map f on its isolated invariant set S as the formal power series

$$\zeta_s(f):=\exp\left(\sum_{n=1}^{\infty} N_n t^n/n\right),$$

where $N_n:=\text{card } S\cap \text{Fix } f^n$. It is well defined whenever $N_n<\infty$ for all $n\in\mathbb{N}$.

Similarly the Lefschetz zeta function $Z_s(f)$, called in Smale's paper [**15**] the false zeta function, is given by

$$Z_s(f):=\exp\left(\sum_{n=1}^{\infty} L_n t^n/n\right),$$

where $L_n:=\text{ind}(f^n, U)$ for any open U isolating S. It is defined if X is an ANR and f is locally compact.

The following theorem concerning the relationality of the Lefschetz zeta function generalizes the Lemma in [7] and also Proposition 5.13 in [**6**].

THEOREM 4. *Assume X is a metric ANR and $S\subseteq X$ is an isolated invariant set of a map of compact attraction $f: X\to X$. Then for any open regular index pair (M, N) of S the Lefschetz zeta function $Z_s(f)$ is a rational function given by*

$$Z_s(f)=\prod_{k=0}^{m}\det(\text{Id}-J_k t)^{(-1)^{k+1}}$$

where $J:=(I_{M,N})': H_*(M, N)/N(I_{M,N})\to H_*(M, N)/N(I_{M,N})$ is an induced map, $N(I_{M,N})$ denotes the generalized kernel of $I_{M,N}$, $m=\dim H_*(M, N)/N(I_{M,N})$ and Id *is the corresponding identity map.*

Proof. We have by an algebraic identity (see [**6**, Lemma 5.2]) that

$$\exp\left(\sum_{n=1}^{\infty}\text{tr}(J_k)^n t^n/n\right)=\det(\text{Id}-J_k t)^{-1}.$$

Since by Theorem 4

$$L_n=\text{ind}(f^n, \text{int}(M\backslash N))=\Lambda(J^n),$$

we obtain

$$\begin{aligned} Z_s(f) &= \exp\left(\sum_{n=1}^{\infty} L_n t^n/n\right) = \exp\left(\sum_{n=1}^{\infty} \Lambda(J^n) t^n/n\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\left(\sum_{k=0}^{m} (-1)^k \operatorname{tr}(J_k)^n\right) t^n/n\right) \\ &= \exp\left(\sum_{k=0}^{m}\left(\sum_{n=1}^{\infty} (-1)^k \operatorname{tr}(J_k)^n t^n/n\right)\right) \\ &= \prod_{k=0}^{m} \exp\left(\sum_{n=1}^{\infty} \operatorname{tr}(J_k)^n t^n/n\right)^{(-1)^k} = \prod_{k=0}^{m} \det(\mathrm{id} - J_k t)^{(-1)^{k+1}}. \qquad \square \end{aligned}$$

We will say that $f: X \to X$ is expansive iff there exists an $\varepsilon > 0$ such that for any two solutions $\sigma, \tau \in \mathrm{Sltn}(f)$

$$\sigma \neq \tau \Rightarrow \exists i \in \mathbb{Z} \ \mathrm{dist}(\sigma(i), \tau(i)) > \varepsilon.$$

It is straightforward to verify that the above definition coincides with the standard definition of expansiveness in the case when f is a homeomorphism.

PROPOSITION 5. *f is expansive iff the diagonal $\Delta \subseteq X \times X$ is an isolated invariant set with respect to $f \times f$.* □

Note that if f is expansive then the periodic points of f of a given period are isolated. If x is an isolated n-periodic point of f we put $i(f^n, x) := i(f^n, V)$, where V is any neighborhood of x isolating it from other periodic points.

Definition 2. We say that an n-periodic point x of f is nondegenerate if $i(f^n, x)$ is 1 or -1.

Obviously, if f is a C^1 map of a C^1 manifold and x is an n-periodic point of f such that the graph of f^n is transversal to the diagonal at x then $i(f^n, x) = \operatorname{sgn}\det(\mathrm{Id} - D \times F^n)$, i.e. x is nondegenerate.

The following theorem generalizes the result of Fried [**4**]. The proof is essentially the same but we use Theorem 3 instead of the lemma in [**4**]. We include the proof for sake of completeness.

THEOREM 5. *Assume X, S and $f: X \to X$ are as in Theorem 4. If $f|_s$ is expansive and all periodic points of f in S are nondegenerate, then $\zeta_s(f)$ is rational.*

Proof. By Proposition 5 the diagonal Δ in $S \times S$ is an isolated invariant set with respect to $f|_s \times f|_s$. Obviously $S \times S$ is an isolated invariant set with respect to $f \times f$, hence so is Δ.

Fixed points of $f^n \times f^n$ in Δ are of the form (x, x), where x is a fixed point of f^n. Hence expansiveness of $f|_s$ implies that $f^n \times f^n$ has only finite number of fixed points in Δ. Moreover, by the multiplicativity property of the fixed point index,

$$\operatorname{ind}(f^n \times f^n, (x, x)) = \operatorname{ind}(f^n, x)^2 = 1,$$

because f has nondegenerate periodic points in S. It follows that $\zeta_\Delta(f \times f) = Z_\Delta(f \times f)$. (Here Δ denotes the diagonal in $S \times S$.) Since obviously $\zeta_S(f) = \zeta_\Delta(f \times f)$, we get $\zeta_S(f) = Z_\Delta(f \times f)$. Obviously, if f is a map of compact attraction, so is $f \times f$, hence the assertion follows from Theorem 4. □

COROLLARY 2. *Let f be a C^1 map of compact attraction on a C^1 manifold and S an isolated invariant set with respect to f with nondegenerate periodic points. If $f|_s$ is expansive then $\zeta_s(f)$ is rational.* □

Remark. The above corollary could be proved without introducing open index pairs if one were able to show that the functions ϕ, γ in Theorem 1 can be chosen C^1. This is possible in case of a flow [**18**, Theorem 2.1] and a diffeomorphism [**13**, Theorem 5.3] but seems to be an open problem for semiflows and C^1 maps, because the proofs in [**18**] and [**13**] rely essentially on the symmetry of the flow or the diffeomorphism with respect to the time variable. (Compare also [**14**, the third paragraph of Remarks on p. 359] and [**17**, Remark on p. 422]).

Example. Take $M = B^2 \times S^1$, where $B^2 := \{z \in \mathbb{C} |\ |z| \le 1\}$ and $S^1 := \{z \in \mathbb{C} |\ |z| = 1\}$. Consider the mapping

$$f: M \ni (x, y) \to (x/4, y^2) \in M.$$

The map f is C^1 but it is not a diffeomorphism. Since M is compact, f is a map of compact attraction. It has nondegenerate periodic points and $S := \{(x, y) \in M \mid x = 0\}$ is an isolated invariant set. Obviously $f|_s$ is expansive and Corollary 2 shows that $\zeta_s(f)$ is rational.

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