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ON VARIETIES OF ABELIAN TOPOLOGICAL GROUPS WITH COPRODUCTS

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Abstract

A class of abelian topological groups was previously defined to be a variety of topological groups with coproducts if it is closed under forming subgroups, quotients, products and coproducts in the category of all abelian topological groups and continuous homomorphisms. This extended research on varieties of topological groups initiated by the second author. The key to describing varieties of topological groups generated by various classes was proving that all topological groups in the variety are a quotient of a subgroup of a product of groups in the generating class. This paper analyses generating varieties of topological groups with coproducts. It focuses on the interplay between forming products and coproducts. It is proved that the variety of topological groups with coproducts generated by all discrete groups contains topological groups which cannot be expressed as a quotient of a subgroup of a product of a coproduct of discrete groups. It is proved that the variety of topological groups with coproducts generated by any infinite-dimensional Hilbert space contains all infinite-dimensional Hilbert spaces, answering an open question. This contrasts with the result that a variety of topological groups generated by a topological group does not contain any infinite-dimensional Hilbert space of greater cardinality.

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1. Introduction

In the 1930s, G.D. Birkhoff and B.H. Neumann defined varieties of groups as the classes of groups satisfying certain laws or equivalently as classes of groups closed under the operations of forming subgroups (S), quotient groups (Q) and arbitrary cartesian products (C). Moreover, H. Neumann in [15] noted that if Ω is any nonempty class of groups and $V(\Omega)$ is the smallest variety of groups containing Ω, then $V(\Omega) = QSC(\Omega)$. Denote by **Ab** the variety of all abelian groups. In the next theorem we summarise relevant results on varieties of groups from [14] (where $\mathbb{Z}$ is the additive group of integers).

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Theorem 1.1. *Let Ω be any nonempty class of abelian groups. Then:*

(i) $V(\Omega) = QSC(\Omega)$;

(ii) $\mathbf{Ab} = V(\mathbb{Z})$, *that is, the variety* $\mathbf{Ab}$ *is* singly generated;

(iii) *the number of distinct varieties of abelian groups is* $\aleph_0$.

The second named author [10, 12] defined a *variety of topological groups* to be a nonempty class $\mathfrak{V}$ of topological groups closed under the operations of forming subgroups (S), (not necessarily Hausdorff) quotient topological groups (Q) and arbitrary cartesian products (C) (with the Tychonoff product topology). For example, the class of all abelian topological groups is a variety of topological groups and will be denoted by **TopAb**. The varieties generated by the most important classes of topological groups (for instance, by the class of locally compact abelian groups, Banach spaces, groups having a subgroup topology and so on) were intensively studied during the last 40 years (see [4, 5, 9–13]). We summarise a few important results for our purposes in the next theorem.

We denote the topological group of all real numbers with the euclidean topology by $\mathbb{R}$, the multiplicative topological group of all complex numbers with modulus one and compact topology induced from the euclidean plane by $\mathbb{T}$, the class of all discrete abelian topological groups by $\mathcal{D}$, the class of all finite abelian topological groups by $\mathcal{F}$, the class of all topological groups underlying Banach spaces by $\mathcal{B}$, the class of all abelian topological groups with a subgroup topology, that is, a basis of open neighbourhoods of the identity being subgroups, by $\mathcal{S}$ and the topological group underlying the Banach space ℓ^1 by ℓ^1.

Definition 1.2. Let X be a completely regular Hausdorff space and $FA(X)$ an abelian topological group which contains X as a subspace such that X contains the identity element of $FA(X)$. The topological group $FA(X)$ is said to be the *Graev free abelian topological group on* X if for every continuous map ϕ of X into any abelian topological group G such that the identity element of $FA(X)$ maps onto the identity element of G, there is a unique continuous homomorphism of $FA(X)$ into G which extends the map ϕ.

It is well known [10] that for every completely regular Hausdorff space X, $FA(X)$ exists and is unique up to topological group isomorphism. Denote by $\mathfrak{s}$ the convergent sequence $0, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$.

Theorem 1.3. *Let Ω be any nonempty class of abelian topological groups. Then:*

(a) $\mathfrak{V}(\Omega) = QSC(\Omega)$ [2];

(b) *the variety* **TopAb** *is not generated by any set of abelian topological groups or, equivalently, is not singly generated* [11];

(c) *there is a proper class of distinct varieties of abelian topological groups* [11, 12];

(d) $\mathfrak{V}(\mathcal{B}) =$ **TopAb** [9, 13];

(e) $\mathfrak{V}(\mathcal{F}) \subsetneq \mathfrak{V}(\mathbb{T}) \subsetneq \mathfrak{V}(\mathbb{R}) \subsetneq \mathfrak{V}(FA(\mathfrak{s})) \subsetneq \mathfrak{V}(\ell_1) \subsetneq$ **TopAb** [9];

(f) $\mathfrak{V}(\mathbb{Z}) \subsetneq \mathfrak{V}(\mathcal{D}) = \mathcal{S} \subsetneq$ **TopAb** [9];

(g) $\mathfrak{V}(\mathbb{R}) \subsetneq \mathfrak{V}(\mathbb{R}, \mathcal{D}) \subsetneq$ **TopAb** [9].

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For a nonempty family $\{G_i\}_{i\in I}$ of groups, the *direct sum* of G_i is denoted by

$$\bigoplus_{i\in I} G_i := \Big\{(g_i)_{i\in I} \in \prod_{i\in I} G_i : g_i = e_i \text{ for almost all } i\Big\},$$

and we denote by j_k the natural embedding of G_k into $\bigoplus_{i\in I} G_i$; that is,

$$j_k(g) = (g_i) \in \bigoplus_{i\in I} G_i, \quad \text{where } g_i = g \text{ if } i = k \text{ and } g_i = e_i \text{ if } i \neq k.$$

If $\{G_i\}_{i\in I}$ is a nonempty family of topological groups, *the final group topology* $\mathcal{T}_f$ on $\bigoplus_{i\in I} G_i$ with respect to the family of canonical homomorphisms $j_k : G_k \to \bigoplus_{i\in I} G_i$ is the finest group topology on $\bigoplus_{i\in I} G_i$ such that all j_k are continuous.

Definition 1.4. Let $\mathcal{G} = \{(G_i, \mathcal{T}_i)\}_{i\in I}$ be a nonempty family of abelian (topological) groups. The (topological) group $(G, \mathcal{T})$ is the *coproduct* of the family $\mathcal{G}$ in the category of abelian (topological) groups and (continuous) homomorphisms if:

(i) for each $i \in I$, there is an embedding $j_i : G_i \to G$;
(ii) for any abelian (topological) group H and each family $\{p_i\}_{i\in I}$ of (continuous) homomorphisms $p_i : G_i \to H$, there exists a unique (continuous) homomorphism $p : G \to H$ such that $p_i = p \circ j_i$ for every $i \in I$.

The underlying group structure of the coproduct $(G, \mathcal{T})$ is the direct sum $\bigoplus_{i\in I} G_i$. The *coproduct topology* $\mathcal{T}$ on G coincides with the final group topology $\mathcal{T}_f$ with respect to the family of canonical homomorphisms $j_i : G_i \to G$. Note that a coproduct of a family of abelian topological groups is unique up to topological group isomorphism.

Since the coproduct of a family of abelian groups in the category of all abelian groups and homomorphisms is just a subgroup of the cartesian product of those groups, abelian coproducts do not feature in the study of varieties of abelian groups. Noting that the operation (K) of forming coproducts of topological groups in the category of all abelian topological groups and continuous homomorphisms is natural from the categorical point of view, we have the following definition.

Definition 1.5 [8]. A nonempty class $\mathfrak{C}$ of abelian topological groups is called a *variety of abelian topological groups with coproducts* if it is closed under the operations of forming subgroups (S), (not necessarily Hausdorff) quotient topological groups (Q), arbitrary cartesian products (C) (with the Tychonoff product topology) and arbitrary coproducts (K) in the category of all abelian topological groups and continuous homomorphisms.

We denote by $\mathfrak{C}(\Omega)$ the smallest variety of abelian topological groups with coproducts generated by a class Ω of abelian topological groups.

Following [8], a topological group G is called *quasilinear* if there is a basis of open neighbourhoods at the identity, $\mathcal{N}(G) = \{U_\alpha : \alpha \in A\}$, such that the subgroup $\langle g\rangle$ generated by any $g \in U_\alpha$ is contained in U_α, in other words,

$$U_\alpha = \bigcup \{\langle g\rangle : g \in U_\alpha\}.$$

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We denote by $\mathcal{QL}$ the class of all quasilinear abelian topological groups. It is clear that $\mathbb{R} \notin \mathcal{QL}$ and $\mathbb{T} \notin \mathcal{QL}$. Every discrete group is quasilinear and, more generally, $\mathcal{S} \subseteq \mathcal{QL}$.

THEOREM 1.6 [8].

(α) *If Ω is a class of abelian topological groups, then*

$$\mathfrak{C}(\Omega) = \bigcup_{n\in\mathbb{N}} QS[C_1K_1.C_2K_2\dots C_nK_n](\Omega). \tag{1.1}$$

(β) $\mathfrak{C}(\mathcal{F}) \subsetneq \mathfrak{C}(\mathbb{T}) \subsetneq \mathfrak{C}(\mathbb{R}) \subsetneq \mathfrak{C}(FA(\mathfrak{s})) = \mathfrak{C}(\ell_1) = \mathbf{TopAb}$*; in particular,* $\mathbf{TopAb}$ *is singly generated.*

(γ) $\mathfrak{C}(\mathbb{Z}) = \mathfrak{C}(\mathcal{S}) \subseteq \mathfrak{C}(\mathcal{QL}) = \mathcal{QL} \subsetneq \mathbf{TopAb}$.

If Ω is a class of abelian groups or abelian topological groups, it is significant that each of the operations (Q), (S) and (C) needs to be used only once for the forming of $V(\Omega)$ and $\mathfrak{V}(\Omega)$ by Theorems 1.1(i) and 1.3(a). For the variety $\mathfrak{C}(\Omega)$ with coproducts, we have only the equality (1.1), which is useful but not as much as we might hope. The next question is posed in [8]: will a *finite* union suffice in (1.1)? Clearly, the answer depends on the interplay between operations (C) and (K), which is considered in Section 2.

2. Interplay between operations (C) and (K)

The minimal number of operations for $\mathfrak{C}(\Omega)$ (see (1.1)) depends on the interplay between operations (C) and (K). In this section we consider the next two natural questions.

QUESTION 2.1. Let Ω be a class of abelian topological groups.

(i) Is $KC(\Omega) \subseteq QSCK(\Omega)$?

(ii) Is $CK(\Omega) \subseteq QSKC(\Omega)$?

In Theorem 2.3 below, we show that there is a negative answer to Question 2.1(i), in particular for the class $\mathcal{D}$ of all abelian discrete groups. However, Question 2.1(ii) has a partial positive answer (see Proposition 2.6).

We need the following description of the coproduct topology $\mathcal{T}_f$ given in [3, Proposition 5] (see also [16]).

PROPOSITION 2.2 [3]. *Let $(G, \mathcal{T}_f)$ be the coproduct of a nonempty family $\{(G_i, \tau_i)\}_{i\in I}$ of abelian topological groups and let $j_k : G_k \to G$ be the canonical isomorphisms. For any countable sequence of open neighbourhoods $(U_{i,n})_n$ of the identity in G_i, we define*

$$\mathrm{U}_f = \bigcup_{N\in\mathbb{N}} \bigcup_{(i_1,\dots,i_N)\in I^N} \sum\nolimits_{n=1}^{N} j_{i_n}(U_{i_n,n}).$$

Then the family of all sets of the form U_f is a basis of open neighbourhoods at the identity in $(G, \mathcal{T}_f)$.

The next theorem is the main result in this section.

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THEOREM 2.3. *Let* $\{G_{i,n} : i \in I, n \in \mathbb{N}\}$ *be a family of nontrivial discrete groups, where* I *is an uncountable set of indices. Set*

$$G := \Big(\bigoplus_{i\in I}\prod_{n\in\mathbb{N}} G_{i,n}, \mathcal{T}_f\Big).$$

Then G *is a quasilinear group whose topology is not a subgroup topology and hence* $\mathcal{S} \subsetneq \mathcal{QL}$*. In particular,* $KC(\mathcal{D}) \not\subset \mathcal{QS}CK(\mathcal{D})$ *and*

$$\mathcal{QS}CK(\mathcal{D}) = \mathcal{S} \subsetneq \mathcal{QS}KC(\mathcal{D}) \subseteq \mathfrak{C}(\mathcal{D}).$$

PROOF. Clearly, for each $i \in I$ and for every sequence $\{V_{i,s}\}_{s\in\mathbb{N}}$ of neighbourhoods of zero in $\prod_{n\in\mathbb{N}} G_{i,n}$, there exists a sequence $k_1^i < k_2^i < \dots$ such that $U_{i,s} \subseteq V_{i,s}$, where

$$U_{i,s} := \underbrace{\{0\}\times\cdots\times\{0\}}_{k_s^i}\times\prod_{n>k_s^i} G_{i,n}. \tag{2.1}$$

So, any neighbourhood of zero in G contains an open neighbourhood U_f of zero of the form (see Proposition 2.2)

$$\mathrm{U}_f = \bigcup_{N\in\mathbb{N}}\ \bigcup_{(i_1,\dots,i_N)\in I^N} \sum\nolimits_{s=1}^{N} j_{i_s}(U_{i_s,s}), \tag{2.2}$$

where $U_{i,s}$ satisfy (2.1). Clearly, the subgroup $\langle \mathrm{U}_f\rangle$ of G generated by U_f is

$$\langle \mathrm{U}_f\rangle = \bigoplus_{i\in I} j_i(U_{i,1}).$$

So, to prove the theorem it is enough to show that each neighbourhood V_f of the form (2.2) does not contain any subgroup of the form $\langle \mathrm{U}_f\rangle$.

Let V_f be defined by sequences $t_1^i < t_2^i < \dots, i \in I$. Since I is uncountable, there are $m \in \mathbb{N}$ and indices $a_1, \dots, a_m$ such that:

(i) $t_j^{a_1} = \cdots = t_j^{a_m} = t_j$, for every $1 \le j \le m$, and $k_1^{a_1} = \cdots = k_1^{a_m} = k_1$;

(ii) $t_m^{a_1} = \cdots = t_m^{a_m} = t_m > k_1$.

Set $A := \{a_1, \dots, a_m\}$. Denote by π_A the projection of G onto $G_{a_1}\times\cdots\times G_{a_m}$, and let π_i denote the projection of G onto G_{a_i}. Clearly,

$$\pi_A(\langle \mathrm{U}_f\rangle) = \bigoplus_{i\in A} j_i(U_{i,1}).$$

For every $1 \le l \le m$, by (ii), take an element $g_l \in U_{a_l,1}\setminus V_{a_l,m}$ and set

$$h := j_{a_1}(g_1) + \cdots + j_{a_m}(g_m).$$

Taking into account that $\{V_{i,s}\}_{s\in\mathbb{N}}$ decreases, (2.2) implies that

$$\pi_A(\mathrm{V}_f) = \bigcup_{(i_1,\dots,i_m)\in A^m} \sum\nolimits_{s=1}^{m} j_{i_s}(V_{i_s,s}). \tag{2.3}$$

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Denote by $\mathbb{S}_m$ the set of all permutations of the set A. Since $V_{i,s}$ are subgroups by (2.1), (2.3) yields

$$\pi_A(\mathrm{V}_f) = \bigcup_{(i_1,\ldots,i_m)\in\mathbb{S}_m} \sum\nolimits_{s=1}^{m} j_{i_s}(V_{i_s,s}).$$

This means that for each element $g \in \pi_A(\mathrm{V}_f)$ there is $a_l \in A$ such that $i_m = a_l$ and the a_l-coordinate of g belongs to $j_{a_l}(V_{a_l,m})$. As $g_l \in U_{a_l,1}\backslash V_{a_l,m}$, this means that $h \notin \pi_A(\mathrm{V}_f)$. Thus, $\pi_A(\langle \mathrm{U}_f\rangle) \not\subset \pi_A(\mathrm{V}_f)$ and hence $\langle \mathrm{U}_f\rangle \not\subset \mathrm{V}_f$.

The last assertion of the theorem follows from the first one and the following equalities:

$$QSCK(\mathcal{D}) = QSC(\mathcal{D}) = \mathfrak{V}(\mathcal{D}) = \mathcal{S}. \qquad \square$$

Theorem 2.3 suggests the following question.

Question 2.4. Is $QSKC(\mathcal{D}) = \mathfrak{C}(\mathcal{D})$?

Remark 2.5. Theorem 2.3 shows that neither (C) nor (K) can be dropped in general from the forming of varieties with coproducts. Indeed, for the variety $\mathfrak{C}(\mathcal{D})$,

$$QSK(\mathcal{D}) = QKS(\mathcal{D}) = KQS(\mathcal{D}) = KSQ(\mathcal{D}) = SKQ(\mathcal{D}) = SQK(\mathcal{D}) = \mathcal{D} \subsetneqq \mathfrak{C}(\mathcal{D})$$

and, by Theorems 1.3(a,f) and 2.3,

$$QSC(\mathcal{D}) = \mathfrak{V}(\mathcal{D}) = \mathcal{S} \subsetneqq \mathfrak{C}(\mathcal{D}).$$

A remark on notation: if Ω is a class of abelian topological groups, then C_c and K_c denote the classes of all abelian topological groups isomorphic to a countable product or a countable coproduct respectively of members of Ω. The operation (C_c) is very useful for varieties $\mathfrak{V}(\Omega)$; see [5, Section 4].

The next proposition partially answers Question 2.1(ii) in the positive.

Proposition 2.6. *Let Ω be a class of abelian topological groups. Then*

$$C_cK_c(\Omega) \subset QKC_c(\Omega).$$

Proof. Throughout the proof of this proposition only, if $\{G_i, i \in I\}$, is a set of abelian topological groups for some index set I, $\bigoplus_{i\in I} G_i$ will denote the coproduct of those topological groups in the category of all abelian topological groups and continuous homomorphisms.

Let $\{G_{a,b} : a, b \in \mathbb{N}\}$ be an arbitrary family in Ω and let

$$G := \prod_{a\in\mathbb{N}}\Big(\bigoplus_{b\in\mathbb{N}} G_{a,b}\Big).$$

We have to show that $G \in QKC_c(\Omega)$.

Let $p_{a,b} : G_{a,b} \to \bigoplus_{b\in\mathbb{N}} G_{a,b}$ be the natural embedding for each $a, b \in \mathbb{N}$. Set $\mathcal{L} := \mathbb{N}^{\mathbb{N}}$ and, for each function $L \in \mathcal{L}$, define:

- $H_{L(a)} := \bigoplus_{b\le L(a)} G_{a,b}$;
- $p_{L(a)} : H_{L(a)} \to \bigoplus_{b\in\mathbb{N}} G_{a,b}$ is the natural embedding;

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- $H_L := \prod_{a\in\mathbb{N}} H_{L(a)}$;
- $p_L : H_L \to G,\ p_L((h_a)_{a\in\mathbb{N}}) := (p_{L(a)}(h_a))_{a\in\mathbb{N}}$.

So, p_L is also a natural embedding. Set

$$X_1 := \bigoplus_{L\in\mathcal{L}} H_L, \quad X_2 := \bigoplus_{a\in\mathbb{N}}\Big(\bigoplus_{b\in\mathbb{N}} G_{a,b}\Big) \quad \text{and } X = X_1 \oplus X_2$$

and define

$$p : X \to G, \quad p := \bigoplus_{L\in\mathcal{L}} p_L \oplus \Big(\bigoplus_{a\in\mathbb{N}}\Big(\bigoplus_{b\in\mathbb{N}} p_{a,b}\Big)\Big).$$

Clearly, p is a continuous surjective homomorphism. We claim that p is open. To this end, it is enough to show that for any neighbourhood U_f of zero in X_1 there is $m \in \mathbb{N}$ such that the subgroup

$$T_m := \prod_{a\le m}\{0_a\} \times \prod_{a>m}\bigoplus_{b\in\mathbb{N}} G_{a,b}$$

is contained in $p(\mathrm{U}_f)$.

We shall use proof by contradiction. Suppose that for every $m \in \mathbb{N}$ there is an element

$$t_m := (0_1, \dots, 0_m, g^m_{m+1}, g^m_{m+2}, \dots) \in T_m \backslash p(\mathrm{U}_f),$$

where $g^m_a \in p_{s^m_a}(H_{s^m_a})$ for $a \ge m + 1$ and $s^m_a \in \mathbb{N}$. We can assume that the neighbourhood U_f is defined by a decreasing sequence $\{V_{L,s}\}_{s\in\mathbb{N}}$ of neighbourhoods of zero in H_L for each $L \in \mathcal{L}$ (see Proposition 2.2). For every $m \in \mathbb{N}$, define $L \in \mathcal{L}$ by $L(1) = 1$ and

$$L(a) := \max\{s^1_a, \dots, s^{a-1}_a\} \quad \text{for } a > 1.$$

Then $g^m_a \in p_{L(a)}(H_{L(a)})$ for every $m \in \mathbb{N}$ and each $a > m$. So, $t_m \in p_L(H_L)$ for every $m \in \mathbb{N}$. Choose $q \in \mathbb{N}$ such that

$$S := \prod_{a\le q}\{0_a\} \times \prod_{a>q} H_{L(a)} \subseteq V_{L,1}.$$

Then $t_m \in p(S) \subset p(\mathrm{U}_f)$ for every $m > q$, which contradicts the choice of elements t_m. Thus, p is open. □

Corollary 2.7. *Let Ω be a class of abelian topological groups. Then, for every $n \in \mathbb{N}$,*

$$QS\,[C_{c,1}K_{c,1}.C_{c,2}K_{c,2}\dots C_{c,n}K_{c,n}](\Omega) \subseteq QS\,KC_c(\Omega).$$

Proof. By Proposition 2.6 and [8, Proposition 2.5],

$$\begin{aligned}
QS\,[C_{c,1}K_{c,1}.C_{c,2}K_{c,2}\dots C_{c,n}K_{c,n}](\Omega) &\subseteq QS\,(QK'C'_{c,1})[C_{c,2}K_{c,2}\dots C_{c,n}K_{c,n}(\Omega)]\\
&\subseteq QS\,(K'C'_{c,2})[K_{c,2}\dots C_{c,n}K_{c,n}(\Omega)]\\
&\subseteq QS\,K'(C'_{c,2}K_{c,2})[C_{c,3}K_{c,3}\dots C_{c,n}K_{c,n}(\Omega)]\\
&\subseteq QS\,K'(QK''C'_{c,3})[C_{c,3}K_{c,3}\dots C_{c,n}K_{c,n}(\Omega)]\\
&\subseteq QS\,Q(K'K'')(C'_{c,3}C_{c,3}K_{c,3})[C_{c,4}\dots C_{c,n}K_{c,n}(\Omega)] \subseteq QS\,KC_c(\Omega).
\end{aligned}$$
□

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COROLLARY 2.8. *Let Ω be a class of abelian topological groups. Then*

$$(C_cK_c)^n(\Omega) \subseteq QKC_c(\Omega) \quad \textit{for all } n \in \mathbb{N}.$$

PROOF. We prove the corollary for $n = 2$. One has

$$\begin{aligned}(C_cK_c)C_cK_c(\Omega) &\subseteq QKC_cC_cK_c(\Omega) = QK(C_cK_c)(\Omega) \\ &\subseteq QKQKC_c(\Omega) \subseteq QQKKC_c(\Omega) = QKC_c(\Omega).\end{aligned}$$ □

3. Embedding of topological groups into direct sums

It is known that if a Banach space B embeds into the product of a family $\mathcal{G}$ of topological groups, then B embeds also into the product of a *finite* subfamily of $\mathcal{G}$ (see [5]). In this section we consider a similar question for coproducts.

Let (G, τ) be a topological group. The filter of all open neighbourhoods of the identity e is denoted by $\mathcal{N}(G)$. The sets of the form

$$V_U^l = \{(x, y) \in G \times G : x^{-1}y \in U\} \quad \text{and} \quad V_U^r = \{(x, y) \in G \times G : yx^{-1} \in U\},$$

where $U \in \mathcal{N}(G)$, form respectively a base of the left $\mathcal{U}_l$ and the right $\mathcal{U}_r$ uniform structures on G. A subset A of G is called *left* (respectively *right*) *uniformly discrete* if there is a $U \in \mathcal{N}(G)$ such that $aU \cap bU = \emptyset$ (respectively $Ua \cap Ub = \emptyset$) for distinct elements $a, b \in A$. *The left uniformly discrete number* $ud_l(G)$ of a subset A of X is defined as follows (see [1]):

$$ud_l(A) = \sup\{|D| : D \text{ is a left uniformly discrete subset of } A\}.$$

The right uniformly discrete number $ud_r(G)$ is defined analogously. Clearly, $ud_l(A) = ud_r(A)$. So, we can define *the uniformly discrete number* of A by $ud(A) := ud_l(A)$.

Let $\{G_i\}_{i \in I}$ be a nonempty family of groups. The natural projection of $\prod_{i\in I} G_i$ onto G_k is denoted by π_k, that is, $\pi_k((g_i)_{i\in I}) = g_k$. If A is a subset of $\prod_{i\in I} G_i$, denote by $\mathrm{supp}(A)$ the set of all indices $k \in I$ for which there exists $a \in A$ such that $\pi_k(a) \neq e_k$.

Let $\{(G_i, \tau_i)\}_{i\in I}$ be a nonempty family of (Hausdorff) topological groups. For every $i \in I$, fix $U_i \in \mathcal{N}(G_i)$ and put

$$\prod_{i\in I} U_i := \Big\{(g_i)_{i\in I} \in \prod_{i\in I} G_i : g_i \in U_i \text{ for all } i \in I\Big\}.$$

Then the sets of the form $\prod_{i\in I} U_i$, where $U_i \in \mathcal{N}(G_i)$ for every $i \in I$, form a neighbourhood basis at the unit of a (Hausdorff) group topology $\mathcal{T}_b$ on $\prod_{i\in I} G_i$ that is called *the box topology*. Clearly, $\mathcal{T}_b \le \mathcal{T}_f$ on $\bigoplus_{i\in I} G_i$.

The next theorem is the main result in this section.

THEOREM 3.1. *Let $\{(G_i, \tau_i)\}_{i\in I}$, where I is a nonempty index set, be a family of Hausdorff topological groups and let τ be an arbitrary group topology on $G = \bigoplus_{i\in I} G_i$ which is finer than the box topology, that is, $\mathcal{T}_b \le \tau$. Then every infinite subset A of G has a left uniformly discrete subset of cardinality* $|\mathrm{supp}(A)|$.

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Proof. If $|\mathrm{supp}(A)|$ is finite, the theorem is trivial because A is infinite and τ is Hausdorff. So, we will assume that $\kappa := |\mathrm{supp}(A)|$ is infinite.

Step 1. Let us show that there are a subset $D = \{d_\alpha\}_{\alpha<\kappa}$ of A and a subset $I_0 = \{i_\alpha\}_{\alpha<\kappa} \subseteq \mathrm{supp}(A)$ such that

$$i_\alpha \in \mathrm{supp}(d_\alpha) \setminus \Big[\bigcup_{\xi<\alpha} \mathrm{supp}(d_\xi)\Big] \quad \text{for every } \alpha < \kappa. \tag{3.1}$$

Indeed, let J be the set of all ordinals α such that their cardinals $|\alpha|$ are strictly less than $|\mathrm{supp}(A)|$. It is well known that J is well ordered and $|J| = |\mathrm{supp}(A)|$.

For $\alpha = 0$, let d_0 be an arbitrary nonzero element of A and choose arbitrarily $i_0 \in \mathrm{supp}(d_0)$. Set $D_0 = \{d_0\}$ and $J_0 = \{i_0\}$.

Fix a nonzero ordinal $\alpha < \kappa$ and assume that for every ordinal $\xi < \alpha$ we built $D_\xi = \{d_\zeta\}_{\zeta\le\xi} \subset A$ and $J_\xi = \{i_\zeta\}_{\zeta\le\xi} \subset \mathrm{supp}(A)$ such that

$$i_\xi \in \mathrm{supp}(d_\xi) \setminus \Big[\bigcup_{\zeta<\xi} \mathrm{supp}(d_\zeta)\Big]. \tag{3.2}$$

Since $\mathrm{supp}(d_\xi)$ is finite for every $\xi < \alpha$, (3.2) implies that

$$\Big|\bigcup_{\zeta<\alpha} \mathrm{supp}(d_\zeta)\Big| = |\alpha| < \kappa.$$

Hence, there exist $d_\alpha \in A$ and $i_\alpha \in \mathrm{supp}(d_\alpha)$ for which (3.1) is fulfilled. Set $D_\alpha := \{d_\zeta\}_{\zeta\le\alpha}$ and $J_\alpha := \{i_\zeta\}_{\zeta\le\alpha}$. Finally, we set

$$D := \bigcup_{\alpha<\kappa} D_\alpha = \{d_\alpha\}_{\alpha<\kappa} \quad \text{and} \quad I_0 = \{i_\alpha\}_{\alpha<\kappa}.$$

By construction, D and I_0 are as desired.

Step 2. For each ordinal $\alpha < \kappa$, choose a symmetric $U_{i_\alpha} \in \mathcal{N}(G_{i_\alpha})$ such that $\pi_{i_\alpha}(d_\alpha) \notin U_{i_\alpha} \cdot U_{i_\alpha}$. Set

$$U = \prod_{i\in I} U_i, \quad \text{where } U_i = U_{i_\alpha} \text{ if } i = i_\alpha \text{ for some } \alpha < \kappa, \text{ and } U_i = G_i \text{ otherwise.}$$

To prove the theorem, it is enough to show that $d_\alpha U \cap d_\beta U = \emptyset$ for every $\beta < \alpha$. By (3.1),

$$\pi_{i_\alpha}(d_\alpha U) = \pi_{i_\alpha}(d_\alpha) U_{i_\alpha} \quad \text{and} \quad \pi_{i_\alpha}(d_\beta U) = \pi_{i_\alpha}(d_\beta) U_{i_\alpha} = U_{i_\alpha}.$$

Hence, $\pi_{i_\alpha}(d_\alpha U) \cap \pi_{i_\alpha}(d_\beta U) = \emptyset$ by the choice of U_{i_α}. Thus, $d_\alpha U \cap d_\beta U = \emptyset$ as well. □

Corollary 3.2. *Let p be a continuous homomorphism of a Hausdorff topological group X into the direct sum $(\bigoplus_{i\in I} G_i, \tau)$ of a family $\{(G_i, \tau_i)\}_{i\in I}$ of Hausdorff topological groups, where $\mathcal{T}_b \le \tau$. Then $|\mathrm{supp}(p(X))| \le ud(X)$.*

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Proof. We use proof by contradiction. Suppose that $|\mathrm{supp}(p(X))| > ud(X)$. By Theorem 3.1, we can find a left U-uniformly discrete subset D of $p(X)$ of cardinality $|\mathrm{supp}(p(X))|$. For every $d \in D$, choose an arbitrary element $a_d \in p^{-1}(d)$. Clearly, the set $A = \{a_d\}_{d\in D}$ has cardinality $|\mathrm{supp}(p(X))|$ and it is $p^{-1}(U)$-left separated. But this contradicts the definition of $ud(X)$. Thus, $|\mathrm{supp}(p(X))| \le ud(X)$. □

Corollary 3.3. *Let p be a continuous homomorphism of a separable infinite Hausdorff topological group X into the direct sum $(\bigoplus_{i\in I} G_i, \tau)$ of a family $\{(G_i, \tau_i)\}_{i\in I}$ of Hausdorff topological groups, where $\mathcal{T}_b \le \tau$. Then $|\mathrm{supp}(p(X))| \le \aleph_0$. Moreover, if X has the Baire property (in particular, if X is a Polish group), then there exists a finite set of indices $J \subset I$ such that*

$$p^{-1}\Big(\bigoplus_{i\in J} G_i\Big)$$

is a clopen subgroup of X.

Proof. Clearly, $ud(X) = \aleph_0$ and the assertion follows from Corollary 3.2. Assume additionally that X has the Baire property. Let $\{i_n\}_{n\in\mathbb{N}}$ be an arbitrary enumeration of $\mathrm{supp}(p(X))$. Then

$$X = \bigcup_{n\in\mathbb{N}} p^{-1}\Big(\bigoplus_{k=1}^{n} G_{i_k}\Big).$$

Since X is Baire, there exists $n \in \mathbb{N}$ such that $p^{-1}(\bigoplus_{k=1}^{n} G_{i_k})$ is an open subgroup of X. □

Corollary 3.4. *Let p be a continuous homomorphism of a separable Banach space X into the direct sum $(\bigoplus_{i\in I} G_i, \tau)$ of Hausdorff topological groups (G_i, τ_i), $i \in I$, where $\mathcal{T}_b \le \tau$. Then there exists a finite set of indices $J \subset I$ such that $p(X) \subseteq \bigoplus_{i\in J} G_i$.*

Proof. Since X has the Baire property and is connected, the assertion immediately follows from Corollary 3.3. □

We do not know whether the separability of X in this corollary can be omitted.

4. On varieties $\mathfrak{C}(\ell_p)$ for $1 \le p < \infty$

The variety $\mathfrak{C}(\ell_1)$ contains all abelian topological groups by [8, Corollary 3.2]. In particular, $\mathfrak{C}(\ell_1)$ contains Banach spaces of arbitrary dimension. Noting that ℓ_1 is a nonreflexive separable Banach space, it was asked in [8, Question 7]: can the variety $\mathfrak{C}(B)$ generated by a *reflexive* Banach space B contain Banach spaces of higher dimension? We note that the answer to the similar question for the variety $\mathfrak{V}(B)$ is negative by [5, Theorem 4.1]. However, for varieties with coproducts the situation changes, as the next theorem shows (recall that $\ell_p(\Gamma)$ is reflexive for every $1 < p < \infty$ and each set Γ).

Theorem 4.1. *For each $1 \le p < \infty$ and each set Γ, $\ell_p(\Gamma) \in \mathfrak{C}(\ell_p)$.*

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PROOF. For each $x = (x_\gamma)_{\gamma\in\Gamma} \in \ell_p(\Gamma)$, set $\mathrm{supp}(x) := \{\gamma \in \Gamma : x_\gamma \neq 0\}$. For each $\boldsymbol{\gamma} = (\gamma_n)_{n\in\mathbb{N}} \in \Gamma^{\leq\mathbb{N}}$, set $L_{\boldsymbol{\gamma}} = \ell_p$ and define $\pi_{\boldsymbol{\gamma}} : L_{\boldsymbol{\gamma}} \to \ell_p(\Gamma)$ by ($\mathbf{x}_{\boldsymbol{\gamma}} = (x_n)_{n\in\mathbb{N}} \in L_{\boldsymbol{\gamma}}$)

$$\pi_{\boldsymbol{\gamma}}(\mathbf{x}_{\boldsymbol{\gamma}}) = (x_\gamma)_{\gamma\in\Gamma}, \quad \text{where } x_\gamma = \begin{cases} x_n & \text{if } \gamma = \gamma_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\pi_{\boldsymbol{\gamma}}$ is an embedding of $L_{\boldsymbol{\gamma}}$ onto its image in $\ell_p(\Gamma)$. Set

$$G = \Big(\bigoplus_{\boldsymbol{\gamma}\in\Gamma^{\leq\mathbb{N}}} L_{\boldsymbol{\gamma}}, \mathcal{T}_f\Big) \quad \text{and} \quad \pi : G \to \ell_p(\Gamma), \pi((\mathbf{x}_{\boldsymbol{\gamma}})) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\leq\mathbb{N}}} \pi_{\boldsymbol{\gamma}}(\mathbf{x}_{\boldsymbol{\gamma}}).$$

Then π is a continuous epimorphism. For the sake of simplicity, we shall identify $L_{\boldsymbol{\gamma}}$ with its image in G.

We show that π is open. Suppose for a contradiction that there is an open neighbourhood U_f of zero in G such that $\pi(\mathrm{U}_f)$ is not a neighbourhood of zero $\mathbf{0}$ in $\ell_p(\Gamma)$. As $\ell_p(\Gamma)$ is metrisable, there is a sequence $\{\mathbf{x}_k\} \subset \ell_p(\Gamma)$ such that $\mathbf{x}_k \notin \pi(\mathrm{U}_f)$, $k \in \mathbb{N}$, and $\mathbf{x}_k \to \mathbf{0}$. Set $\boldsymbol{\gamma} := \bigcup_{k\in\mathbb{N}} \mathrm{supp}(\mathbf{x}_k)$. Then $\boldsymbol{\gamma} \in \Gamma^{\leq\mathbb{N}}$. So, there is an open neighbourhood $U_{\boldsymbol{\gamma}}$ of zero in $L_{\boldsymbol{\gamma}}$ such that $U_{\boldsymbol{\gamma}} \subset \mathrm{U}_f$. Clearly, $\mathbf{x}_k \in \pi_{\boldsymbol{\gamma}}(U_{\boldsymbol{\gamma}}) \subset \pi(\mathrm{U}_f)$ for all sufficiently large k. This contradicts the choice of $\mathbf{x}_k$. Thus, π is open and $\ell_p(\Gamma) \in \mathfrak{C}(\ell_p)$. □

Denote by $\mathcal{H}$ the class of all Hilbert spaces. It is well known that any Hilbert space has the form $\ell_2(\Gamma)$ for some set Γ. So, Theorem 4.1 implies the following result.

COROLLARY 4.2. $\mathfrak{C}(\ell_2) = \mathfrak{C}(H) = \mathfrak{C}(\mathcal{H})$ *for any infinite-dimensional Hilbert space* H.

In the next proposition we show that, for the two generators $FA(\mathfrak{s})$ and ℓ_1 of **TopAb**, it is enough to use each of the operations (Q), (S), (C) and (K) only once to generate **TopAb**.

PROPOSITION 4.3. $QSCK(FA(\mathfrak{s})) = QSCK(\ell_1) = \mathbf{TopAb}$.

PROOF. The first equality repeats the proof of [8, Theorem 3.1]: since any metrisable abelian topological group X belongs to $QK(FA(\mathfrak{s}))$ by [6, Theorem 1.14] and [7, Theorem 1.18],

$$\mathbf{TopAb} = QSC(\mathcal{B}) = QSC(QK(FA(\mathfrak{s}))) = QSCK(FA(\mathfrak{s})).$$

To prove the second equality, let us recall that any Banach space is a quotient space of $\ell_1(\Gamma)$ for some set Γ (see [17, Proposition 11.4.6]). So, Theorem 4.1 implies that

$$\mathbf{TopAb} = QSC(\mathcal{B}) = QSC(QK(\ell_1)) = QSCK(\ell_1).$$ □

This proposition motivates the following question.

QUESTION 4.4. Let Ω be a class of abelian topological groups. Is $QS(CK \cup KC)(\Omega) = \mathfrak{C}(\Omega)$? Is $QSCKCK(\Omega) = \mathfrak{C}(\Omega)$?

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