BI-EMBEDDINGS OF GRAPHS

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Let γ and γ' be non-negative integers. We say that the graph G is (γ, γ') *bi-embeddable* if G can be embedded in a surface of genus γ and the complement $\bar{G}$ of G can be embedded in a surface of genus γ'. Let $N(\gamma, \gamma')$ be the least integer such that every graph with at least $N(\gamma, \gamma')$ points is not (γ, γ') bi-embeddable. It has been shown in [1] and [5] that $N(0, 0) = 9$; this result was also obtained by John R. Ball of the Carnegie Institute of Technology. Our object here is to obtain upper and lower bounds for $N(\gamma, \gamma')$.

Let S_p denote the closed orientable 2-manifold of genus p. The genus $\gamma(G)$ of the graph G is the minimum value of p for which G may be embedded in S_p. Youngs [7] has shown that for such a minimal embedding every face is a 2-cell, and so we may apply Euler's formula. If $\gamma \geqq \gamma(G)$, then clearly G can be embedded in a surface of genus γ, although the faces need not be 2-cells.

THEOREM 1. $$N(\gamma, \gamma') \leqq 8+\sqrt{(18+12(\gamma+\gamma'))}.$$

Proof. Let the connected components of G be $H_1, H_2, \ldots$. Let γ_i be the genus of H_i and suppose that H_i has p_i points, q_i lines and r_i faces. Since every face of H_i has at least 3 lines on its boundary, we have $2q_i \geqq 3r_i$. Substituting this in Euler's formula, we obtain

$$2-2\gamma_i = p_i - q_i + r_i \leqq p_i - q_i/3,$$

so that

$$q_i/p_i \leqq 3+6(\gamma_i-1)/p_i.$$

The average degree $d(H_i)$ of the points of H_i is $2q_i/p_i$ and the average degree $d(G)$ of the points of G is given by

$$d(G) = p^{-1}\sum_i d(H_i)p_i \leqq p^{-1}\sum_i p_i(6+12(\gamma_i-1)/p_i),$$

where p is the number of points of G, so that $p = \sum_i p_i$. Thus

$$d(G) \leqq 6+12p^{-1}\sum_i(\gamma_i-1).$$

From Corollary 2 of [2] we have

$$\gamma(G) = \sum_i \gamma(H_i),$$

so that, if G can be embedded in a surface of genus γ, then

$$d(G) \leqq 6+12(\gamma-1)/p.$$

Similarly, if $\bar{G}$ can be embedded in a surface of genus γ', then

$$d(\bar{G}) \leqq 6+12(\gamma'-1)/p.$$

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We now observe that $d(G)+d(\bar{G}) = p-1$, so that, if G is (γ, γ') bi-embeddable, we have

$$p-1 \leqq 12+12(\gamma+\gamma'-2)/p,$$

which gives

$$p \leqq \tfrac{1}{2}(13+\sqrt{(73+48(\gamma+\gamma'))}).$$

Therefore

$$N(\gamma, \gamma') \leqq \tfrac{1}{2}(15+\sqrt{(73+48(\gamma+\gamma'))}) \leqq 8+\sqrt{(18+12(\gamma+\gamma'))},$$

which completes the proof.

Before considering lower bounds for $N(\gamma, \gamma')$, we observe that $N(\gamma, \gamma') = N(\gamma', \gamma)$; so we may suppose that $\gamma \geqq \gamma'$.

THEOREM 2. *If* $\gamma \geqq \gamma' \geqq 5\gamma/6$, *then*

$$N(\gamma, \gamma') \geqq \sqrt{(8\gamma+12\gamma')}.$$

Proof. We put $h = \gamma/(\gamma+\gamma')$, so that $\frac{1}{2} \leqq h \leqq 6/11$. Let

$$m = \sqrt{(3-h)}+\sqrt{(3-5h)}, \qquad n = \sqrt{(3-h)}-\sqrt{(3-5h)},$$
$$x = \{m\sqrt{(\gamma+\gamma')}\}+1, \qquad y = \{n\sqrt{(\gamma+\gamma')}\}+1,$$

where $\{x\}$ is the least integer not less than x. We take G to be the complete bipartite graph $K_{x,y}$. Using a result of Ringel [3], we see that

$$\gamma(G) = \{(x-2)(y-2)/4\} \leqq mn(\gamma+\gamma')/4 = \gamma.$$

Also, $\bar{G} = K_x \cup K_y$; so, using the well-known result of Ringel and Youngs [4], we see that

$$\gamma(\bar{G}) \leqq \{(x-3)(x-4)/12\}+\{(y-3)(y-4)/12\} \leqq (m^2+n^2)(\gamma+\gamma')/12 = (1-h)(\gamma+\gamma') = \gamma'.$$

Therefore $N(\gamma, \gamma') > x+y \geqq 2\sqrt{(3-h)}\sqrt{(\gamma+\gamma')} = \sqrt{(8\gamma+12\gamma')}$.

COROLLARY 1. $\sqrt{(20\gamma)} \leqq N(\gamma, \gamma) \leqq 8+\sqrt{(18+24\gamma)}$.

THEOREM 3. *If* $5\gamma/6 \geqq \gamma' \geqq \frac{1}{2}\gamma$, *then*

$$N(\gamma, \gamma') \geqq \sqrt{(8\gamma+12\gamma')}.$$

Proof. We put $m = \{\sqrt{(2\gamma)}\}$ and take G to be the complete tripartite graph $K_{m,m,m}$ so that $\bar{G} = 3K_m$. From a theorem of Ringel and Youngs which has since been generalized by White [6], we see that

$$\gamma(G) = \tfrac{1}{2}(m-2)(m-1) < \tfrac{1}{2}(m-1)^2 \leqq \gamma.$$

Using the estimate of Ringel and Youngs [4], we have

$$\gamma(\bar{G}) \leqq 3\{(m-3)(m-4)/12\} \leqq (m-1)^2/4 \leqq \tfrac{1}{2}\gamma \leqq \gamma'.$$

Hence, $N(\gamma, \gamma') > 3m = 3\{\sqrt{(2\gamma)}\} \geqq \sqrt{(18\gamma)}$.

Theorem 3 is of particular interest when $\gamma' = \frac{1}{2}\gamma$, since the coefficient of $\sqrt{\gamma}$ is then best possible.

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COROLLARY 2. $\sqrt{(18\gamma)} \leqq N(\gamma, \frac{1}{2}\gamma) \leqq 8+\sqrt{(18+18\gamma)}$.

THEOREM 4. *If* $\frac{1}{2}\gamma \geqq \gamma' \geqq 0$, *then*

$$N(\gamma, \gamma') \geqq 3+H(\gamma),$$

where $H(\gamma) = [\frac{1}{2}(7+\sqrt{(1+48\gamma)})]$ *and* $[x]$ *denotes the integer part of* x.

Proof. Let $H = H(\gamma)$; from [4] we see that the complete graph K_H can be embedded in a surface of genus γ. We take $G = K_H \cup K_2$. Since K_2 can be placed in one of the faces in the embedding of K_H we see that G may also be embedded in a surface of genus γ. Then $\bar{G}$ is the complete bipartite graph $K_{H,2}$, which is planar. Hence

$$N(\gamma, \gamma') \geqq N(\gamma, 0) > H+2,$$

which completes the proof.

When $\gamma' = 0$, the coefficient of $\sqrt{\gamma}$ in the lower bound is $\sqrt{12}$, which is best possible.

COROLLARY 3. $[\frac{1}{2}(13+\sqrt{(1+48\gamma)}] \leqq N(\gamma, 0) \leqq 8+\sqrt{(18+12\gamma)}$.

White [6] proved that the genus of the complete tripartite graph $K_{mn,n,n}$ is given by

$$\gamma(K_{mn,n,n}) = \tfrac{1}{2}(mn-2)(n-1) < \tfrac{1}{2}mn^2,$$

and we shall use this to obtain lower bounds for $N(\gamma, \gamma')$ which improve on Theorem 4 when γ' is greater than a suitable multiple of γ.

THEOREM 5. *Let* $m \geqq 12$ *be an integer; then, for* $\gamma \geqq \gamma' \geqq 6m\gamma/(m^2+2)$, *we have*

$$N(\gamma, \gamma') \geqq (m+2)[(m^2+2)^{-\frac{1}{2}}\sqrt{(12\gamma)}].$$

Proof. We put $n = [\sqrt{(12\gamma/(m^2+2))}]$ and take $\bar{G} = K_{mn,n,n}$, so that

$$\gamma(\bar{G}) < \tfrac{1}{2}mn^2 \leqq 6m\gamma/(m^2+2) \leqq \gamma'.$$

Then $G = K_{mn} \cup 2K_n$ so that

$$\gamma(G) \leqq \{(mn-3)(mn-4)/12\}+2\{(n-3)(n-4)/12\}$$
$$\leqq (m^2+2)n^2/12 \leqq \gamma.$$

Hence $N(\gamma, \gamma') > mn+2n = (m+2)[(m^2+2)^{-\frac{1}{2}}\sqrt{(12\gamma)}]$.

Finally we remark that the cases $m = 5$ and $m = 6$ may be used to obtain the following improvements on Theorem 2. Since the details are similar to those in the proof of Theorem 5, the proofs are omitted.

COROLLARY 4. *If* $\gamma \geqq \gamma' \geqq 18\gamma/19$, *then*

$$N(\gamma, \gamma') \geqq 8[\sqrt{(6\gamma/19)}] > \sqrt{(20.2\ldots\gamma)}, \quad \textit{as} \quad \gamma \to \infty.$$

COROLLARY 5. *If* $18\gamma/19 \geqq \gamma' \geqq 9\gamma/10$, *then*

$$N(\gamma, \gamma') \geqq 7[\sqrt{(2\gamma/5)}].$$

Note added in proof. B. L. Garman of Western Michigan University has recently proved that $\gamma(K_{n,n,n,n}) = (n-1)^2$ when $n \equiv 2 \pmod 4$. It follows from this result that $N(\gamma, \gamma') \geqq 4\sqrt{\gamma}$ for $\gamma' \geqq \gamma/3$, so that the bound of Theorem 1 is also best possible at $\gamma' = \gamma/3$.

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