ON A TRANSFORMATION GROUP

S. K. KAUL

0. Let Γ denote a group of *real* linear fractional transformations (the constants defining any element of Γ are real numbers); see (**3**, § 2, p. 10). Then it is known that Γ *is discontinuous if and only if it is discrete* (**3**, Theorem 2F, p. 13).

Now Γ may also be regarded, equivalently, as a group of homeomorphisms of a disc D onto itself; and if Γ is discrete, then, except for elements of finite order, each element of Γ is either of type 1 or type 2 (see Definitions 0.1 and 0.2 below).

We wish to generalize the result quoted above in purely topological terms. Thus, throughout this paper we denote by X a compact metric space with metric d, and by G a topological transformation group on X each element of which, except the identity e, is either of type 1 or type 2. Let $L = \{a \in X\colon g(a) = a \text{ for some } g \text{ in } G - e\}$, and $O = X - \bar{L}$. We assume furthermore that O is non-empty.

Let g be a homeomorphism of X onto itself.

Definition 0.1. g is said to be of type 2 if there exist distinct points p, q in X such that $g(p) = p$, $g(q) = q$, and, furthermore, one of the points, say p, has the property that for any compact set $C \subset X - \{q\}$, $\text{Lim}_{n\to\infty} g^n(C) = p$, and the other q that for any compact set $C \subset X - \{p\}$, $\text{Lim}_{n\to\infty} g^{-n}(C) = q$.

Definition 0.2. g is said to be of type 1 if there exists a single point p in X such that $g(p) = p$, and for any compact set $C \subset X - \{p\}$, $\text{Lim}_{n\to\infty} g^n(C) = p$ and $\text{Lim}_{n\to\infty} g^{-n}(C) = p$.

For these definitions see also (**2**).

Definition 0.3. G is said to be *discontinuous* if for each x in O all the accumulation points of $G(x) = \{g(x)\colon g \in G\}$ lie in $\bar{L}$. G is said to be *discrete* if there does not exist any sequence of distinct elements in G converging pointwise to the identity e in G.

Remark 1. Suppose that Γ is discrete or equivalently discontinuous (in the sense of (**3**)). Then from (**3**, p. 18, Theorem 3A), for any limit point (**3**, p. 10) λ, there is for any given ordinary point (**3**, p. 10) z, a sequence $\{v_n\}$ of distinct elements of Γ such that $\{v_n(z)\}$ converges to λ. Since z is an ordinary point, in the proof of (**3**, p. 18, Theorem 3B), case 2 cannot occur and $\{v_n(\infty)\}$ converges to λ. Consequently, for any other ordinary point z', the sequence $\{v_n(z')\}$ must also converge to λ. For if this is not the case, then for some subsequence $\{v_{ni}\}$

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the sequence $\{v_{n_i}(z')\}$ converges to a point λ' other than λ, and by the same reasoning as above, $\{v_{n_i}(\infty)\}$ converges to λ' $(\neq\lambda)$. However, this is impossible.

Remark 2. Γ may be regarded equivalently as a group of homeomorphisms of a disc D onto itself. It is well known that under the hyperbolic metric on the interior D° of D, each element of Γ is an isometry. Hence, Γ is regular on D° (see Definition 1.1). If, however, Γ is discrete, then Γ has property K (see Definition 1.1) at each ordinary point of Γ from Remark 1. Hence, from Theorem 1.1, it is regular on the set of all ordinary points of Γ, with respect to any metric consistent with the topology on D. Thus, for the so-called groups of the second kind (**3**, § 3E, p. 21), Γ is regular on a set that properly contains D°.

Definition 0.4. We say that G is quasi-discrete if for no sequence $\{g_n\}$ of distinct elements of G does the sequence $\{g_n|O\}$, of g_n's restricted to O, converges to $e|O$.

We shall prove the following result.

THEOREM A. *Suppose that G has property K on O, and O is connected and locally connected. If G is quasi-discrete, then it is discontinuous.*

Remark 3. Of course, the converse of Theorem A is always true. Actually, it is easy to see that if G is discontinuous, then it is discrete. Theorem A does indeed give the classical result for Γ quoted in the first paragraph. To see this, it is enough to check that *if Γ is discrete, then it is quasi-discrete*:

Suppose, on the contrary, that Γ is discrete but that there exists a sequence $\{v_n\}$ of distinct elements of Γ such that $\{v_n|O\}$ converges to $e|O$. Since Γ is discrete, the set of its ordinary points contains the upper half plane (**3**, p.13, Theorem 2F). Thus, from (**4**, p. 73, Theorem 9H) we see that $\{g\}$ converges to a linear fractional transformation which, being the identity on the upper half plane, is the identity transformation. However, this contradicts that Γ is discrete.

Definition 0.5. G is said to be *properly discontinuous* if for any $x \in O$ there exists an open set U containing x such that $g[U] \cap U = \emptyset$ for any $g \in G - e$. G is said to satisfy Sperner's condition (see **1**) if for any compact subset C of O the set $\{g \in G\colon g[C] \cap C \neq \emptyset\}$ is finite.

THEOREM B. *If G is regular on O* (see Definition 1.1), *then the following are equivalent:*

(a) *G is discontinuous,*

(b) *G satisfies Sperner's condition,*

(c) *G is properly discontinuous.*

1. Before proceeding with the problem at hand we prove a result, in a slightly more general setting, which may be of interest in itself.

Definition 1.1. Let Y and Z be metric spaces, and, for a positive real number r and a point x of a metric space, let $U(x, r)$ denote the open r-sphere about x.

Let F be a family of continuous functions from Z to Y. F is said to be *regular* at a point p of Z if given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$f[U(p, \delta)] \subset U(f(p), \epsilon)$$

for each f in F. F is said to be regular on $A \subseteq Z$ if it is regular at each point of A. We say that F has *property* K at a point p in Z if for any sequence $\{f_n\}$ of distinct elements of F such that $\{f_n(p)\}$ converges to, say, q in Y, given any open set V containing q, there exists an open set U containing p such that for any $z \in U, f_n(z) \in V$ for $n \geqq N(z)$ for some positive integer $N(z)$ depending upon z. F has property K on Z if it has the property K at each point of Z.

It is clear from the definition of regularity that if F is regular at some point, then it has property K at that point. We give sufficient conditions for the converse.

THEOREM 1.1. *Suppose that Z is locally compact and locally connected, Y is compact and each member of F is an open map. If F has property* K *at a point, then it is regular at that point.*

Proof. Suppose that F has property K at a point p but is not regular at p Then there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exists an infinite subset $F(\delta)$ of F such that for any f in $F(\delta)$, $\operatorname{diam} f[U(p, \delta)] > \epsilon$, where diam denotes diameter. Since $F(n^{-1})$ is infinite for each $n = 1, 2, \ldots$, there exists a sequence $\{f_n\}$ of distinct elements of F such that $f_n \in F(n^{-1})$. Since Y is compact we may assume, without loss of generality, that the sequence $\{f_n(p)\}$ converges to, say, q. Assume also, for convenience, that $f_n(p) \in U(q, \epsilon)$ for all n. Since F satisfies property K and X is locally compact, there exists a $\delta > 0$ such that for any $z \in \overline{U(p, \delta)} = D$ there exists an $N(z)$ such that for $n \geqq N(z), f_n(z) \in U(q, \epsilon)$.

Consider the subspace D. For each $n, f_n^{-1}[\operatorname{bdry} U(q, \epsilon)] = B_n$ is a closed set in Z, where bdry denotes the boundary. We claim that for $n = 1, 2, \ldots$, the closed set $A_n = D \cap B_n$ does not contain an interior point with respect to D. Suppose the contrary. Let U be an open set in D contained in A_n. If U does not lie in $U(p, \delta)$, then it contains a non-empty open set W which does. But then W is open in Z, and therefore, since f_n is an open map, $f_n[W]$ is open in Y. However, since $W \subset A_n, f_n[W] \subset \operatorname{bdry}[U(q, \epsilon)]$, which is impossible. This establishes the claim that the A_n's are nowhere dense in D. Since D is a compact Hausdorff space, by Baire's theorem

$$D - \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (D - A_n) = E$$

is an open, everywhere dense subset of D. By the assumption that $f_n(p) \in U(q, \epsilon)$ for all n, $p \in E$. Hence, there exists an $\eta > 0$ such that $U(p, \eta) \subset E$. Since Z is locally connected, there exists a connected open set V such that $p \in V \subset U(p, \eta)$. From the choice of f_n ($\in F(n^{-1})$), there exists a positive integer N such that

$U(p, n^{-1}) \subset V$ for $n \geqq N$. Hence, diam$f_n[V] > \epsilon$ for $n \geqq N$. But then, since V is connected and $f_n[V] \cap U(q, \epsilon) \neq \emptyset, f_n[V] \cap \text{bdry}[U(q, \epsilon)] \neq \emptyset$. Hence, $V \cap A_n \neq \emptyset$. However, this is a contradiction, since $V \subset E$. Thus, the assumption that F is not regular leads to a contradiction, and this completes the proof.

Remark 4. In view of the Baire theorem for complete metric spaces, Theorem 1.1 is still true if we assume X to be complete instead of locally compact.

2. Let A and B be two functions, taking $G - e$ into X, defined as follows: Let $g \in G - e$; if g is of type 1, then $A(g) = B(g) = p$, where p is the fixed point of g. If g is of type 2, then $A(g) = p$ and $B(p) = q$, where p and q are as in Definition 0.2. That is, $A(g)$ is the fixed point of g to which $\{g^n(C)\}$ converges for any compact subset C of X not containing the other fixed point of g and $B(g)$ is the fixed point of g to which $\{(g^{-1})^n(C)\}$ converges for any compact subset C of X not containing the other fixed point, $n = 1, 2, \ldots$. Clearly then $A(g) = B(g^{-1})$ for any $g \in G - e$.

LEMMA 2.1. *Let $g, h \in G$. Then $A(hgh^{-1}) = h(A(g))$ and $B(hgh^{-1}) = h(B(g))$.*

Proof. It is easy to see that the only fixed points of hgh^{-1} are $h(A(g))$ and $h(B(g))$, and since h is a homeomorphism, the fixed points are distinct if and only if $A(g) \neq B(g)$. Let $C \subseteq X - h(B(g))$ be compact. Then

$$\begin{aligned}\lim_{n\to\infty} (hgh^{-1})^n(C) &= \lim_{n\to\infty} (hg^nh^{-1})(C)\\ &= h \lim_{n\to\infty} g^n(h^{-1}(C))\\ &= h[A(g)], \quad \text{since } h^{-1}[C] \subset X - B(g).\end{aligned}$$

Hence, $A(hgh^{-1}) = h[A(g)]$. Similarly, one can show that $B(hgh^{-1}) = h[B(g)]$.

LEMMA 2.2. *If $g \in G$, then $g[L] = L$. Consequently, $g[\bar{L}] = \bar{L}$ and $g[0] = 0$.*

Proof. To prove the lemma it is enough to show that for any $g \in G, g[L] \subset L$. Let $a \in L$. Then there exists an $h \in G$ such that $A[h] = a$ or $B[h] = a$. From Lemma 2.1, $g(a) = A(ghg^{-1})$ or $B(ghg^{-1})$. Since $ghg^{-1} \in G$, $g(a) \in L$. Hence, $g[L] \subset L$. This completes the proof.

THEOREM 2.1. *If C satisfies Sperner's condition, then G is discontinuous.*

Proof. Suppose that G is not discontinuous. Then there exists a point $x \in O$ and a sequence $\{g_n\}$ of distinct elements of G such that $\text{Lim}_{n\to\infty} g_n(x) = y \in O$. From Lemma 2.2, $C = \{g_n(x): n = 1, 2, \ldots\} \cup \{x, y\}$ is contained in O. Clearly, C is compact and the set $\{g \in G: g[C] \cap C \neq \emptyset\}$ contains $\{g_n\}$, contradicting that G satisfies Sperner's condition. Hence, G is discontinuous.

LEMMA 2.3. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in O converging to points x and y in O, respectively, and let $\{g_n\}$ be a sequence in G such that $g_n(x_n) = y_n$, $n = 1, 2, \ldots$. If G is regular on O, then $\text{Lim}_{n\to\infty} g_n(x) = y$.*

Proof. Since G is regular on O and $x \in O$, given an $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x, z) < \delta$, then for any $g \in G$, $d(g(x), g(z)) < \epsilon/2$. Since $\text{Lim}_{n\to\infty} x_n = x$, there exists an N_1 such that $d(x, x_n) < \delta$ for $n \geqq N_1$. Again, since $\text{Lim}_{n\to\infty} y_n = y$, there exists an N_2 such that $d(y, y_n) < \epsilon/2$ for $n \geqq N_2$. Thus, if $n \geqq \max(N_1, N_2)$, then

$$\begin{aligned} d(g_n(x), y) &\leqq d(g_n(x), g_n(x_n)) + d(g_n(x_n), y) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Hence, $\text{Lim}_{n\to\infty} g_n(x) = y$.

LEMMA 2.4. *Let $x \in O$, let $\{g_n\}$ be any sequence in G, and let $\{g_n(x)\}$ converge to $y \in O$. If G is regular on O, then for any given $\epsilon > 0$ there exists a positive integer N such that $d(g_m^{-1} \cdot g_n(x), x) < \epsilon$ for any $m, n > N$.*

Proof. Since G is regular, given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, z) < \delta$ implies that $d(g(x), g(z)) < \epsilon/2$ for any g in G. From Lemma 2.3 we can deduce that $\text{Lim}_{n\to\infty} g_n(x) = y$ implies that $\text{Lim}_{n\to\infty} g_n^{-1}(y) = x$. Hence, there exists a positive integer N_1 such that $d(x, g_n^{-1}(y)) < \epsilon/2$ for $n \geqq N_1$. Let N_2 be such that, for $n \geqq N_2$, $d(g_n(x), y) < \delta$. This yields (since $g_n(x) \in O$, from Lemma 2.2): $d(g_m^{-1} \cdot g_n(x), g_m^{-1}(y)) < \epsilon/2$ for $n \geqq N_2$ and any m. If $n, m > N = \max(N_1, N_2)$, then

$$\begin{aligned} d(g_m^{-1} \cdot g_n(x), x) &\leqq d(g_m^{-1} \cdot g_n(x), g_m^{-1}(y)) + d(g_m^{-1}(y), x) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

This completes the proof.

THEOREM 2.2. *If G is regular on O and properly discontinuous, then G is discontinuous.*

Proof. Suppose that G is not discontinuous. Then there exists an $x \in O$ and a sequence $\{g_n\}$ of distinct elements of G such that $\{g_n(x)\}$ converges to, say, $y \in O$. Applying Lemma 2.3 with $x_n = x$ for each n, and $y_n = g_n(x)$, we obtain $\text{Lim}_{n\to\infty} g_n(x) = y$. Thus, from Lemma 2.4, we see that for any open set U containing x there exists an infinite number of distinct elements $g_m^{-1} \cdot g_n$ in G for which $g_m^{-1} \cdot g_n[U] \cap U \neq \emptyset$, contradicting the fact that G is properly discontinuous. This completes the proof.

THEOREM 2.3. *Suppose that X is locally connected and G satisfies property* K *on O, then G is regular on O.*

Proof. The proof follows directly from Theorem 1.1.

THEOREM 2.4. *If G is discontinuous and regular on O, then G satisfies Sperner's condition.*

Proof. If G does not satisfy Sperner's condition, then there exists a compact set $C \subset O$ such that the set $\{g \in G\colon g[C] \cap C \neq \emptyset\}$ is infinite. This implies that there exist sequences $\{x_n\}$ and $\{y_n\}$ in C, converging to say x and y, respectively, in C, and a sequence $\{g_n\}$ of distinct elements of G such that $g_n(x_n) = y_n$. From Lemma 2.3, $\text{Lim}_{n\to\infty} g_n(x) = y$. This contradicts the hypothesis that G is discontinuous and completes the proof.

Proof of Theorem B. Since G is regular on O, from Theorem 2.4 we see that (a) implies (b).

If G is not properly discontinuous, then for some x in O and any open set U containing x, the set $\{g \in G: g[U] \cap U \neq \emptyset\}$ is infinite. Let U be an open set containing x such that $\bar{U} \subset 0$. Then $\{g \in G: g[\bar{U}] \cap \bar{U} \neq \emptyset\}$ is infinite and G does not satisfy Sperner's condition. Hence, (b) implies (c).

From Theorem 2.3 we see that (c) implies (a). This completes the proof.

3. Assume, for the purposes of this section, that G is regular on O.

LEMMA 3.1. *Let $x \in O$ and let $\{g_n\}$ be a sequence in G such that*

$$\text{Lim sup}_{n\to\infty} g_n(x) \subseteq 0.$$

Then there exists a $\delta > 0$ such that if $d(x, y) < \delta$, then

$$\text{Lim sup}_{n\to\infty} g_n(y) \subseteq 0.$$

Proof. Let $\text{Lim sup}_{n\to\infty} g_n(x) = F$. Then F is closed (7). Since $F \subset O$ and O is open, there exists an $\epsilon > 0$ such that for any $p \in F$, $U(p, \epsilon) \subset O$. Since G is regular on O, given $x \in O$ and $\epsilon > 0$ there exists a $\delta > 0$ such that if $y \in U(x, \delta)$, then $g(y) \in U(g(x), \epsilon/4)$ for any $g \in G$.

Suppose that $y \in O$ and $d(x, y) < \delta$. If $\text{Lim sup}_{n\to\infty} g_n(y) \cap \bar{L} \neq \emptyset$, then there exists a subsequence $\{h_n\}$ of $\{g_n\}$ such that $\text{Lim}_{n\to\infty} h_n(y) = a \in \bar{L}$. But then $h_n(x) \in U(a, \epsilon/2)$ for all except a finite number of n's. Hence, there exists a point $p \in F$ such that $d(a, p) < \epsilon$, contradicting the choice of ϵ. Hence, $\text{Lim sup}_{n\to\infty} g_n(y) \cap \bar{L} = \emptyset$. This completes the proof.

The proof of the following lemma is trivial.

LEMMA 3.2. *Let $x \in O$, $\{g_n\} \subset G$ and* $\text{Lim}_{n\to\infty} g_n(x) = a \in \bar{L}$ *and* $\text{Lim}_{n\to\infty} g_n^{-1}(a) = b$. *Then there exist subsequences $\{g_{n(k)}\}$ and $\{g_{m(k)}\}$, $k = 1, 2, \ldots,$ of $\{g_n\}$, such that* $\text{Lim}_{k\to\infty} g_m^{-1} \cdot g_{n(k)}(x) = b$.

THEOREM 3.1. *Suppose that O is connected. If for some $x \in O$ there exists a sequence $\{g_n\}$ in G such that* $\text{Lim sup}_{n\to\infty} g_n(x) \subset O$, *then for any $y \in O$,*

$$\text{Lim sup}_{n\to\infty} g_n(y) \subset O.$$

Proof. From Lemma 3.1 the set $O' = \{z \in O: \text{Lim sup}_{n\to\infty} g_n(z) \subset O\}$ is open. It is non-empty since x lies in it. We shall show that O' is closed in O. Suppose that $p \in O - O'$ is an accumulation point of O', and $\text{Lim sup}_{n\to\infty} g_n(p) \cap \bar{L} \neq \emptyset$. Then there exists a subsequence $\{h_n\}$ of $\{g_n\}$ such that $\text{Lim}_{n\to\infty} h_n(p) = a \in \bar{L}$. We may assume, without loss of generality, that $\{h_n^{-1}(a)\}$ converges to $b \in \bar{L}$

since $h_n^{-1}(a) \in \bar{L}$ for all n [Lemma 2.2] and $\bar{L}$ is compact. Since $p \neq b$, there exists an $\epsilon > 0$ such that $U(p, \epsilon) \cap U(b, \epsilon) = \emptyset$ and $U(p, \epsilon) \subset O$. Now from Lemma 3.2 there exist subsequences $\{h_{n(k)}\}$ and $\{h_{m(k)}\}$, $k = 1, 2, \ldots$, of $\{h_n\}$ such that $\mathrm{Lim}_{k\to\infty} h_{m(k)}^{-1} \cdot h_{n(k)}(p) = b$. Since p is an accumulation point of O' and G is regular on O, there exists a $z \in O'$ such that $h_{m(k)}^{-1} \cdot h_{n(k)}(z) \in U(b, \epsilon)$ for all $k \geqq N$ for some positive integer N. However, this contradicts Lemma 2.4. Hence, O' is closed in O. Since O is connected, $O' = O$ and the proof is complete.

Using the regularity of G on O, it is easy to prove the following result.

LEMMA 3.3. *Let* $\{x_n\} \subset O$ *be a sequence converging to* x *in* O, *and let* $\{g_n\} \subset G$ *be such that* $\mathrm{Lim}_{m\to\infty} g_m(x_n) = x_n$ *for each* $n = 1, 2, \ldots$; *then* $\mathrm{Lim}_{n\to\infty} g_n(x) = x$.

Proof of Theorem A. Since G has property K on O and O is locally connected and locally compact and X is compact from Theorem 1.1, G is regular on O. Suppose that G is not discontinuous. Then there exists an $x \in O$ and a sequence $\{g_n\}$ of distinct elements of G such that $\mathrm{Lim}_{n\to\infty} g_n(x) \in O$. Thus, from Theorem 3.1, $\mathrm{Lim\,sup}_{n\to\infty} g_n(y) \subset O$ for any $y \in O$. Since X is separable, there exists a countable set $\{p_i\}$, $i = 1, 2, \ldots$, in O which is dense in X. For each i, let R_i be the closure of the set $\{g_n(p_i) : n = 1, 2, \ldots\}$. Then by the above observation $R_i \subset O$ for each $i = 1, 2, \ldots$ and is compact. Hence, by Cantor's diagonal process, there exists an increasing sequence $n(k)$, $k = 1, 2, \ldots$, of natural numbers, such that for each i, the sequence $\{g_{n(k)}(p_i)\}$ converges in R_i; see (**5**, p. 45, Theorem 9). From Lemma 2.4 the sequence

$$\{g_{n(k+1)}^{-1} \cdot g_{n(k)}(p_i)\}, \qquad k = 1, 2, \ldots,$$

converges to p_i, $i = 1, 2, \ldots$. Since $\{p_i\}$ is dense in O, by Lemma 3.3, if $y \in O$, then $g_{n(k+1)}^{-1} \cdot g_{n(k)}(y) = y$. Hence, $\mathrm{Lim}_{k\to\infty} g_{n(k+1)}^{-1} \cdot g_{n(k)}|O = e|O$, where e is the identity in G. However, this is a contradiction, since G is quasidiscrete. Hence, G is discontinuous on O, and the proof is complete.

This work was motivated by some problems suggested in (**1**).

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University of Saskatchewan,
Regina, Saskatchewan