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Periodic solutions of four-order degenerate differential equations with finite delay in vector-valued function spaces

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In this paper, we mainly investigate the well-posedness of the four-order degenerate differential equation (P_4): $(Mu)''''(t) + \alpha(Lu)'''(t) + (Lu)''(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t)$, $(t \in [0, 2\pi])$ in periodic Lebesgue–Bochner spaces $L^p(\mathbb{T}; X)$ and periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$, where A, B, L and M are closed linear operators on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(L)$ and $\alpha, \beta, \gamma \in \mathbb{C}$, G and F are bounded linear operators from $L^p([-2\pi, 0]; X)$ (respectively $B^s_{p,q}([-2\pi, 0]; X)$) into X, $u_t(\cdot) = u(t + \cdot)$ and $u'_t(\cdot) = u'(t + \cdot)$ are defined on $[-2\pi, 0]$ for $t \in [0, 2\pi]$. We completely characterize the well-posedness of (P_4) in the above two function spaces by using known operator-valued Fourier multiplier theorems.

Keywords: well-posedness; degenerate differential equation; Fourier multiplier; Lebesgue-Bochner spaces; Besov spaces

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1. Introduction

The characterizations of the well-posedness for abstract degenerate differential equations with periodic initial conditions have been studied extensively in the last years. See e.g. [**5**–**11**], [**14**–**20**] and the references therein. For examples, Lizama and Ponce [**16**] considered the first-order degenerate equation:

$$(Mu)'(t) = Au(t) + f(t), \quad (t \in \mathbb{T} := [0, 2\pi]), \tag{1.1}$$

they gave necessary and sufficient conditions to guarantee the well-posedness of (1.1) in Lebesgue–Bochner spaces $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ under some appropriate assumptions on the modified resolvent operator determined by (1.1). Moreover, they also investigated

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the first-order degenerate equation with infinite delay [17]:

$$(Mu)'(t) = \alpha Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\mathrm{d}s + f(t), \quad (t \in \mathbb{T}), \tag{1.2}$$

where A and M are closed linear operators defined on a Banach space X with $D(A) \subseteq D(M)$, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel, $\alpha \in \mathbb{R}\backslash\{0\}$ and f an X-valued function defined on $\mathbb{T}$.

Bu [9] considered a new second-order degenerate equation and gave necessary or sufficient conditions for this equation to be L^p-well-posed (respectively $B^s_{p,q}$-well-posed and $F^s_{p,q}$-well-posed), which recover some known results presented in [5, 6, 10] in the simpler case $M = I_X$. We notice that third-order differential equations also describe some kinds of models arising from natural phenomena, such as flexible space structures with internal damping, the well-posedness of third-order differential equations has been investigated extensively by many authors. See [1–3, 7, 8, 13, 14, 19] for more information and references therein. For example, Poblete and Pozo [19] studied the well-posedness for the abstract third-order equation:

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), \ (t \in \mathbb{T}), \tag{1.3}$$

where A and B are closed linear operators defined on a Banach space X with $D(A) \cap D(B) \neq \emptyset$, the constants α, β, $\gamma \in \mathbb{R}^+$ and f belong to either the Lebesgue–Bochner spaces, or periodic Besov spaces, or periodic Triebel–Lizorkin spaces. They give necessary and sufficient conditions for (1.3) to be L^p-well-posed (respectively $B^s_{p,q}$-well-posed and $F^s_{p,q}$-well-posed) by using vector-valued Fourier theorems in the vector-valued function spaces.

In this paper, we study the following four-order degenerate differential equation:

$$\begin{aligned}(Mu)''''(t) &+ \alpha(Lu)'''(t) + (Lu)''(t) \\ &= \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad (t \in \mathbb{T}),\end{aligned} \tag{P_4}$$

where A, B, L and M are closed linear operators on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(L)$ and α, β, $\gamma \in \mathbb{C}$, G and F are bounded linear operators from $L^p([-2\pi, 0]; X)$ (respectively $B^s_{p,q}([-2\pi, 0]; X)$) into X, $u_t(\cdot) = u(t + \cdot)$ and $u'_t(\cdot) = u'(\cdot + t)$ are defined on $[-2\pi, 0]$ for $t \in [0, 2\pi]$.

Let $f \in L^p(\mathbb{T}; X)$ be given, a function $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X) \cap L^p(\mathbb{T}; D(A))$ is called a strong L^p-solution of (P_4), if $Mu \in W^{4,p}_{\mathrm{per}}(\mathbb{T}; X)$, $Lu \in W^{3,p}_{\mathrm{per}}(\mathbb{T}; X)$, $u' \in L^p(\mathbb{T}; D(B))$ and (P_4) is satisfied a.e. on $\mathbb{T}$, here we consider $D(A)$ and $D(B)$ as Banach spaces equipped with the graph norms. We say that (P_4) is L^p-well-posed, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of (P_4). We introduce similarly the $B^s_{p,q}$-well-posedness of (P_4).

The main purpose of this paper is to give some characterizations of the well-posedness of (P_4) in Lebesgue–Bochner spaces $L^p(\mathbb{T}; X)$ and periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. The characterizations of the well-posedness of (P_4) involve the Rademacher boundedness (or norm boundedness) of the M-resolvent of A, B and L defined by (P_4). More precisely, we show that when X is a UMD Banach space and $1 < p < \infty$, if $\{k(G_{k+1} - G_k) : \ k \in \mathbb{Z}\}$ is Rademacher-bounded, then (P_4) is

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L^p-well-posed if and only if $\rho_M(A, B, L) = \mathbb{Z}$ (the M-resolvent set of A, B and L defined by (P_4)) and the sets

$$\{k^4 M N_k : k \in \mathbb{Z}\}, \quad \{k^3 L N_k : k \in \mathbb{Z}\}, \quad \{k B N_k : k \in \mathbb{Z}\}, \quad \{k N_k : k \in \mathbb{Z}\}$$

are Rademacher-bounded, where

$$N_k = [(k^4 M - (i\alpha k^3 + k^2)L - \beta A - i\gamma k B - ikG_k - F_k]^{-1},$$

$G_k,\ F_k,\ H_k \in \mathcal{L}(X)$ are defined by $G_k x = G(e_k x)$, $F_k x = F(e_k x)$, $x \in X$. Since this characterization of the L^p-well-posedness of (P_4) does not depend on the space parameter $1 < p < \infty$, we deduce that when X is a UMD Banach space and the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is Rademacher-bounded, then (P_4) is L^p-well-posed for some $1 < p < \infty$ if and only if it is L^p-well-posed for all $1 < p < \infty$.

We also give a similar characterization for the $B^s_{p,q}$-well-posedness of (P_4): let X be a Banach space, $1 \leqslant p, q \leqslant \infty$, $s > 0$, assume that the sets $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ and $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$ are norm-bounded, then the problem (P_4) is $B^s_{p,q}$-well-posed if and only if $\subset \rho_M(A, B, L) = \mathbb{Z}$ and the sets

$$\{k^4 M N_k : k \in \mathbb{Z}\}, \quad \{k^3 L N_k : k \in \mathbb{Z}\}, \quad \{k B N_k : k \in \mathbb{Z}\}, \quad \{k N_k : k \in \mathbb{Z}\}$$

are norm-bounded, where N_k, F_k, G_k and H_k are defined as in the L^p-well-posedness case. Since this characterization of the $B^s_{p,q}$-well-posedness of (P_4) does not depend on the parameters $1 \leqslant p, q \leqslant \infty$, $s > 0$, we deduce that when the sets $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ and $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$ are norm-bounded, then (P_4) is $B^s_{p,q}$-well-posed for some $1 \leqslant p, q \leqslant \infty$, $s > 0$ if and only if it is $B^s_{p,q}$-well-posed for all $1 \leqslant p, q \leqslant \infty$, $s > 0$.

Our main tools in the investigation of the well-posedness of (P_4) are the operator-valued Fourier multiplier theorems obtained by Arendt and Bu [**5**, **6**] on $L^p(\mathbb{T}; X)$ and $B^s_{p,q}(\mathbb{T}; X)$. In fact, our main idea is to transform the well-posedness of (P_4) to an operator-valued Fourier multiplier problem in the corresponding vector-valued function space.

This work is organized as follows: in § 2, we study the well-posedness of (P_4) in vector-valued Lebesgue–Bochner spaces $L^p(\mathbb{T}; X)$. In § 3, we consider the well-posedness of (P_4) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. In the last section, we give some examples of degenerate differential equations with finite delay to which our abstract results may be applied.

2. Well-posedness of (P_4) in Lebesgue–Bochner spaces

Let X and Y be complex Banach spaces and let $\mathbb{T} := [0, 2\pi]$. We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. For $1 \leqslant p < \infty$, we denote by $L^p(\mathbb{T}; X)$ the space of all equivalent class of X-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\|f\|_{L^p} := \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(t)\|^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

For $f \in L^1(\mathbb{T}; X)$, the k-th Fourier coefficient of f is defined by

$$\hat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)\,\mathrm{d}t,$$

where $k \in \mathbb{Z}$ and $e_k(t) = e^{ikt}$ when $t \in \mathbb{T}$.

Definition 2.1. *Let X and Y be complex Banach spaces and $1 \leqslant p < \infty$, we say that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique $u \in L^p(\mathbb{T}; Y)$ such that $\hat{u}(k) = M_k\hat{f}(k)$ when $k \in \mathbb{Z}$.*

From the closed graph theorem, if $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier, then there exists a unique bounded linear operator $T \in \mathcal{L}(L^p(\mathbb{T}; X), L^p(\mathbb{T}; Y))$ satisfying $(Tf)^\wedge(k) = M_k\hat{f}(k)$ when $f \in L^p(\mathbb{T}; X)$ and $k \in \mathbb{Z}$. The operator-valued Fourier multiplier theorem on $L^p(\mathbb{T}; X)$ obtained in [5] involves the Rademacher boundedness for sets of bounded linear operators. Let γ_j be the j-th Rademacher function on $[0, 1]$ defined by $\gamma_j(t) = \mathrm{sgn}(\sin(2^{\mathrm{j}-1}\mathrm{t}))$ when $j \geqslant 1$. For $x \in X$, we denote by $\gamma_j \otimes x$ the vector-valued function $t \to r_j(t)x$ on $[0, 1]$.

Definition 2.2. *Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X, Y)$ is said to be Rademacher-bounded (R-bounded, in short), if there exists $C > 0$ such that*

$$\left\|\sum_{j=1}^n \gamma_j \otimes T_j x_j\right\|_{L^1([0,1];Y)} \leqslant C \left\|\sum_{j=1}^n \gamma_j \otimes x_j\right\|_{L^1([0,1];X)}$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

Remark 2.3.

(i) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then it can be shown easily from the definition that $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still R-bounded.

(ii) Let X be a UMD Banach space and let $M_k = m_k I_X$ with $m_k \in \mathbb{C}$, where I_X is the identity operator on X, if $\sup_{k\in\mathbb{Z}} |m_k| < \infty$ and $\sup_{k\in\mathbb{Z}} |k(m_{k+1} - m_k)| < \infty$, then $(M_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier whenever $1 < p < \infty$ [5].

The main tool in our study of L^p-well-posedness of (P_4) is the L^p-Fourier multiplier theorem established in [5]. The following results will be very important in the proof of our main result of this section. For the concept of UMD Banach spaces, we refer the readers to [5] and references therein.

Theorem 2.4 [5, Theorem 1.3]. *Let X, Y be* UMD *Banach spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. If the sets $\{M_k :\ k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) :\ k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

Proposition 2.5 [5, Proposition 1.11]. *Let X, Y be Banach spaces, $1 \leqslant p < \infty$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ be an L^p-Fourier multiplier, then the set $\{M_k :\ k \in \mathbb{Z}\}$ is R-bounded.*

Now we consider the following four-order degenerate differential equations with finite delays:

$$\begin{aligned}(Mu)''''(t) &+ \alpha(Lu)'''(t) + (Lu)''(t)\\ &= \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad (t \in \mathbb{T}) \qquad (P_4)\end{aligned}$$

where A, B, M and L are closed linear operators on a Banach space X satisfying $D(A) \cap D(B) \subset D(M) \cap D(L)$, $\alpha, \beta, \gamma \in \mathbb{C}$ are given and $F, G : L^p([-2\pi, 0]; X) \to X$ are bounded linear operators (F and G are known as the delay operators). Moreover, for fixed $t \in \mathbb{T}$, the functions u_t and u'_t are elements in $L^p([-2\pi, 0]; X)$ defined by $u_t(s) = u(t + s)$, $u'_t(s) = u'(t + s)$ for $-2\pi \leqslant s \leqslant 0$, here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

Let $F, G \in \mathcal{L}((L^p[-2\pi, 0]; X), X)$ and $k \in \mathbb{Z}$. We define the linear operators $F_k, G_k \in \mathcal{L}(X)$ by

$$F_k x := F(e_k x), \quad G_k x := G(e_k x), \tag{2.1}$$

for $x \in X$, where $e_k(t) = e^{ikt}$ when $t \in \mathbb{T}$. It is clear that $\|F_k\| \leqslant \|F\|$ and $\|G_k\| \leqslant \|G\|$ as $\|e_k\|_p = 1$. It is easy to see that when $u \in L^p(\mathbb{T}; X)$, then

$$\widehat{Fu.}(k) = F_k\hat{u}(k), \quad \widehat{Gu.}(k) = G_k\hat{u}(k) \tag{2.2}$$

for $k \in \mathbb{Z}$. This implies that $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as

$$\|Fu_t\| \leqslant \|F\|\|u.\|_{L^p([-2\pi,0];X)} = \|F\|\|u\|_{L^p},$$

and

$$\|Gu_t\| \leqslant \|G\|\|u.\|_{L^p([-2\pi,0];X)} = \|G\|\|u\|_{L^p},$$

for $t \in \mathbb{T}$ so that $Fu.$, $Gu.$, $Hu. \in L^p(\mathbb{T}; X)$.

Now we define the resolvent set of (P_4) by

$$\begin{aligned}\rho_M(A, B, L) := \big\{&k \in \mathbb{Z} : k^4M - (\alpha ik^3 + k^2)L\\ &- \beta A - i\gamma kB - ikG_k - F_k \text{ is invertible from}\\ D(A) \cap D(B) &\text{ onto } X \quad \text{and} \quad [k^4M - (\alpha ik^3 + k^2)L - \beta A\\ &- i\gamma kB - ikG_k - F_k]^{-1} \in \mathcal{L}(X)\big\}.\end{aligned}$$

For the sake of simplicity, when $k \in \rho_M(A, B, L)$, we will use the following notation:

$$N_k = [a_kM - b_kL - \beta A - c_kB - ikG_k - F_k]^{-1}, \quad (k \in \mathbb{Z}), \tag{2.3}$$

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where

$$a_k = k^4, \quad b_k = \alpha i k^3 + k^2, \quad c_k = i\gamma k, \quad (k \in \mathbb{Z}). \tag{2.4}$$

If $k \in \rho_M(A,\, B,\, L)$, then $MN_k,\ \ LN_k,\ \ AN_k$ and BN_k make sense as $D(A) \cap D(B) \subset D(M) \cap D(L)$ by assumption, and they belong to $\mathcal{L}(X)$ by the closed graph theorem and the closedness of $A,\ B,\ M$ and L.

Let $(L_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,\, Y)$ be a given sequence of operators. We define

$$(\triangle^0 L)_k = L_k, \quad (\triangle L)_k = L_{k+1} - L_k, \quad (k \in \mathbb{Z})$$

and for $n = 2,\, 3,\, \ldots,$ set

$$(\triangle^n L)_k = \triangle(\triangle^{n-1} L)_k, \quad (k \in \mathbb{Z}).$$

Definition 2.6. *A sequence $(d_k)_{k\in\mathbb{Z}} \subseteq \mathbb{C}\backslash\{0\}$ is called 1-regular if the sequence $(k\frac{\triangle^1 d_k}{d_k})_{k\in\mathbb{Z}}$ is bounded; it is called 2-regular if it is 1-regular and the sequence $(k^2\frac{\triangle^2 d_k}{d_k})_{k\in\mathbb{Z}}$ is bounded; it is called 3-regular if it is 2-regular and the sequence $(k^3\frac{\triangle^3 d_k}{d_k})_{k\in\mathbb{Z}}$ is bounded.*

Remark 2.7. It is easy to see that $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ are 3-regular.

Definition 2.8. *Let $1 \leqslant p < \infty$, $n \geqslant 1$ be an integer and let X be a Banach space, we define the the following vector-valued function spaces:*

$$\begin{aligned} W^{n,p}_{per}(\mathbb{T}; X) := \big\{&u \in L^p(\mathbb{T}; X):\ \textit{there exists } v \in L^p(\mathbb{T}; X), \textit{such that } \hat{v}(k) \\ &= (ik)^n \hat{u}(k) \textit{ for all } k \in \mathbb{Z}\big\}. \end{aligned}$$

$W^{n,p}_{per}(\mathbb{T}; X)$ is the n-th X-valued periodic Sobolev space.

Remark 2.9. We have the following two useful properties concerning these spaces:

(i) Let $m,\, n \in \mathbb{N}$. If $n \leqslant m$, then $W^{m,p}_{\rm per}(\mathbb{T}; X) \subseteq W^{n,p}_{\rm per}(\mathbb{T}; X)$.

(ii) If $u \in W^{n,p}_{\rm per}(\mathbb{T}; X)$, then for any $0 \leqslant k \leqslant n-1$, we have $u^{(k)}(0) = u^{(k)}(2\pi)$.

Let $1 \leqslant p < \infty$, we define the solution space of the L^p-well-posedness of (P_4) by

$$\begin{aligned} S_p(A,B,M,L) := \big\{&u \in W^{1,p}_{\rm per}(\mathbb{T}; X) \cap L^p(\mathbb{T}; D(A)) : Mu \in W^{4,p}_{\rm per}(\mathbb{T}; X), \\ &Lu \in W^{3,p}_{\rm per}(\mathbb{T}; X),\ u' \in L^p(\mathbb{T}; D(B))\big\}, \end{aligned}$$

here we consider $D(A)$ and $D(B)$ as Banach spaces equipped with their graph norms. The space $S_p(A,\, B,\, M,\, L)$ is complete equipped with the norm

$$\begin{aligned} \|u\|_{S_p(A,B,M,L)} := {}& \|u\|_{L^p} + \|Au\|_{L^p} + \|(Mu)'\|_{L^p} + \|(Mu)''\|_{L^p} + \|(Mu)'''\|_{L^p} \\ &+ \|(Mu)''''\|_{L^p} + \|(Lu)'\|_{L^p} + \|(Lu)''\|_{L^p} + \|(Lu)'''\|_{L^p} + \|Bu'\|_{L^p}. \end{aligned}$$

If $u \in S_p(A,\, B,\, M,\, L)$, then Mu, $(Mu)'$, $(Mu)''$ and $(Mu)'''$ are X-valued continuous functions on $\mathbb{T}$, and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$, $(Mu)''(0) = (Mu)''(2\pi)$, $(Mu)'''(0) = (Mu)'''(2\pi)$ by [5, Lemma 2.1].

Definition 2.10. *Let $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{T}; X)$, $u \in S_p(A, B, M, L)$ is called a strong L^p-solution of (P_4), if (P_4) is satisfied a.e. on $\mathbb{T}$. We say that (P_4) is L^p-well-posed, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of (P_4).*

If (P_4) is L^p-well-posed, then there exists a constant $C > 0$, such that for each $f \in L^p(\mathbb{T}; X)$, if $u \in S_p(A, B, M, L)$ is the unique strong L^p-solution of (P_4), we have

$$\|u\|_{S_p(A,B,M,L)} \leqslant C \|f\|_{L^p}. \tag{2.5}$$

This follows easily from the closed graph theorem.

In order to prove our main result of this section, we need the following preparations.

Proposition 2.11. *Let A, B, M and L be closed linear operators defined on a UMD Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(L)$, $1 < p < \infty$ and α, $\beta, \gamma \in \mathbb{C}$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$. Assume that $\rho_M(A, B, L) = \mathbb{Z}$ and the sets $\{a_k M N_k : k \in \mathbb{Z}\}$, $\{b_k L N_k : k \in \mathbb{Z}\}$, $\{c_k B N_k : k \in \mathbb{Z}\}$, $\{k \triangle G_k : k \in \mathbb{Z}\}$ and $\{k N_k : k \in \mathbb{Z}\}$ are R-bounded, then $(a_k M N_k)_{k\in\mathbb{Z}}$, $(b_k L N_k)_{k\in\mathbb{Z}}$, $(c_k B N_k)_{k\in\mathbb{Z}}$ and $(k N_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.*

Proof. We only need to show that the set $\{k(N_k^{-1} - N_{k+1}^{-1})N_k : k \in \mathbb{Z}\}$ is R-bounded by [**11**, Theorem 1.1] and theorem 2.4, here we have used the facts that $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ are 1-regular sequences. It follows from the definition of N_k that

$$\begin{aligned}
&(N_k^{-1} - N_{k+1}^{-1})N_k \\
&\quad = [a_k M - b_k L - \beta A - c_k B - ikG_k - F_k - a_{k+1}M + b_{k+1}L + \beta A + c_{k+1}B \\
&\qquad + i(k+1)G_{k+1} + F_{k+1}]N_k \\
&\quad = [-\triangle a_k M + \triangle b_k L + \triangle c_k B + ik\triangle G_k + iG_{k+1} + \triangle F_k]N_k,
\end{aligned} \tag{2.6}$$

which implies

$$\begin{aligned}
&k(N_k^{-1} - N_{k+1}^{-1})N_k \\
&\quad = -\frac{k\triangle a_k}{a_k}(a_k M N_k) + \frac{k\triangle b_k}{b_k}(b_k L N_k) + \frac{k\triangle c_k}{c_k}(c_k B N_k) \\
&\qquad + i(k\triangle G_k)(kN_k) + iG_{k+1}(kN_k) + \triangle F_k(kN_k),
\end{aligned} \tag{2.7}$$

when $k \neq 0$. It follows from remark 2.3 that the products and sums of R-bounded sets are still R-bounded. Thus, the set $\{k(N_k^{-1} - N_{k+1}^{-1})N_k : k \in \mathbb{Z}\}$ is R-bounded. This completes the proof. □

The following statement is the main result of this section which gives a necessary and sufficient condition for the L^p-well-posedness of (P_4).

Theorem 2.12. *Let X be a UMD Banach space, $1 < p < \infty$ and let A, B, L and M be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M) \cap$*

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$D(L)$ and $\alpha,\ \beta, \gamma \in \mathbb{C}$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$ be such that the set $\{k\Delta G_k :\ k \in \mathbb{Z}\}$ is R-bounded. Then the following assertions are equivalent:

(i) *(P_4) is L^p-well-posed;*

(ii) *$\rho_M(A, B, L) = \mathbb{Z}$, the sets $\{k^4 M N_k : k \in \mathbb{Z}\}$, $\{k^3 L N_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where N_k is defined by (2.3), the operators F_k and G_k are defined by (2.1).*

Proof. First we show that the implication $(i) \Rightarrow (ii)$ holds true. We assume that (P_4) is L^p-well-posed and let $k \in \mathbb{Z}$ and $y \in X$ be fixed, we consider the function f defined by $f(t) = e^{ikt}y$ when $t \in \mathbb{T}$. Then it is clear that $f \in L^p(\mathbb{T}; X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ when $n \neq k$. Since (P_4) is L^p-well-posed, there exists a unique $u \in S_p(A, B, L, M)$ satisfying

$$(Mu)''''(t) + \alpha(Lu)'''(t) + (Lu)''(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t) \tag{2.8}$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A) \cap D(B)$ when $n \in \mathbb{Z}$ by [**5**, Lemma 3.1] as $u \in L^p(\mathbb{T}; D(A)) \cap L^p(\mathbb{T}; D(B))$. Taking Fourier transforms on both sides of (2.8), we obtain

$$[k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]\hat{u}(k) = y \tag{2.9}$$

and $[n^4 M - (\alpha in^3 + n^2)L - \beta A - i\gamma nB - inG_n - F_n]\hat{u}(n) = 0$ when $n \neq k$. This implies that the operator $k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k$ defined on $D(A) \cap D(B)$ with values in X is surjective. To show that it is also injective, we let $x \in D(A) \cap D(B)$ be such that

$$[k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]x = 0.$$

Let u be the function given by $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$, then it is clear that $u \in S_p(A, B, M, L)$ and (P_4) is satisfied a.e. on $\mathbb{T}$ when $f = 0$. Thus, u is a strong L^p-solution of (P_4) when taking $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that the operator $k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k$ from $D(A) \cap D(B)$ into X is injective. Therefore, $k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k$ is bijective from $D(A) \cap D(B)$ onto X.

Next we show that $[k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let $u \in S_p(A, B, M, L)$ be the unique strong L^p-solution of (P_4). Then

$$\hat{u}(n) = \begin{cases} 0 & n \neq k, \\ [k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]^{-1}y & n = k, \end{cases}$$

by (2.9). This implies that u is given by

$$u(t) = e^{ikt}[k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]^{-1}y$$

when $t \in \mathbb{T}$. By (2.5), there exists a constant $C > 0$ independent from y and k, such that $\|u\|_{L^p} \leqslant C\|f\|_{L^p}$. This implies that

$$\left\|[k^4 M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]^{-1}y\right\| \leqslant C\|y\|$$

when $y \in X$, or equivalently

$$\left\|[k^4 M - (\alpha i k^3 + k^2)L - \beta A - i\gamma k B - ikG_k - F_k]^{-1}\right\| \leqslant C.$$

We have shown that $k \in \rho_M(A,\, B,\, L)$ for all $k \in \mathbb{Z}$. Thus, $\rho_M(A,\, B,\, L) = \mathbb{Z}$.

Finally, we show that $(k^4 M N_k)_{k\in\mathbb{Z}}$, $(k^3 L N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$ and $(kBN_k)_{k\in\mathbb{Z}}$ define L^p-Fourier multipliers. Let $f \in L^p(\mathbb{T}; X)$, then there exists $u \in S_p(A,\, B,\, M,\, L)$, a strong L^p-solution of (P_4) by assumption. Taking Fourier transforms on both sides of (P_4), we get that $\hat{u}(k) \in D(A) \cap D(B)$ by [**5**, Lemma 3.1] and

$$[k^4 M - (\alpha i k^3 + k^2)L - \beta A - i\gamma k B - ikG_k - F_k]\hat{u}(k) = \hat{f}(k)$$

for $k \in \mathbb{Z}$. Since $k^4 M - (\alpha i k^3 + k^2)L - \beta A - i\gamma k B - ikG_k - F_k$ is invertible, we have

$$\hat{u}(k) = [k^4 M - (\alpha i k^3 + k^2)L - \beta A - i\gamma k B - ikG_k - F_k]^{-1}\hat{f}(k) = N_k \hat{f}(k)$$

when $k \in \mathbb{Z}$. It follows from $u \in S_p(A,\, B,\, M,\, L)$ that $u \in L^p(\mathbb{T}; D(A)) \cap W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$, $Mu \in W^{4,p}_{\mathrm{per}}(\mathbb{T}; X)$, $Lu \in W^{3,p}_{\mathrm{per}}(\mathbb{T}; X)$ and $u' \in L^p(\mathbb{T}; D(B))$. We have

$$\begin{aligned}\widehat{(Mu)''''}(k) = k^4 M\hat{u}(k), \quad \widehat{(Lu)'''}(k) = -ik^3 L\hat{u}(k), \quad \widehat{Bu'}(k)\\ = ikB\hat{u}(k), \quad \widehat{u'}(k) = ik\hat{u}(k)\end{aligned}$$

when $k \in \mathbb{Z}$. We conclude that $(k^4 M N_k)_{k\in\mathbb{Z}}, (k^3 L N_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ define L^p-Fourier multipliers as $(Mu)''''$, $(Lu)'''$, Bu', $u' \in L^p(\mathbb{T}; X)$. It follows from proposition 2.5 that the sets $\{k^4 M N_k : k \in \mathbb{Z}\}$, $\{k^3 L N_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded. We have shown that the implication $(i) \Rightarrow (ii)$ is true.

Next we show that the implication $(ii) \Rightarrow (i)$ is valid. Assume that $\rho_M(A,\, B,\, L) = \mathbb{Z}$ and the sets $\{k^4 M N_k : k \in \mathbb{Z}\}$, $\{k^3 L N_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ and $\{kBN_k : k \in \mathbb{Z}\}$ are R-bounded. It follows from proposition 2.11 that $(k^4 M N_k)_{k\in\mathbb{Z}}$, $(k^3 L N_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. This implies that the sequences $(N_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k^2 L N_k)_{k\in\mathbb{Z}}$, $(MN_k)_{k\in\mathbb{Z}}$, $(LN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multiplier. Here we have used the easy fact that $(d_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier and the fact that the product of two L^p-Fourier multipliers is still an L^p-Fourier multiplier, where d_k is defined by $d_k = 1/k$ when $k \neq 0$ and $d_0 = 0$. In particular, considering $N_k \in \mathcal{L}(X,\, D(B))$, the sequence $(N_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. Then for all $f \in L^p(\mathbb{T}; X)$, there exist $u_i \in L^p(\mathbb{T}; X)$ $(1 \leqslant i \leqslant 7)$ and $u \in L^p(\mathbb{T}; D(B))$ satisfying

$$\begin{aligned}&\hat{u}_1(k) = k^4 M N_k \hat{f}(k), \quad \hat{u}_2(k) = ikN_k\hat{f}(k),\\ &\hat{u}_3(k) = MN_k\hat{f}(k), \quad \hat{u}_4(k) = LN_k\hat{f}(k)\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\hat{u}_5(k) = ikBN_k\hat{f}(k), \quad \hat{u}_6(k) = -ik^3 LN_k\hat{f}(k),\\ &\hat{u}_7(k) = -k^2 LN_k\hat{f}(k), \hat{u}(k) = N_k\hat{f}(k)\end{aligned} \tag{2.11}$$

for $k \in \mathbb{Z}$. Hence, $\hat{u}_2(k) = ik\hat{u}(k)$ for $k \in \mathbb{Z}$ by (2.10). This implies that $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$. It follows from (2.11) that $\widehat{u'}(k) = ik\hat{u}(k) = ikN_k\hat{f}(k)$ when $k \in \mathbb{Z}$.

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This together with $\hat{u}_5(k) = ikBN_k\hat{f}(k)$ when $k \in \mathbb{Z}$ implies that $u' \in L^p(\mathbb{T}; D(B))$ [**5**, Lemma 3.1]. By (2.10) and (2.11), we have $\hat{u}_3(k) = M\hat{u}(k)$ when $k \in \mathbb{Z}$. Hence, $u \in L^p(\mathbb{T}; D(M))$ and $Mu = u_3$. Similarly, by using (2.10) and (2.11), we have $\hat{u}_4(k) = L\hat{u}(k)$ when $k \in \mathbb{Z}$. Thus, $u \in L^p(\mathbb{T}; D(L))$ and $Lu = u_4$ [**5**, Lemma 3.1]. By (2.10) and the fact that $Mu = u_3$, we deduce $\hat{u}_1(k) = (ik)^4\widehat{Mu}(k) = (ik)^4\hat{u}_3(k)$ when $k \in \mathbb{Z}$. Thus, $Mu \in W^{4,p}_{\mathrm{per}}(\mathbb{T}; X)$. Similarly, using (2.11) and the fact hat $Lu = u_4$, we deduce that $Lu \in W^{3,p}_{\mathrm{per}}(\mathbb{T}; X)$.

We note that $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by (2.2), where G_k, F_k and H_k are defined by (2.1). Thus, $(ikG_kN_k)_{k\in\mathbb{Z}}$ and $(F_kD_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as the product of two L^p-Fourier multipliers is still an L^p-Fourier multiplier. We have

$$\beta AN_k = k^4MN_k - (\alpha ik^3 + k^2)LN_k - i\gamma kBN_k - ikG_kN_k - F_kN_k - I_X$$

for $k \in \mathbb{Z}$. It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also an L^p-Fourier multiplier as the sum of L^p-Fourier multipliers is an L^p-Fourier multiplier. We deduce from (2.11) and [**5**, Lemma 3.1] that $u \in L^p(\mathbb{T}; D(A))$. We have shown that $u \in S_p(A, B, M, L)$. This shows the existence of strong L^p-solution.

To show uniqueness of strong L^p-solution, we let $u \in S_p(A, B, M, L)$ be such that

$$(Mu)'''(t) + \alpha(Lu)'''(t) + (Nu)''(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t$$

a.e. on $\mathbb{T}$. Taking the Fourier transforms on both sides, we deduce that

$$[k^4M - (\alpha ik^3 + k^2)L - \beta A - i\gamma kB - ikG_k - F_k]\hat{u}(k) = 0$$

when $k \in \mathbb{Z}$. Since $\rho_M(A, B, L) = \mathbb{Z}$, this implies that $\hat{u}(k) = 0$ when $k \in \mathbb{Z}$ and thus $u = 0$. This shows that the solution is unique. This completes the proof. □

We notice that the assumption that the underlying Banach space X is a UMD space in theorem 2.12 was only used in the implication $(ii) \Rightarrow (i)$. Since the second statement of theorem 2.12 does not depend on the space parameter $1 < p < \infty$, theorem 2.12 has the following immediate consequence.

Corollary 2.13. *Let X be a* UMD *Banach space, let A, B, L and M be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M) \cap D(L)$, and α, β, $\gamma \in \mathbb{C}$. Then if (P_4) is L^p-well-posed for some $1 < p < \infty$, then it is L^p-well-posed for all $1 < p < \infty$.*

3. Well-posedness of (P_4) in Besov spaces

In this section, we consider the well-posedness of (P_4) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. Firstly, we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [**6**]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = \sup_{x\in\mathbb{T}} \left|f^{(\alpha)}(x)\right|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let

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$\mathcal{D}'(\mathbb{T};X) := \mathcal{L}(\mathcal{D}(\mathbb{T}),X)$ be the space of all continuous linear operator from $\mathcal{D}(\mathbb{T})$ to X. We consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \leqslant 2\}, I_k = \left\{t \in \mathbb{R} : 2^{k-1} < |t| \leqslant 2^{k+1}\right\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $\mathrm{supp}(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$, $\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1$ for $x \in \mathbb{R}$, and for each $\alpha \in \mathbb{N}_0$, $\sup_{x\in\mathbb{R},\, k\in\mathbb{N}_0} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty$. Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \phi(\mathbb{R})$ be fixed. For $1 \leqslant p$, $q \leqslant \infty$, $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T};X) = \Bigg\{ f \in \mathcal{D}'(\mathbb{T};X) : \|f\|_{B^s_{p,q}} := \left(\sum_{j\geqslant 0} 2^{sjq}\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\Big\|_p^q\right)^{1/q} < \infty\Bigg\}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T};X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms on $B^s_{p,q}(\mathbb{T};X)$. $B^s_{p,q}(\mathbb{T};X)$ equipped with the norm $\|\cdot\|_{B^s_{p,q}}$ is a Banach space. See [**6**, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T};X)$. It is well known that if $s_1 \leqslant s_2$, then $B^{s_1}_{p,q}(\mathbb{T};X) \subset B^{s_2}_{p,q}(\mathbb{T};X)$ and the embedding is continuous [**6**, Theorem 2.3]. When $s > 0$, it is shown in [**6**, Theorem 2.3] that $B^s_{p,q}(\mathbb{T};X) \subset L^p(\mathbb{T};X)$, $f \in B^{s+1}_{p,q}(\mathbb{T};X)$ if and only if f is differentiable a.e. on $\mathbb{T}$ and $f' \in B^s_{p,q}(\mathbb{T};X)$. This implies that if $u \in B^s_{p,q}(\mathbb{T};X)$ is such that there exists $v \in B^s_{p,q}(\mathbb{T};X)$ satisfying $\hat{v}(k) = ik\hat{u}(k)$ when $k \in \mathbb{Z}$, then $u \in B^{s+1}_{p,q}(\mathbb{T};X)$ and $u' = v$.

Let $1 \leqslant p$, $q \leqslant \infty$, $s > 0$ be fixed. We consider the following four-order degenerate differential equations with finite delay:

$$\begin{aligned}(Mu)''''(t) &+ \alpha(Lu)'''(t) + (Lu)''(t) \\ &= \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad (t \in \mathbb{T})\end{aligned} \tag{P_4}$$

where A, B, M and L are closed linear operators on a Banach space X satisfying $D(A) \cap D(B) \subset D(M) \cap D(L)$ and $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $f \in B^s_{p,q}(\mathbb{T};X)$ is given, and $F, G : B^s_{p,q}([-2\pi,0];X) \to X$ are bounded linear operators. Moreover, for fixed $t \in \mathbb{T}$, $u_t \in B^s_{p,q}([-2\pi,0];X)$ is defined by $u_t(s) = u(t+s)$ for $-2\pi \leqslant s \leqslant 0$, here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

Let $F, G \in \mathcal{L}(B^s_{p,q}[-2\pi,0];X), X)$ and $k \in \mathbb{Z}$. We define the linear operators $F_k,\ G_k$ by

$$F_k x := F(e_k * \otimes x), \quad G_k x := G(e_k \otimes x) \tag{3.1}$$

when $x \in X$. It is clear that there exists a constant $C > 0$ such that $\|e_k \otimes x\|_{B^s_{p,q}(\mathbb{T};X)} \leqslant C\|x\|$ when $k \in \mathbb{Z}$. Thus,

$$\|F_k\| \leqslant C\|F\|, \quad \|G_k\| \leqslant C\|G\| \tag{3.2}$$

whenever $k \in \mathbb{Z}$. It can be seen easily that when $u \in B^s_{p,q}(\mathbb{T};X)$, then

$$\widehat{Fu_\cdot}(k) = F_k\hat{u}(k), \quad \widehat{Gu_\cdot}(k) = G_k\hat{u}(k)$$

for $k \in \mathbb{Z}$. The resolvent set of (P_4) in the $B^s_{p,q}$-well-posedness setting is defined by

$$\begin{aligned}\rho_M(A,B,L) := \big\{k \in \mathbb{Z} : k^4M - (\alpha i k^3 + k^2)L - \beta A \\ - i\gamma k B - ikG_k - F_k \text{ is invertible from} \\ D(A)\cap D(B) \text{ onto } X \text{ and } [k^4M - (\alpha i k^3 + k^2)L \\ - \beta A - i\gamma k B - ikG_k - F_k]^{-1} \in \mathcal{L}(X)\big\}.\end{aligned}$$

For the sake of simplicity, when $k \in \rho_M(A,\, B,\, L)$, we will use the following notation:

$$N_k = [k^4M - (\alpha i k^3 + k^2)L - \beta A - i\gamma k B - ikG_k - F_k]^{-1}. \tag{3.3}$$

If $k \in \rho_M(A,\, B,\, L)$, then MN_k, LN_k, AN_k and BN_k make sense as $D(A)\cap D(B) \subset D(M)\cap D(L)$ by assumption, and they belong to $\mathcal{L}(X)$ by the closed graph theorem and the closedness of A, B, M and L.

Let $1 \leqslant p,\, q \leqslant \infty$, $s > 0$. It is noted that that the functions $Gu_{\cdot}$ and $Fu'_{\cdot}$ are uniformly bounded on $\mathbb{T}$, but they are not necessarily in $B^s_{p,q}(\mathbb{T};X)$. We define the solution space of $B^s_{p,q}$-well-posedness for (P_4) by

$$\begin{aligned}S_{p,q,s}(A,B,M,L) := \big\{u \in B^s_{p,q}(\mathbb{T};D(A)) \cap B^{1+s}_{p,q}(\mathbb{T};X) \\ : \ Mu \in B^{4+s}_{p,q}(\mathbb{T};X), Lu \in B^{2+s}_{p,q}(\mathbb{T};X), \\ u' \in B^s_{p,q}(\mathbb{T};D(B)) \text{ and } Fu_{\cdot}, Gu'_{\cdot} \in B^s_{p,q}(\mathbb{T};X)\big\}.\end{aligned}$$

Here again we consider $D(A)$ and $D(B)$ as Banach spaces equipped with their graph norms. $S_{p,q,s}(A,\, B,\, M,\, L)$ is a Banach space with the norm

$$\begin{aligned}\|u\|_{S_{p,q,s}(A,B,M,L)} := \|u\|_{B^{1+s}_{p,q}(\mathbb{T};X)} + \|u\|_{B^s_{p,q}(\mathbb{T};D(A))} \\ + \|Mu\|_{B^{4+s}_{p,q}(\mathbb{T};X)} + \|Lu\|_{B^{3+s}_{p,q}(\mathbb{T};X)} \\ + \|u'\|_{B^s_{p,q}(\mathbb{T};D(B))} + \|Fu_{\cdot}\|_{B^s_{p,q}(\mathbb{T};X)} + \|Gu'_{\cdot}\|_{B^s_{p,q}(\mathbb{T};X)}.\end{aligned}$$

If $u \in S_{p,q,s}(A,\, B,\, M,\, L)$, then Mu, $(Mu)'$, $(Mu)''$ and $(Mu)'''$ are X-valued continuous function on $\mathbb{T}$, and $Mu(0) = Mu(2\pi)$,$(Mu)'(0) = (Mu)'(2\pi)$, $(Mu)''(0) = (Mu)''(2\pi)$ and $(Mu)'''(0) = (Mu)'''(2\pi)$ by [**5**, Lemma 2.1].

Now we give the definition of the $B^s_{p,q}$-well-posedness of (P_4).

Definition 3.1. *Let $1 \leqslant p,\, q \leqslant \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T};X)$, $u \in S_{p,q,s}(A,\, B,\, M,\, L)$ is called a strong $B^s_{p,q}$-solution of (P_4), if (P_4) is satisfied a.e. on $\mathbb{T}$. We say that (P_4) is $B^s_{p,q}$-well-posed, if for each $f \in B^s_{p,q}(\mathbb{T};X)$, there exists a unique strong $B^s_{p,q}$-solution of (P_4).*

If (P_4) is $B^s_{p,q}$-well-posed and $u \in S_{p,q,s}(A,\, B,\, M,\, L)$ is the unique strong $B^s_{p,q}$-solution of (P_4), there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T};X)$, we have

$$\|u\|_{S_{p,q,s}(A,B,M,L)} \leqslant C\,\|f\|_{B^s_{p,q}}. \tag{3.4}$$

This is an easy result that can be obtained by the closedness of the operators A, B, M and L and the closed graph theorem.

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Next we give the definition of operator-valued Fourier multipliers in the context of periodic Besov spaces, which is important in the proof of our main result of this section.

DEFINITION 3.2. *Let* X, Y *be Banach spaces,* $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ *and let* $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. *We say that* $(M_k)_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$*-Fourier multiplier, if for each* $f \in B^s_{p,q}(\mathbb{T}; X)$, *there exists* $u \in B^s_{p,q}(\mathbb{T}; Y)$, *such that* $\hat{u}(k) = M_k \hat{f}(k)$ *for all* $k \in \mathbb{Z}$.

The following result has been obtained in [**6**, Theorem 4.5] which gives a sufficient condition for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier. For the notions of B-convex Banach spaces, we refer the readers to [**6**] and references therein.

THEOREM 3.3. *Let* X, Y *be Banach spaces and let* $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. *We assume that*

$$\sup_{k\in\mathbb{Z}} \left(\|M_k\| + \left\|k \mathop{\triangle} M_k\right\| \right) = \sup_{k\in\mathbb{Z}} \left(\|M_k\| + \|k(M_{k+1} - M_k)\| \right) < \infty, \tag{3.5}$$

$$\sup_{k\in\mathbb{Z}} \left\|k^2 \mathop{\triangle}^2 M_k\right\| = \sup_{k\in\mathbb{Z}} \left\|k^2 \left(M_{k+2} - 2M_{k+1} + M_k\right)\right\| < \infty. \tag{3.6}$$

Then for $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ *is an* $B^s_{p,q}$*-multiplier. If* X *is* B*-convex, then the first-order condition* (3.5) *is already sufficient for* $(M_k)_{k\in\mathbb{Z}}$ *to be a* $B^s_{p,q}$*-multiplier.*

REMARK 3.4.

(i) If $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, then there exists a bounded linear operator T from $B^s_{p,q}(\mathbb{T}; X)$ to $B^s_{p,q}(\mathbb{T}; Y)$ satisfying $\widehat{Tf}(k) = M_k \hat{f}(k)$ when $k \in \mathbb{Z}$. This implies in particular that $(M_k)_{k\in\mathbb{Z}}$ must be bounded.

(ii) If $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers, it can be seen easily that the product sequence $(M_k N_k)_{k\in\mathbb{Z}}$ and the sum sequence $(M_k + N_k)_{k\in\mathbb{Z}}$ are still $B^s_{p,q}$-Fourier multipliers.

(iii) Let $c_k = \frac{1}{k}$ when $k \neq 0$ and $c_0 = 1$, then it is easy to see that the sequence $(c_k I_X)_{k\in\mathbb{Z}}$ satisfies the conditions (3.2) and (3.3). Thus, the sequence $(c_k I_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier by theorem 3.3.

In order to prove our main result, we need the following facts.

PROPOSITION 3.5. *Let* A, B, M *and* L *be closed linear operators defined on a Banach space* X *satisfying* $D(A) \cap D(B) \subset D(M) \cap D(L)$, α, β, $\gamma \in \mathbb{C}$ *and let* F, $G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$, *where* $1 \leqslant p, q \leqslant \infty$ *and* $s > 0$. *Assume that* $\rho_M(A, B, L) = \mathbb{Z}$ *and the sets* $\{k\Delta^2 F_k : k \in \mathbb{Z}\}$, $\{k\Delta G_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 G_k : k \in \mathbb{Z}\}$, $\left\{k^4 M N_k : k \in \mathbb{Z}\right\}$, $\{k^3 L N_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ *and* $\{kN_k : k \in \mathbb{Z}\}$ *are norm-bounded, where* N_k *is defined by* (3.3), *the operators* F_k, G_k, H_k *are defined by* (3.1). *Then* $(k^4 M N_k)_{k\in\mathbb{Z}}$, $(k^3 L N_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$, $(F_k N_k)_{k\in\mathbb{Z}}$ *and* $(kG_k N_k)_{k\in\mathbb{Z}}$ *are* $B^s_{p,q}$*-Fourier multipliers.*

Proof. It follows immediately from the norm boundedness of the set $\{kN_k : k \in \mathbb{Z}\}$ that the set $\{N_k : k \in \mathbb{Z}\}$ is norm-bounded. Let $L_k = (N_k^{-1} - N_{k+1}^{-1})N_k$ when $k \in \mathbb{Z}$. Then the set $\{kL_k : k \in \mathbb{Z}\}$ is norm-bounded by the proof of proposition 2.11. Since remark 2.7 and the sequence $(k^j)_{k\in\mathbb{Z}}$ is 2-regular when $0 \leqslant j \leqslant 3$, to show that $(k^4MN_k)_{k\in\mathbb{Z}}$, $(k^3LN_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers, we only need to show that the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is norm-bounded by [**11**, Theorem 1.1] and theorem 3.3. We have

$$L_k = L_k^{(1)} + L_k^{(2)},$$

where

$$\begin{aligned} L_k^{(1)} &:= -\Delta a_k MN_k + \Delta b_k LN_k + \Delta c_k BN_k, \\ L_k^{(2)} &:= ik\Delta G_k N_k + iG_{k+1}N_k + \Delta F_k N_k, \end{aligned}$$

when $k \in \mathbb{Z}$ by (2.6). We observe that

$$\begin{aligned} \Delta L_k^{(1)} &= -\Delta a_{k+1}MN_{k+1} + \Delta b_{k+1}LN_{k+1} \\ &\quad + \Delta c_{k+1}BN_{k+1} + \Delta a_k MN_k - \Delta b_k LN_k - \Delta c_k BN_k \\ &= -\Delta^2 a_k MN_{k+1} - \Delta a_k M\Delta N_k + \Delta^2 b_k LN_{k+1} \\ &\quad + \Delta b_k L\Delta N_k + \Delta^2 c_k BN_{k+1} + \Delta c_k B\Delta N_k \\ &= -\Delta^2 a_k MN_{k+1} - \Delta a_k MN_{k+1}L_k + \Delta^2 b_k LN_{k+1} \\ &\quad + \Delta b_k LN_{k+1}L_k + \Delta^2 c_k BN_{k+1} + \Delta c_k BN_{k+1}L_k, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \Delta L_k^{(2)} &= i(k+1)\Delta G_{k+1}N_{k+1} + iG_{k+2}N_{k+1} \\ &\quad + \Delta F_{k+1}N_{k+1} - ik\Delta G_k N_k - iG_{k+1}N_k - \Delta F_k N_k \\ &= ik\Delta^2 G_k N_{k+1} + ik\Delta G_k \Delta N_k + i\Delta G_{k+1}N_{k+1} + i\Delta G_{k+1}N_{k+1} \\ &\quad + iG_{k+1}\Delta N_k + \Delta^2 F_k N_{k+1} + \Delta F_k \Delta N_k \\ &= ik\Delta^2 G_k N_{k+1} + ik\Delta G_k \Delta N_k + 2i\Delta G_{k+1}N_{k+1} + iG_{k+1}\Delta N_k \\ &\quad + \Delta^2 F_k N_{k+1} + \Delta F_k \Delta N_k \\ &= ik\Delta^2 G_k N_{k+1} + ik\Delta G_k N_{k+1}L_k + 2i\Delta G_{k+1}N_{k+1} \\ &\quad + iG_{k+1}N_{k+1}L_k + \Delta^2 F_k N_{k+1} + \Delta F_k \Delta N_k, \end{aligned} \tag{3.8}$$

when $k \in \mathbb{Z}$. It follows from (3.7) and (3.8) that the sets$\{k^2\Delta L_k^{(1)} : k \in \mathbb{Z}\}$ and $\{k^2\Delta L_k^{(2)} : k \in \mathbb{Z}\}$ are norm-bounded by the norm boundedness of the sets $\{kL_k : k \in \mathbb{Z}\}$ and the assumptions that the sets $\{k\Delta^2 F_k : \ k \in \mathbb{Z}\}$, $\{k\Delta G_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 G_k : \ k \in \mathbb{Z}\}$, $\left\{k^4 MN_k : \ k \in \mathbb{Z}\right\}$, $\{k^3 LN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : \ k \in \mathbb{Z}\}$ are norm-bounded.

It remains to show that the sequences $(F_k N_k)_{k\in\mathbb{Z}}$ and $(kG_k N_k)_{k\in\mathbb{Z}}$ satisfy (3.5) and (3.6). This follows easily from the norm boundedness of the sets $\{k\Delta^2 F_k :$

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$k \in \mathbb{Z}\}$, $\{k\Delta G_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta^2 G_k : \ k \in \mathbb{Z}\}$. We omit the details. The proof is completed. □

Next we give a necessary and sufficient condition for $B^s_{p,q}$-well-posedness of (P_4). Its proof is just an easy adaptation of the proof of theorem 2.12 by using proposition 3.5. We omit the detail.

THEOREM 3.6. *Let X be a Banach space, $1 \leqslant p, q \leqslant \infty$, $s > 0$, let A, B, M and L be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M) \cap D(L)$ and $\alpha, \beta, \ \gamma \in \mathbb{C}$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that the sets $\{k\Delta^2 F_k : \ k \in \mathbb{Z}\}$, $\{k\Delta G_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta^2 G_k : \ k \in \mathbb{Z}\}$ are norm-bounded. Then the following assertions are equivalent:*

(i) *(P_4) is $B^s_{p,q}$-well-posed.*

(ii) *$\rho_M(A, B, L) = \mathbb{Z}$ and the sets $\left\{k^4 M N_k : k \in \mathbb{Z}\right\}$, $\{k^3 L N_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : \ k \in \mathbb{Z}\}$ are norm-bounded, where N_k is defined by (3.3).*

When the underlying Banach space X is B-convex, the first-order Marcinkiewicz-type condition (3.5) is already sufficient for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier. This remark together with the proof of theorem 2.12 gives immediately the following result which gives an characterization of the $B^s_{p,q}$-well-posedness of (P_4) under a weaker condition on the sequence $(G_k)_{k\in\mathbb{Z}}$ when the underlying Banach space is B-convex.

THEOREM 3.7. *Let X be a B-convex Banach space, $1 \leqslant p, q \leqslant \infty$, $s > 0$, let A, B, M and L be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M) \cap D(L)$ and $\alpha, \ \beta, \ \gamma \in \mathbb{C}$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that $\{k\Delta G_k : k \in \mathbb{Z}\}$ is norm-bounded. Then the following assertions are equivalent:*

(i) *(P_4) is $B^s_{p,q}$-well-posed.*

(ii) *$\rho_M(A, B, L) = \mathbb{Z}$ and the sets $\left\{k^4 M N_k : \ k \in \mathbb{Z}\right\}$, $\{k^3 L N_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are norm-bounded, where N_k is defined by (3.3).*

Since the second statement of theorem 3.6 does not depend on the parameters $1 \leqslant p, q \leqslant \infty$, $s > 0$, theorem 3.6 has the following immediate consequence.

COROLLARY 3.8. *Let X be a Banach space, $1 \leqslant p, q \leqslant \infty$, $s > 0$, let A, B, M and L be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M) \cap D(L)$ and $\alpha, \beta, \ \gamma \in \mathbb{C}$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that the sets $\{k\Delta^2 F_k : \ k \in \mathbb{Z}\}$, $\{k\Delta G_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta^2 G_k : \ k \in \mathbb{Z}\}$ are norm-bounded. Then if (P_4) is $B^s_{p,q}$-well-posed for some $1 \leqslant p, q \leqslant \infty$, $s > 0$, then it is $B^s_{p,q}$-well-posed for all $1 \leqslant p, q \leqslant \infty$, $s > 0$.*

4. Applications

EXAMPLE 4.1. Let Ω be a bounded domain in $\mathbb{R}^k$ with smooth boundary, m be a given non-negative-bounded measurable function on Ω and let $\alpha, \gamma \in \mathbb{C}$, $\ \beta > 0$ be

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given. We let X be the Hilbert space $H^{-1}(\Omega)$, and let $F,\, G \in \mathcal{L}(L^p([-2\pi,\, 0]; X),\, X)$ for some $1 < p < \infty$. We consider the problem

$$\begin{cases} \dfrac{\partial^4}{\partial t^4}(m(x)u(t,x)) + \alpha\dfrac{\partial^3}{\partial t^3}(m(x)u(t,x)) + \dfrac{\partial^2}{\partial t^2}(m(x)u(t,x)) \\ = \beta\Delta u(t,x) + \gamma\Delta\dfrac{\partial}{\partial t}u(t,x) + Gu_t'(\cdot,x) + Fu_t(\cdot,x) + f(t,x),\ (t,x) \in \mathbb{T}\times\Omega, \\ u(t,x) = 0,\ (t,x) \in \mathbb{T}\times\partial\Omega. \end{cases}$$

where f is defined on $\mathbb{T}\times\Omega$ and the Laplacian Δ only acts on the space variable $x \in \Omega$, u_t' and u_t are defined by $u_t'(s,\, x) = u'(t+s,\, x)$ and $u_t'(s,\, x) = u(t+s,\, x)$ when $t \in \mathbb{T}$, $\ s \in [-2\pi,\, 0]$ and $x \in \Omega$.

Let M be the multiplication operator on X by m, then there exist constants $C > 0$, $\beta > 0$, such that

$$\left\|M(zM+\Delta)^{-1}\right\| \leqslant \frac{C}{1+|z|} \tag{4.1}$$

whenever $Re(z) \leqslant \beta(1+|Im(z)|)$ by [**12**, Section 3.7], where Δ is the Laplacian on $H^{-1}(\Omega)$ with Dirichlet boundary condition. Let $A = \Delta$ and we assume that $D(A) \subset D(M)$. Then the above equation may be rewritten in the form

$$\begin{aligned} &(Mu)''''(t) + \alpha(Mu)'''(t) + (Mu)''(t) \\ &\quad = \beta Au(t) + \gamma Au'(t) + Gu_t' + Fu_t + f(t), \quad (t \in \mathbb{T}) \end{aligned} \tag{P_1}$$

a differential equation on $\mathbb{T}$ with values in X, where $f \in L^p(\mathbb{T}; X)$ and the solution $u \in W^{1,p}_{\text{per}}(\mathbb{T}; D(A))$ satisfies $\ Mu \in W^{4,p}_{\text{per}}(\mathbb{T}; X)$.

We assume that $\rho_M(A,\, A,\, M) = \mathbb{Z}$ and the set $\{k\Delta G_k : k \in \mathbb{Z}\}$ is norm-bounded. Furthermore, we assume that $m > 0$ a.e. on Ω and m is regular enough so that the multiplication operator by m^{-1} is bounded on $H^{-1}(\Omega)$, then

$$\left\|(zM+\Delta)^{-1}\right\| \leqslant \frac{C}{1+|z|} \tag{4.2}$$

whenever $Rez \leqslant \beta(1+|Imz|)$ by (4.1). We claim that (P_1) is L^p-well-posed. Indeed, the operator $(k^4 - \alpha ik^3 - k^2)M - (\beta + ik)A - ikG_k - F_k : D(A) \to X$ is bijective and $[(k^4 - \alpha ik^3 - k^2)M - (\beta + ik)A - ikG_k - F_k]^{-1} \in \mathcal{L}(X)$ whenever $k \in \mathbb{Z}$ by the assumption $\rho_M(A,\, A,\, M) = \mathbb{Z}$. It follows that the sets

$$\{k^2MN_k : k \in \mathbb{Z}\},\ \{\Delta N_k : k \in \mathbb{Z}\}, \quad \{kN_k : k \in \mathbb{Z}\}$$

are norm-bounded by (4.1) and (4.2), where $N_k = [(k^4 - \alpha ik^3 - k^2)M - (\beta + ik)A - ikG_k - F_k]^{-1}$. Here we have used the uniform boundedness of the sequences $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$. Thus, the problem (P_1) is L^p-well-posed by theorem 2.12. Here we have used the fact that $H^{-1}(\Omega)$ is a Hilbert space and the fact that every norm-bounded subset of $\mathcal{L}(X)$ is R-bounded when X is isomorphic to a Hilbert space [**5**].

Under the same assumptions, we obtain the $B^s_{p,q}$-well-posedness of (P_1) when $1 \leqslant p,\, q \leqslant \infty$ by corollary 3.8.

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EXAMPLE 4.2. Let H be a Hilbert space, P be a densely defined positive self-adjoint operator on H with $P \geqslant \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = \sum_{i=0}^{k} a_i P^i$ with $a_i \geqslant 0,\ \ a_k > 0$, where k is an integer $\geqslant 2$. Then there exists $C > 0$ and $\beta > 0$ such that

$$\left\| M(zM + A)^{-1} \right\| \leqslant \frac{C}{1 + |z|} \tag{4.3}$$

whenever $Rez \geqslant -\beta(1 + |Imz|)$ by [**12**, page 73]. If M is regular enough so that $0 \in \rho(M)$, then

$$\left\| (zM + A)^{-1} \right\| \leqslant \frac{C}{1 + |z|} \tag{4.4}$$

whenever $Rez \geqslant -\beta(1 + |Imz|)$ by (4.3).

Let $\Omega = (0,\, 1)$ and let $H = L^2(\Omega)$. It is clear that the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ generates a contraction semigroup on H and $P = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is positive and self-adjoint in H [**4**, Example 3.4.7]. Hence, $1 \in \rho(\frac{\mathrm{d}^2}{\mathrm{d}x^2})$, or equivalently $M = I_X + P$ has a bounded inverse. Let $\alpha,\, \gamma \in \mathbb{C}$ and $\beta < 0$ be fixed and let $F,\, G \in \mathcal{L}(L^p([-2\pi,\, 0]; X),\, X)$ for some $1 < p < \infty$, we consider the following equations:

$$\begin{cases} \dfrac{\partial^4}{\partial t^4}(1 - \dfrac{\partial^2}{\partial x^2})u(t,x) + \alpha \dfrac{\partial^3}{\partial t^3}(1 - \dfrac{\partial^2}{\partial x^2})u(t,x) + \dfrac{\partial^2}{\partial t^2}(1 - \dfrac{\partial^2}{\partial x^2})u(t,x) \\ = \beta \dfrac{\partial^4}{\partial x^4}u(t,x) + \gamma \dfrac{\partial^4}{\partial x^4}\dfrac{\partial}{\partial t}u(t,x) \\ \quad + Gu_t'(\cdot,x) + Fu_t(\cdot,x) + f(t,x), \quad (t,x) \in \mathbb{T} \times \Omega, \\ u(t,0) = u(t,1) = \dfrac{\partial^2}{\partial x^2}u(t,0) = \dfrac{\partial^2}{\partial x^2}u(t,1) = 0,\ t \in \mathbb{T}. \end{cases}$$

This equation can be rewritten in the compact form:

$$\begin{aligned} &(Mu)''''(t) + \alpha(Mu)'''(t) + (Mu)''(t) \\ &\quad = \beta Au(t) + \gamma Au'(t) + Gu_t' + Fu_t + f(t), \quad (t \in \mathbb{T}) \end{aligned} \tag{P_2}$$

a differential equation on $\mathbb{T}$ with values in H, where $f \in L^p(\mathbb{T}; H)$ and the solution u is in $u \in W^{1,p}_{\text{per}}(\mathbb{T}; D(A))$, satisfies $Mu \in W^{4,p}_{\text{per}}(\mathbb{T}; H)$, where $M = 1 - \frac{\partial^2}{\partial x^2}$ and $A = \Delta^2$, here we consider Δ as the Laplacian on $L^2(\Omega)$ with Dirichlet boundary condition. If $\rho_M(A,\, A,\, M) = \mathbb{Z}$, one can obtain the L^p-well-posedness of (P_2) by using (4.3), (4.4) and theorem 2.12 under suitable assumption on the delay operator G. Here again we have used the fact that $L^2(\Omega)$ is a Hilbert space and the fact that every norm-bounded subset of $\mathcal{L}(X)$ is R-bounded when X is isomorphic to a Hilbert space [**5**]. One can also obtain the $B^s_{p,q}$-well-posedness pf (P_2) when $1 \leqslant p,\, q \leqslant \infty$ by using theorem 3.6 or corollary 3.8.

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