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AN EFFECTIVE SEVEN CUBE THEOREM

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It is well-known that every sufficiently large positive integer is the sum of seven cubes. Both proofs of this result, due to Linnik and Watson, are ineffective. Here we show that Watson's proof may be made effective.

1. Introduction

It is well-known that every sufficiently large positive integer may be represented as the sum of seven cubes of positive integers. This was first proved by Linnik [3] in 1943, and a much simpler proof was provided by Watson [4]. Recently Hooley [2] has announced a conditional proof of the asymptotic formula for the number of representations, his proof depending on conjectured properties of the Hasse-Weil L-functions.

The methods of Linnik and Watson are ineffective, incapable of providing an explicit value n_0 such that all $n > n_0$ are the sum of seven cubes. The purpose of this note is to show that Watson's proof can be modified to give an effective result. Thus it may now be possible, with sufficient diligence and computer time, to prove that every integer $n > 454$ is representable as the sum of seven cubes of positive integers. I am grateful to Professor Watson for pointing out to me that no effective

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proof of the result was known.

THEOREM. *There exists an effectively computable number* n_0 *such that every integer* $n > n_0$ *is the sum of seven cubes of positive integers.*

2. Primes in arithmetic progressions

We use the notation of Davenport [1], on p. 123 of which is the following effective version of the prime number theorem for an arithmetic progression $a \bmod q$, where $(a, q) = 1$.

LEMMA 1. *Let* $\delta > 0$ *and*

(1) $$q \le (\log x)^{1-\delta}$$

then, for some absolute constant c,

(2) $$\psi(x, q, a) = \frac{x}{\varphi(q)} + O\{x \exp[-c(\log x)^{\frac{1}{2}}]\}$$

or equivalently

(3) $$\pi(x, q, a) = \frac{\operatorname{li} x}{\varphi(q)} + O\{x \exp[-c(\log x)]^{\frac{1}{2}}\} .$$

The range (1) is inadequate for our purposes and the restriction is caused by the possible existence of Siegel zeros. However the moduli q for which Siegel zeros exist are scarce and we can avoid them. Then, from [1, p. 123, equation (9)] we obtain a version of Lemma 1 effective and uniform for

(4) $$q \le (\log x)^{100} ,$$

or indeed over a larger range.

Let $q_1, q_2, \ldots$ be the (possible) sequence of positive integers for which there exists a real primitive character $\chi \pmod q$ for which $L(s, \chi)$ has a real zero β satisfying

(5) $$\beta > 1 - \tau/\log q , \quad \tau > 0 .$$

Choosing $\tau > 0$ as a suitably small constant we have, see [1, p. 94], $q_{j+1} > q_j^2$ and so at most $\log x$ values $q_j \le x$. We shall call a modulus q "good" if q and all its divisors are not in the sequence $\{q_j\}$.

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Then the characters mod q, primitive or imprimitive, do not have Siegel zeros and the corresponding term in equation (9) on p. 123 of Davenport [1] may be omitted. Hence we obtain

LEMMA 2. *For any good modulus* q *satisfying* (4) *we have* (2) *and* (3) *holding effectively and uniformly in* q.

From this we obtain, as in Lemma 2 of Watson [4],

LEMMA 3. *If* X *is sufficiently large,* q *is a good modulus satisfying* (4) *and* $(a, q) = 1$ *then for some absolute constant* $A > 0$ *there are at least* $A\,\mathrm{li}(x)/\varphi(q)$ *primes* $p \equiv a \bmod q$ *in the interval* $X < p < 1.01\,X$.

In the case $q = 6$ we can deduce Lemma 3 from Lemma 1.

3. Proof of theorem

We begin by quoting Lemma 3 of Watson [4].

LEMMA 4. *Let* N *be a positive integer, and suppose there exist distinct primes* p, q, r *such that*

$$p \equiv q \equiv r \equiv -1 \bmod 6 , \tag{6}$$

$$r < q < 1.01\,r , \tag{7}$$

$$\tfrac{3}{4}q^{18}p^3 < N < q^{18}p^3 , \tag{8}$$

$$N \equiv 3p \bmod 6p , \tag{9}$$

$$4N \equiv r^{18}p^3 \bmod q^6 , \tag{10}$$

$$2N \equiv q^{18}p^3 \bmod r^6 . \tag{11}$$

Then N *is representable as the sum of six positive integral cubes.*

LEMMA 5. *For some constant* $B > 0$ *and* $n > n_0(B)$ *there are at least* $B(\log n)^4/(\log\log n)^2$ *pairs of primes* q, r, *neither of which divides* n *and which satisfy* (6), (7) *and*

$$\tfrac{1}{2}(\log n)^2 < q,\ r < (\log n)^2 . \tag{12}$$

Proof. We apply Lemma 3 with $a = -1$, $q = 6$ and $X = (\log n)^2$ and $100(\log n)^2/101$ successively. This gives

$$C(\log n)^2/\log\log n \quad (C > 0)$$

primes $p \equiv -1 \bmod 6$, any two of which satisfy (9) and (14). Now the number of primes $p > \frac{1}{2}(\log n)^2$ which divide n is at most $\log n/\log\log n$, for $n \geq 10$. Thus we can choose $B(\log n)^4/(\log\log n)^2$ pairs of primes q, r, neither of which divides n.

For each such pair q, r consider the number

$$k = 6q^6r^6 < 6(\log n)^{24} . \tag{13}$$

Each such k has 196 divisors and the number of moduli $q < 6(\log n)^{24}$ with a real primitive character χ having a Siegel zero is $O(\log\log n)$. Thus we can choose a pair q, r, and so k, such that k and its divisors are not moduli corresponding to Siegel zeros. Thus k is a good modulus. Let

$$X = n^{1/3}q^{-6} , \tag{14}$$

then $\log X > \frac{1}{4}\log n$ if n is sufficiently large. Thus $k < (\log X)^{100}$. Now every number prime to qr is congruent to a cube to the modulus q^6 and also the modulus r^6, so we can find a number l such that

$$4n \equiv r^{18}l^3 \bmod q^6 , \quad 2n \equiv q^{18}l^3 \bmod r^6 . \tag{15}$$

Now we apply Lemma 3 with the good modulus $k < (\log X)^{100}$. There exists a prime p satisfying

$$p \equiv -1 \bmod 6 , \quad p \equiv l \bmod q^6r^6 , \tag{16}$$

$$X < p < 1.01\, X . \tag{17}$$

The remainder of the proof follows Watson exactly, but we include it for completeness. By (15) we have

$$4n \equiv r^{18}p^3 \bmod q^6 , \quad 2n \equiv q^{18}p^3 \bmod r^6 .$$

Since every integer is congruent to a cube $\bmod 6p$ there is an integer t satisfying

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$$0 < t \leq 6pq^2r^2 ,$$

$$n - 3p \equiv t^3 \bmod 6p , \quad t \equiv 0 \bmod q^2r^2 .$$

Now $N = n - t^3$ satisfies (9), (10) and (11). Also

$$n - t^3 < n = q^{18}X^3 < q^{18}p^3 ,$$

$$n - t^3 \geq n - 216p^3q^6r^6 = q^{18}X^3 - 216p^3q^6r^6$$
$$> (1.01)^{-3}q^{18}p^3 - 216q^6r^6p^3 > \tfrac{3}{4}q^{18}p^3 .$$

Thus all the conditions of Lemma 4 are satisfied, since (14) and (17) show that p is different from q and r. Hence $N = n - t^3$ is the sum of six cubes and so n is the sum of seven cubes.

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