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Abstract

We prove that formal Fourier Jacobi expansions of degree two are Siegel modular forms. As a corollary, we deduce modularity of the generating function of special cycles of codimension two, which were defined by Kudla. A second application is the proof of termination of an algorithm to compute Fourier expansions of arbitrary Siegel modular forms of degree two. Combining both results enables us to determine relations of special cycles in the second Chow group.

1. Introduction and statement of results

There is a close interplay between Siegel modular forms and Jacobi forms, which, for example, was employed to prove the Saito–Kurokawa conjecture [And79, Maa79a, Maa79b, Maa79c, Zag81]. This connection is founded on the Fourier Jacobi expansion of Siegel modular forms, which can be turned into a formal notion. Formal Fourier Jacobi expansions of degree two and weight $k \in \mathbb{Z}$, in the simplest case, are formal series with respect to q' of the form

$$\sum_{0 \leqslant m \in \mathbb{Z}} \phi_m \, {q'}^m$$

such that:

(i) the ϕ_m are Jacobi forms of weight k and index m (cf. [EZ85] and § 4 for a definition);

(ii) the Fourier coefficients of ϕ_m satisfy $c(\phi_m; n, r) = c(\phi_n; m, r)$.

A more precise definition will be given later.

Ibukiyama *et al.* studied such expansions in [IPY13], and at the end of the introduction they asked the question whether every Fourier Jacobi expansion is convergent. Theorem 1.5 answers this question in the affirmative.

Theorem 1.1. *Every degree-two formal Fourier Jacobi expansion converges and hence is the Fourier Jacobi expansion of a Siegel modular form.*

A more detailed statement is presented in Theorem 1.5.

The significance of formal Fourier Jacobi expansions stems from their relation to families of certain algebraic cycles on Shimura varieties of orthogonal type. The study of cycles, that is, subvarieties of varieties, is a prominent theme in algebraic geometry, pursued, e.g., in [Lec86, Fab90, Zha97]. Codimension-r cycles on a variety X are classified up to rational equivalence

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by the rth Chow group $\mathrm{CH}^r(X)_{\mathbb{C}}$, first studied in [Cho56]. In most cases, little is known about it. Using definite subspaces in indefinite quadratic spaces, Kudla produced cycles of arbitrary codimension on Shimura varieties of orthogonal type, which he called 'special cycles' [Kud97] – see also [Sch10]. Special cycles generalize interesting geometric configurations. For example, CM points on modular curves and Hirzebruch's and Zagier's curves T_N on Hilbert modular surfaces [HZ76] are included in this notion. As Kudla's special cycles are related explicitly to geometry of lattices, one hopes that they are easier to study than cycles that do not come with this additional information. Kudla conjectured the following result.

Conjecture 1.2 (Kudla). The generating function of special cycles of codimension r is a Siegel modular form of degree r.

Theorem 1.3 (Borcherds [Bor99, Bor00]). *Conjecture 1.2 is true for $r = 1$.*

As our main application, we establish a further case.

Theorem 1.4. *Conjecture 1.2 is true for $r = 2$.*

A more specific description of this modularity result is contained in Theorem 1.6.

Our affirmation of Kudla's prediction in the case $r = 2$ yields rich geometric information. It allows us to compute relations that were inaccessible before. Compare this with results obtained in the 1980s by Kudla and Millson. They examined modularity of generating functions of intersection numbers of special cycles, which, they found, arise from (additive) theta lifts [KM86, KM87, KM90]. In this way, they discovered relations of such intersection numbers.

1.1 Siegel modular forms

We now discuss the theory of formal Fourier Jacobi series in greater detail. Afterward, we turn our attention to implications for geometry.

In order to explain Theorem 1.5, which implies Theorem 1.6 and hence Theorem 1.4, we need to make several definitions. For the time being, we restrict ourselves to the case of integral weights. Fix $k, l \in \mathbb{Z}$, and let ρ be a finite-dimensional, unitary representation of the symplectic group $\mathrm{Sp}_2(\mathbb{Z}) \subset \mathrm{Mat}_4(\mathbb{Z})$ with representation space V_ρ and finite index kernel. Denote the lth symmetric power of the canonical representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ by σ_l, and write V_l for its representation space. A (doubly vector valued) Siegel modular form is a holomorphic function Φ from the Siegel upper half space $\mathbb{H}^{(2)}$ to $V_l \otimes V_\rho$ that is invariant under the modular transformations

$$(\det{}^k \otimes \sigma_l)(CZ+D)^{-1}\,\rho(\gamma)^{-1}\,\Phi((AZ+B)(CZ+D)^{-1}) \quad \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z})\right).$$

Every Siegel modular form Φ admits an absolutely convergent Fourier Jacobi expansion

$$\sum_{0 \leqslant m \in \mathbb{Q}} \phi_m(\tau, z)\, q'^{\,m},$$

where $\phi_m \in \mathrm{J}_{k,m}(\sigma_l \otimes \rho)$ is a (doubly vector valued) Jacobi form (cf. [EZ85, Sko08, IK11] and § 4). Jacobi forms have likewise Fourier expansions

$$\phi_m(\tau, z) = \sum_{0 \leqslant n \in \mathbb{Q},\, r \in \mathbb{Q}} c(\phi_m; n, r)\, q^n \zeta^r.$$

By modularity of Φ, their Fourier coefficients satisfy

$$c(\phi_m; n, r) = (\det{}^k \otimes \sigma_l)(S)^{-1}\,\rho(\mathrm{rot}(S))^{-1}\,c(\phi_n; m, r) \tag{1.1}$$

for all $0 \leqslant m, n \in \mathbb{Q}$ and all $r \in \mathbb{Q}$, where

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \quad \text{and} \quad \mathrm{rot}(S) = \begin{pmatrix} S & \\ & S \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}).$$

Our main theorem can be considered as a converse.

Theorem 1.5. *Let*

$$\sum_{0 \leqslant m \in \mathbb{Q}} \phi_m(\tau, z)\, q'^{\,m}$$

be a formal expansion in q' with coefficients in $\mathrm{J}_{k,m}(\sigma_l \otimes \rho)$, where $k \in \frac{1}{2}\mathbb{Z}$ and $0 \leqslant l \in \mathbb{Z}$. If the Fourier coefficients of all ϕ_m satisfy (1.1) for all $0 \leqslant m, n \in \mathbb{Q}$ and all $r \in \mathbb{Q}$, then this series converges absolutely and defines a Siegel modular form.

The proof of Theorem 1.5, which is an unrefined version of Theorem 5.16, relies on comparing dimensions and on a regularity result for meromorphic functions in two variables. In the case of even weight k, we bound the asymptotic dimension of spaces of formal Fourier Jacobi expansions as $k \to \infty$. It equals, we find, the asymptotic dimension of corresponding spaces of Siegel modular forms. Regularity implies that the graded module of expansions as in Theorem 1.5 is free over the graded ring of classical Siegel modular forms. In conjunction with the dimension estimate, we obtain the result for even k. For $k \in \frac{1}{2}\mathbb{Z}$, we obtain an even-weight Siegel modular form by multiplication (or tensoring) with another suitable Siegel modular form. This allows us to reduce the general case to the case of even weights.

The special case of $l = 0$ and trivial ρ was considered in [IPY13]. Ibukiyama *et al.* showed straightforwardly that dimensions of the space of formal Fourier Jacobi expansions and the space of Siegel modular forms are equal. Their technique, however, relies on very detailed knowledge of dimension formulas, which is not available in the more general setting.

1.2 Special cycles

Zhang proved that the generating functions of special cycles, as presented later, are formal Fourier Jacobi expansions [Zha09]. He pulled back along embeddings of Shimura curves and employed Borcherds' previous result. For codimension-two cycles, Theorem 1.5 implies modularity. In this manner, our approach shows that Kudla's modularity conjecture can be examined by combining a detailed analysis of the codimension-one case – already pursued by Borcherds – and rigidity results for formal expansions of Siegel modular forms.

We give details of special cycles and their generating functions. Let $\mathcal{L}$ be a lattice of signature $(n, 2)$; write $\mathcal{L}^{\#}$ for its dual. We consider Shimura varieties X_Γ of orthogonal type associated to subgroups Γ of $\mathrm{O}(\mathcal{L})$ that act trivially on $\mathrm{disc}\, \mathcal{L} = \mathcal{L}^{\#}/\mathcal{L}$. By definition, we have

$$X_\Gamma = \Gamma \backslash \mathrm{Gr}^-(\mathcal{L} \otimes \mathbb{R}),$$

where Gr^- denotes the Grassmannian of maximal negative definite subspaces. Note that for our choice of $\mathcal{L}$, X_Γ carries the structure of a complex, quasi-projective variety.

Given a positive semi-definite, symmetric 'moment matrix' $0 \leqslant T \in \mathrm{Sym}_r(\mathbb{Q})$ and $\mu \in \mathrm{disc}^r \mathcal{L} = (\mathrm{disc}\, \mathcal{L})^r$, there is a special cycle $\{Z(T, \mu)\} \in \mathrm{CH}^\bullet(X_\Gamma)_\mathbb{C}$ whose codimension equals the rank of T, denoted by $\mathrm{rk}(T)$. In § 6, we give a precise definition. Write $\omega^\vee$ for the anti-canonical bundle on X_Γ, and denote the canonical basis elements of $\mathbb{C}[\mathrm{disc}^r \mathcal{L}]$ by $\mathfrak{e}_\mu$ ($\mu \in \mathrm{disc}^r \mathcal{L}$). We prove the following result.

THEOREM 1.6. *Let $\Gamma \subset \mathrm{O}(\mathcal{L})$ be a subgroup that fixes* $\operatorname{disc} \mathcal{L}$ *elementwise. Then*

$$\sum_{\substack{\mu \in \operatorname{disc}^2 \mathcal{L} \\ 0 \leqslant T \in \operatorname{Sym}_2(\mathbb{Q})}} \{Z(T,\mu)\} \cdot \{\omega^\vee\}^{2-\operatorname{rk}(T)} \exp(2\pi i \operatorname{trace}(TZ))\, \mathfrak{e}_\mu$$

is a vector-valued Siegel modular form with coefficients in $\mathrm{CH}^2(X_\Gamma)_\mathbb{C} \otimes \mathbb{C}[\operatorname{disc}^2 \mathcal{L}]$, *which has weight* $1 + n/2$.

As explained in § 1.1, there are two notions of vector-valued Siegel modular forms. In Theorem 1.6, we refer to the one that comes from representations of $\mathrm{Sp}_2(\mathbb{Z})$ (or its double cover). Note also that X_Γ is not compact. Its Chow ring can be viewed as the quotient $\mathrm{CH}^\bullet(Y_\Gamma)_\mathbb{C}/\mathrm{CH}^\bullet(\partial Y_\Gamma)$, where Y_Γ denotes the toroidal compactification of X_Γ and ∂Y_Γ are the boundary components.

As an outcome of Theorem 1.6, we are able to prove the following statement, which was shown by Zhang using *ad hoc* methods [Zha09].

COROLLARY 1.7. *The span of special cycles* $\{Z(T,\mu)\}$ *in* $\mathrm{CH}^2(X_\Gamma)_\mathbb{C}$ *is finite dimensional.*

1.3 Computing relations of special cycles

Let $\rho_{\mathcal{L},2}$ denote the type of the generating function in Theorem 1.6. Its representation space is the group algebra $\mathbb{C}[\operatorname{disc}^2 \mathcal{L}]$. Denote by Φ_μ ($\mu \in \operatorname{disc}^2 \mathcal{L}$) the components of a vector-valued Siegel modular form Φ of type $\rho_{\mathcal{L},2}$, and write $c(\Phi_\mu; T)$ for its Tth Fourier coefficient.

COROLLARY 1.8. *Suppose that* $b : \operatorname{Sym}_2(\mathbb{Q}) \times \operatorname{disc}^2 \mathcal{L} \to \mathbb{C}$ *is a function with finite support such that*

$$\sum_{T,\mu} b(T,\mu)\, c(\Phi_\mu; T) = 0$$

for every Siegel modular form of weight $1 + n/2$ *and type* $\rho_{\mathcal{L},2}$. *Then we have*

$$\sum_{T,\mu} b(T,\mu)\, \{Z(T,\mu)\} = 0 \in \mathrm{CH}^2(X_\Gamma)_\mathbb{C}.$$

Explicit knowledge of Fourier coefficients of Siegel modular forms thus yields results on relations in $\mathrm{CH}^2(X_\Gamma)_\mathbb{C}$. This observation motivates our considerations in Appendix A. However, we warn the reader that relations computed in this way do not necessarily exhaust all relations that hold for the $\{Z(T,\mu)\}$, even though this is conceivable, if $\mathcal{L}$ satisfies some mild hypotheses. The computational investigation of relations for $r = 1$ was resolved by the author in [Rau12], building on work of Bruinier. According to [Bru14], if $r = 1$ and $\mathcal{L}$ splits off a hyperbolic plane and a unimodular hyperbolic plane, all relations that hold for the $\{Z(T,\mu)\}$ ($0 \leqslant T \in \mathbb{Q}$, $\mu \in \operatorname{disc} \mathcal{L}$) can be described by the analog of Corollary 1.8.

We prove correctness and termination of an algorithm to compute Fourier expansions of Siegel modular forms of degree two. It is very much inspired, and in a sense that we make clear in a moment, generalizes ideas conveyed in [Poo11, IPY13].

THEOREM 1.9. *There is an explicit, terminating algorithm that computes Fourier expansions of Siegel modular forms of arbitrary weight and type.*

The proof of this finding is very similar to the proof of Theorem 1.5. However, in order to give a precise criterion on which precision implies correctness of the algorithm, we need effective versions of all involved statements.

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Combining the above and Corollary 1.8, we obtain the following result.

Corollary 1.10. *There is an explicit, terminating algorithm that computes infinitely many relations of special cycles* $\{Z(T,\mu)\} \in \mathrm{CH}^2(X_\Gamma)_\mathbb{C}$.

In [Poo11, IPY13], Ibukiyama *et al.* – following a suggestion by Armand Brumer – hinted at an algorithm to compute paramodular forms, which is very similar to ours. But they were not able to prove that theirs terminates, except in four cases. Since paramodular forms can be considered as Siegel modular forms for congruence subgroups of $\mathrm{Sp}_2(\mathbb{Z})$, our algorithm also applies to their problem after inducing the trivial representation of this subgroup to $\mathrm{Sp}_2(\mathbb{Z})$. In this sense, we resolve the question raised at the end of the introduction of [IPY13], asking whether 'the growth condition is superfluous'. In practice, however, our algorithm will most certainly be slower than Poor and Yuen's, and proving that it terminates without employing induction of representations would be useful. We remark that due to the Atkin–Lehner-like involutions on paramodular forms our algorithm yields correctness and termination of Poor and Yuen's algorithm in the squarefree case.

1.4 Final remarks

In Theorem 1.5, we have also treated the case of vector-valued weight $l \neq 0$, which we have not yet made use of. We have included it in our considerations, because of work by Bergström *et al.* [FvdG04a, FvdG04b, BFvdG08]. They described a method that computes quite efficiently Hecke eigenvalues of Siegel modular forms for all k and l if ρ is trivial. These computations rely on extensive precomputations, which would have to be redone if one wanted to study Siegel modular forms for congruence subgroups other than those that they examined. Our method then provides an alternative path to follow, which even yields further information.

The paper is organized as follows. In § 2, we briefly fix notation. Sections 3 and 4 contain general discussions of Siegel modular forms and Jacobi forms. Section 5 is the heart of this paper. Formal Fourier Jacobi expansions are defined in § 5.1. We show that they form a free module in § 5.2. Dimension estimates established in § 5.3 lead to the proof of Theorem 1.5, which is contained in § 5.4. Our main application is then discussed in § 6. We consider computations of Fourier expansions in Appendix A.

2. Notation

We write I_n for the $n \times n$ identity matrix. The zero matrix of size $n \times m$ is denoted by $0^{(n,m)}$. If $m = 1$, we suppress the corresponding superscript. The module of $n \times n$ matrices with entries in a ring R is denoted by $\mathrm{Mat}_n(R)$. We write $\cdot^{\mathrm{T}}$ for transposition of matrices. The space of symmetric matrices of size n is denoted by $\mathrm{Sym}_n(R)$.

The group of affine transformations on $\mathbb{C}^n$ is denoted by $\mathrm{Aff}_n(\mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$. It can be embedded into $\mathrm{GL}_{n+1}(\mathbb{C})$ via

$$(\gamma, \mu) \mapsto \begin{pmatrix} \gamma & \mu \\ 0^{(1,n)} & 1 \end{pmatrix}. \tag{2.1}$$

Restrictions of representations of $\mathrm{GL}_2(\mathbb{C})$ to $\mathrm{Aff}_1(\mathbb{C})$ that appear in this paper are taken with respect to this embedding. When referring to representations, we always mean finite-dimensional representations. The scalar representation of $\mathrm{GL}_n(\mathbb{C})$ that lets g act as $\det(g)^k$ is denoted by $\det^k$. The lth symmetric power of the canonical representation of $\mathrm{GL}_n(\mathbb{C})$ ($n \geqslant 2$) on $\mathbb{C}^n$ is denoted by σ_l.

Both $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{Aff}_n(\mathbb{C})$ have connected double covers $\widetilde{\mathrm{GL}}_n(\mathbb{C})$ and $\widetilde{\mathrm{Aff}}_n(\mathbb{C})$. We think of elements in $\widetilde{\mathrm{GL}}_n(\mathbb{C})$ as pairs $(A, \det(A)^{1/2})$, where the second component is one choice of root. The one-dimensional representation of $\widetilde{\mathrm{GL}}_n(\mathbb{C})$ which lets $(A, \det(A)^{1/2})$ act as $\det^{1/2}(A)$ is denoted by $\det^{1/2}$. We have $\widetilde{\mathrm{Aff}}_n(\mathbb{C}) \cong \widetilde{\mathrm{GL}}_n(\mathbb{C}) \ltimes \mathbb{C}^n$, and we will use this isomorphism without further mentioning it. The above embedding of $\mathrm{Aff}_n(\mathbb{C})$ into $\mathrm{GL}_{n+1}(\mathbb{C})$ extends to an embedding of $\widetilde{\mathrm{Aff}}_n(\mathbb{C})$ into $\widetilde{\mathrm{GL}}_{n+1}(\mathbb{C})$.

3. Siegel modular forms

3.1 Preliminaries on Siegel modular forms

Let $J_g = \begin{pmatrix} 0^{(g,g)} & -I_g \\ I_g & 0^{(g,g)} \end{pmatrix}$. The full Siegel modular group of degree g is

$$\mathrm{Sp}_g(\mathbb{Z}) = \{\gamma \in \mathrm{Mat}_{2g}(\mathbb{Z}) : \gamma^{\mathrm{T}} J_g \gamma = J_g\}$$

with typical elements $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for $A, B, C, D \in \mathrm{Mat}_g(\mathbb{Z})$. One kind of elements, that we use frequently, is

$$\mathrm{rot}(U) = \begin{pmatrix} U & \\ & (U^{-1})^{\mathrm{T}} \end{pmatrix}$$

for $U \in \mathrm{GL}_g(\mathbb{Z})$. The metaplectic double cover $\widetilde{\mathrm{Sp}}_g(\mathbb{Z})$ of $\mathrm{Sp}_g(\mathbb{Z})$ is the nontrivial central extension of $\mathrm{Sp}_g(\mathbb{Z})$ by the cyclic group of order two.

The full Siegel modular group acts on the Siegel upper half space of degree g

$$\mathbb{H}^{(g)} = \{Z = X + iY \in \mathrm{Mat}_g(\mathbb{C}) : Y > 0\}$$

via

$$\gamma Z = (AZ + B)(CZ + D)^{-1}.$$

We obtain an action of $\widetilde{\mathrm{Sp}}_g(\mathbb{Z})$ on $\mathbb{H}^{(g)}$ by composing with the projection $\widetilde{\mathrm{Sp}}_g(\mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_g(\mathbb{Z})$. In this paper, we will focus on the case $g \leqslant 2$, and so we fix special notation in both cases. We usually write τ for elements of $\mathbb{H}^{(1)}$, and $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ for elements of $\mathbb{H}^{(2)}$. Further, we set $q = \exp(2\pi i\,\tau)$, $\zeta = \exp(2\pi i\,z)$ and $q' = \exp(2\pi i\,\tau')$.

We can use the upper half space to find a concrete description of elements in $\widetilde{\mathrm{Sp}}_g(\mathbb{Z})$. Elements of $\widetilde{\mathrm{Sp}}_g(\mathbb{Z})$ can and will be thought of as pairs

$$(\gamma,\, ((CZ + D), \det(CZ + D)^{1/2})).$$

The first component is an element of $\mathrm{Sp}_g(\mathbb{Z})$, and the second component is a holomorphic map $\mathbb{H}^{(g)} \to \widetilde{\mathrm{GL}}_g(\mathbb{C})$, which we typically denote by ω.

Let σ be a representation of $\widetilde{\mathrm{GL}}_g(\mathbb{C})$, the double cover of $\mathrm{GL}_g(\mathbb{C})$, with representation space V_σ. Let ρ be a representation of $\widetilde{\mathrm{Sp}}_g(\mathbb{Z})$ with representation space V_ρ. The Siegel slash action of weight σ and type ρ is defined as

$$(\Phi|_{\sigma,\rho}\,(\gamma, \omega))(Z) = \sigma(\omega(Z))^{-1} \rho(\gamma)^{-1}\, \Phi(\gamma Z)$$

for all $\Phi : \mathbb{H}^{(g)} \to V_\sigma \otimes V_\rho$ and all $(\gamma, \omega) \in \widetilde{\mathrm{Sp}}_g(\mathbb{Z})$.

Definition 3.1. Let σ be a representation of $\widetilde{\mathrm{GL}}_g(\mathbb{C})$ with representation space V_σ, and let ρ be a representation of $\widetilde{\mathrm{Sp}}_g(\mathbb{Z})$ with representation space V_ρ. A function $\Phi : \mathbb{H}^{(g)} \to V_\sigma \otimes V_\rho$ is called a Siegel modular form of weight σ and type ρ if

$$\Phi|_{\sigma,\rho}\,(\gamma,\omega) = \Phi \quad \text{for all } (\gamma,\omega) \in \widetilde{\mathrm{Sp}}_g(\mathbb{Z})$$

and, in addition, if $g = 1$, Φ has Fourier expansion

$$\Phi(\tau) = \sum_{0 \leqslant n \in \mathbb{Q}} c(\Phi; n)\, q^n, \quad c(\Phi; n) \in V_\sigma \otimes V_\rho.$$

If Φ is meromorphic and satisfies $\Phi|_{\sigma,\rho}\,(\gamma,\omega) = \Phi$ for all $(\gamma,\omega) \in \widetilde{\mathrm{Sp}}_g(\mathbb{Z})$, we call it a meromorphic Siegel modular form.

We write $\mathrm{M}^{(g)}(\sigma,\rho)$ for the space of Siegel modular forms of weight σ and type ρ.

Irreducible representations of $\widetilde{\mathrm{GL}}_g(\mathbb{C})$ either factor through $\mathrm{GL}_g(\mathbb{C})$, or they are of the from $\det^{1/2} \otimes \sigma'$, where σ' factors through $\mathrm{GL}_g(\mathbb{C})$. The meaning of $\det^k$ with $k \in \frac{1}{2}\mathbb{Z}$ thus becomes clear. The space of Siegel modular forms of degree one (i.e., elliptic modular forms) of weight $\det^k$, $k \in \frac{1}{2}\mathbb{Z}$ and type ρ is denoted by $\mathrm{M}^{(1)}_k(\rho)$. The space of degree-two Siegel modular forms of weight $\det^k \otimes \sigma_l$, $k \in \frac{1}{2}\mathbb{Z}$, $0 \leqslant l \in \mathbb{Z}$ and type ρ is denoted by $\mathrm{M}^{(2)}_{k,l}(\rho)$. If $l = 0$, we suppress it and, if ρ is trivial, we abbreviate $\mathrm{M}^{(1)}_k(\rho) = \mathrm{M}^{(1)}_k$ and $\mathrm{M}^{(2)}_{k,l}(\rho) = \mathrm{M}^{(2)}_{k,l}$.

Remark 3.2. Suppose that σ and ρ are irreducible. If σ factors through $\mathrm{GL}_g(\mathbb{C})$, one can show that for those ρ which do not factor through $\mathrm{Sp}_g(\mathbb{Z})$ we have $\mathrm{M}^{(g)}(\sigma,\rho) = \{0\}$. On the other hand, we have $\mathrm{M}^{(g)}(\sigma,\rho) = \{0\}$ if ρ factors through $\mathrm{Sp}_g(\mathbb{Z})$ and σ does not.

More concretely, this means that $\mathrm{M}^{(g)}_{k,l}(\rho) = \{0\}$ if:

(i) $k \in \mathbb{Z}$ and ρ does not factor through $\mathrm{Sp}_g(\mathbb{Z})$; or

(ii) $k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ and ρ factors through $\mathrm{Sp}_g(\mathbb{Z})$.

3.2 Structure theorems and dimension formulas

Let ρ be any (finite-dimensional) representation of $\mathrm{Sp}_1(\mathbb{Z})$. By results of Mason and Marks [MM10], $\bigoplus_{k\in\mathbb{Z}} \mathrm{M}^{(1)}_k(\rho)$ is a free module over $\bigoplus_{k\in\mathbb{Z}} \mathrm{M}^{(1)}_k$ of rank $\dim\rho$. Hence, there is a polynomial $p_{\mathrm{g}}(\rho; t)$ with $p_{\mathrm{g}}(\rho; 1) = \dim\rho$ such that

$$\sum_{k\in\mathbb{Z}} \dim \mathrm{M}^{(1)}_k(\rho)\, t^k = \frac{p_{\mathrm{g}}(\rho; t)}{(1-t^4)(1-t^6)}. \tag{3.1}$$

Lemma 3.3. *Fix $0 \leqslant l \in \mathbb{Z}$ and a representation ρ of $\mathrm{Sp}_2(\mathbb{Z})$ with finite index kernel. Then we have*

$$\mathrm{M}^{(2)}_{k,l}(\rho) = \dim\sigma_l \cdot \dim\rho \cdot \dim \mathrm{M}^{(2)}_k + O_{l,\rho}(k^2)$$

as $k \to \infty$, $k \in 2\mathbb{Z}$.

Proof. Fix a sufficiently small principal congruence subgroup $\Gamma \subset \mathrm{Sp}_2(\mathbb{Z})$ such that ρ is trivial on Γ. Write $\mathrm{M}_{k,l}(\Gamma)$ for the space of Siegel modular forms with respect to Γ. We have

$$\begin{aligned}\dim \mathrm{M}_{k,l}(\rho) &= \frac{1}{\#(\Gamma\backslash\mathrm{Sp}_2(\mathbb{Z}))} \dim \mathrm{M}_{k,l}(\rho,\Gamma) + O(k^2)\\ &= \frac{\dim\rho}{\#(\Gamma\backslash\mathrm{Sp}_2(\mathbb{Z}))} \cdot \dim \mathrm{M}_{k,l}(\Gamma) + O(k^2)\\ &= \dim\rho \cdot \dim \mathrm{M}_{k,l} + O(k^2) = \dim\rho \cdot \dim\sigma_l \cdot \dim \mathrm{M}_k + O(k^2).\end{aligned}$$

We have used that ρ restricted to Γ is trivial in the second step. The last step is due to Tsushima [Tsu83]. The first and the third steps follow along the lines of [Tsu82, § 1]. We give details for the first equality; the third one follows in an analogous way. We express elements of $\mathrm{M}_{k,l}(\rho, \Gamma)$ as sections of a vector bundle $V_{k,l,\rho} \cong \det^k \otimes \sigma_l \otimes \rho$ on the toroidal compactification X_Γ of the Siegel modular threefold attached to Γ. Since det corresponds to the determinant of the Hodge bundle (see [BvdGHZ08, p. 203]), higher cohomology of $V_{k,l,\rho}$ vanishes for sufficiently large k by the Kawamata–Viehweg vanishing theorem [Kaw82, Vie82]. By virtue of [Tsu82, Theorems 1.1 and 1.2], we have

$$\dim M_{k,l}(\rho) = \frac{1}{\#(\Gamma \backslash \mathrm{Sp}_2(\mathbb{Z}))} \Bigl(\dim M_{k,l}(\rho, \Gamma) + \sum_{1 \not\equiv \gamma : \Gamma \backslash \mathrm{Sp}_2(\mathbb{Z})} \tau(\gamma) \Bigr),$$

where $\tau(\gamma)$ was defined by Tsushima [Tsu82, p. 845]. It suffices to bound the growth of $\tau(\gamma)$ in k by k^2. However, this is immediate, since it is the evaluation of essentially the Chern character of $V_{k,l,\rho}$ restricted to $\mathrm{fix}_\gamma(X_\Gamma)$, the fixed point set of γ. Since γ acts nontrivially on X_Γ, it has dimension at most two. $\square$

4. Jacobi forms

Let

$$\mathbb{H}^{\mathrm{J}} = \mathbb{H}^{(1)} \times \mathbb{C}$$

be the Jacobi upper half space. The extended Jacobi group

$$\Gamma^{\mathrm{J}} = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \mathbin{\widetilde{\times}} \mathbb{Z}$$

has group law

$$(\gamma, (\lambda, \mu), \kappa) \cdot (\gamma', (\lambda', \mu'), \kappa') = (\gamma\gamma',\, (\lambda, \mu)\gamma' + (\lambda', \mu'),\, \kappa + \kappa' + \lambda\mu' - \mu\lambda').$$

It can be viewed as a subgroup of $\mathrm{Sp}_2(\mathbb{Z})$ via the embedding

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu), \kappa \right) \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote the double cover of Γ^{J} by $\widetilde{\Gamma}^{\mathrm{J}}$. Its elements are pairs $(\gamma^{\mathrm{J}}, \omega)$ with $\gamma^{\mathrm{J}} \in \Gamma^{\mathrm{J}}$ and $\omega : \mathbb{H}^{(1)} \to \widetilde{\mathrm{GL}}_1(\mathbb{C})$ a holomorphic square root of $c\tau + d$.

The action of Γ^{J} (and thus $\widetilde{\Gamma}^{\mathrm{J}}$) on $\mathbb{H}^{\mathrm{J}}$ is given by

$$\gamma^{\mathrm{J}}(\tau, z) = (\gamma, (\lambda, \mu), \kappa)(\tau, z) = \left(\gamma\tau, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

For $m \in \mathbb{Q}$ and $x \in \mathbb{C}$, we set $e^m(x) = \exp(2\pi i m\, x)$. Given $k \in \frac{1}{2}\mathbb{Z}$, $m \in \mathbb{Q}$ and a representation ρ of $\widetilde{\mathrm{Aff}}_1(\mathbb{C}) \times \widetilde{\Gamma}^{\mathrm{J}}$ with representation space V_ρ, there is a slash action

$$\begin{aligned} (\phi|^{\mathrm{J}}_{k,m,\rho}\, (\gamma^{\mathrm{J}}, \omega))(\tau, z) = \omega(c\tau + d)^{-2k}\, e^m \left(\frac{-c(z + \lambda\tau + \mu)^2}{c\tau + d} + 2\lambda z + \lambda^2\tau + \kappa \right) \\ \times \rho((\omega(\tau), c(z + \mu) - d\lambda), (\gamma^{\mathrm{J}}, \omega))^{-1} \phi(\gamma^{\mathrm{J}}(\tau, z)) \end{aligned}$$

on functions $\phi : \mathbb{H}^{\mathrm{J}} \to V_\rho$. We will often use representations of $\widetilde{\Gamma}^{\mathrm{J}}$ and $\widetilde{\mathrm{Aff}}_1(\mathbb{C})$, which we then consider as representations of $\widetilde{\mathrm{Aff}}_1(\mathbb{C}) \times \widetilde{\Gamma}^{\mathrm{J}}$ by tensoring with the trivial representations of $\widetilde{\Gamma}^{\mathrm{J}}$ and $\widetilde{\mathrm{Aff}}_1(\mathbb{C})$, respectively. Note that $|_{k,m,\det^{k'} \otimes \rho} = |_{k+k',m,\rho}$ for any $k' \in \frac{1}{2}\mathbb{Z}$.

DEFINITION 4.1. Let ρ be a representation of $\widetilde{\mathrm{Aff}}_1(\mathbb{C}) \times \widetilde{\Gamma}^{\mathrm{J}}$ with representation space V_ρ. A holomorphic function $\phi : \mathbb{H}^{\mathrm{J}} \to V_\rho$ is called a Jacobi form of weight k, index m and type ρ if:

(i) $\phi|^{\mathrm{J}}_{k,m,\rho}\,(\gamma^{\mathrm{J}}, \omega) = \phi$ for all $(\gamma^{\mathrm{J}}, \omega) \in \Gamma^{\mathrm{J}}$;

(ii) for all $\alpha, \beta \in \mathbb{Q}$, $\phi(\tau, \alpha\tau + \beta)$ is bounded (with respect to any norm on V_ρ) as $y \to \infty$.

We denote the space of Jacobi forms of weight k, index m and type ρ by $\mathrm{J}_{k,m}(\rho)$. If ρ is trivial, we suppress it. Note that $J_{k,m} = \{0\}$ if $m \in \mathbb{Q}\backslash\mathbb{Z}$. Except for Proposition 4.5, in this section, we restrict our attention to $k \in \mathbb{Z}$. Note that we do not need to assume that ρ has finite index kernel.

The last component $\mathbb{Z}$ of Γ^{J} is central. Hence, for irreducible representations ρ, it acts by scalars. We have the following connection between m and ρ.

PROPOSITION 4.2. *Let $k \in \mathbb{Z}$ and $m \in \mathbb{Q}$. If $\rho : \mathrm{Aff}_1(\mathbb{C}) \otimes \Gamma^{\mathrm{J}} \to \mathrm{GL}(V_\rho)$ is irreducible and $\rho(I_2, (0,0), 1) \neq \exp(2\pi i\, m)$, then $\mathrm{J}_{k,m}(\rho) = \{0\}$.*

Proof. This follows when inspecting the definition of $|^{\mathrm{J}}_{k,m,\rho}$. □

Jacobi forms have a Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{0 \leqslant n \in \mathbb{Q} \\ r \in \mathbb{Q}}} c(\phi; n, r)\, q^n \zeta^r.$$

By Condition (i), $c(\phi; n, r)$ depends only on $4mn - r^2$ and $r \pmod{(2m\mathbb{Z})}$. From Condition (ii), one infers that $c(\phi; n, r) = 0$ if $4mn - r^2 < 0$. This is actually equivalent to Condition (ii). The vanishing order of $\phi \in \mathrm{J}_{k,l}(\rho)$ is defined as

$$\mathrm{ord}\, \phi = \inf\{n \in \mathbb{Q} : c(\phi; n, r) \neq 0\}.$$

The spaces of Jacobi forms with vanishing order at least $0 \leqslant d \in \mathbb{Q}$ play a central role in § 5. Define

$$\mathrm{J}_{k,m}(\rho)[d] = \{\phi \in \mathrm{J}_{k,m}(\rho) : \mathrm{ord}\, \phi \geqslant d\}. \tag{4.1}$$

The definitions of the maps $\mathcal{D}_\nu$ that were given on [EZ85, p. 29] carry over to our setting if ρ is trivial on $\mathrm{Aff}_1(\mathbb{C})$. Given a Jacobi form ϕ of even weight k, index m and type $\rho : \Gamma^{\mathrm{J}} \to \mathrm{GL}(V_\rho)$, set

$$\mathcal{D}_{2\nu}(\phi) := \sum_{0 \leqslant \mu \leqslant \nu} \frac{(-1)^\mu\, (2\nu)!\, (k + 2\nu - \mu - 2)!}{\mu!\, (2\nu - 2\mu)!\, (k + \nu - 2)!} \left(\frac{\partial_z}{2\pi i}\right)^{2\nu - 2\mu} \left(\frac{\partial_\tau}{2\pi i}\right)^{\mu} \phi$$

and

$$\mathcal{D}_{2\nu+1}(\phi) := \sum_{0 \leqslant \mu \leqslant \nu} \frac{(-1)^\mu\, (2\nu+1)!\, (k + 2\nu - \mu - 1)!}{\mu!\, (2\nu + 1 - 2\mu)!\, (k + \nu - 1)!} \left(\frac{\partial_z}{2\pi i}\right)^{2\nu + 1 - 2\mu} \left(\frac{\partial_\tau}{2\pi i}\right)^{\mu} \phi.$$

Note that either $\mathcal{D}_{2\nu}(\phi)$ or $\mathcal{D}_{2\nu+1}(\phi)$ is zero depending on k and ρ. Combining the above maps, we define $\mathcal{D}(k, \rho)$ for irreducible ρ.

$$\mathcal{D}(k, \rho) = \begin{cases} \mathcal{D}_0 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_{2\lfloor m \rfloor} & \text{if } (-1)^k \rho(-I_2) \text{ is trivial,} \\ \mathcal{D}_1 \oplus \mathcal{D}_3 \oplus \cdots \oplus \mathcal{D}_{2\lfloor m \rfloor - 1} & \text{otherwise.} \end{cases}$$

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If $\rho = \bigoplus_i \rho_i$ for irreducible ρ_i, then we set

$$\mathcal{D}(k,\rho) = \bigoplus_i \mathcal{D}(k,\rho_i).$$

For irreducible ρ, it is not hard to see that the image of $\mathcal{D}(k,\rho)$ restricted to $\mathrm{J}_{k,m}(\rho)[d]$ is contained in $\bigoplus_{\nu=0}^{\lfloor m \rfloor} \mathrm{M}_{k+2\nu}(\rho)[d]$ or $\bigoplus_{\nu=0}^{\lfloor m \rfloor - 1} \mathrm{M}_{k+2\nu+1}(\rho)[d]$.

The next theorem is mostly due to Eichler and Zagier [EZ85]. We simply adapt it to the case of vector-valued Jacobi forms.

THEOREM 4.3. *Let $k \in \mathbb{Z}$ and $m \in \mathbb{Q}$, and suppose that $\rho : \mathrm{Aff}_1(\mathbb{C}) \otimes \Gamma^{\mathrm{J}} \longrightarrow \mathrm{GL}(V_\rho)$ is trivial on $\mathrm{Aff}_1(\mathbb{C})$. Then the map $\mathcal{D}(k,\rho)$ is injective.*

Proof. We may restrict to the case of irreducible ρ. Further, we only treat the case of trivial $(-1)^k \rho(-I_2)$. The proof in the other case works completely analogously.

Choose $0 < N \in \mathbb{Z}$ such that ρ is trivial on $(N\mathbb{Z})^2 \mathbin{\widetilde{\times}} N\mathbb{Z} \subseteq \Gamma^{\mathrm{J}}$. In particular, we have $Nm \in \mathbb{Z}$. Given any Jacobi form ϕ of weight k and type $\rho : \Gamma^{\mathrm{J}} \to V_\rho$, odd Taylor coefficients of ϕ vanish by assumption on $(-1)^k \rho(-I_2)$. This follows directly from the Jacobi transformation law applied to $-I_2 \in \mathrm{SL}_2(\mathbb{Z})$.

Observe that $\mathcal{D}(k,\rho)$ injectively maps the set of all 2νth Taylor coefficients $(0 \leqslant \nu \leqslant \lfloor m \rfloor)$ to $\bigoplus_{\nu=0}^{\lfloor m \rfloor} \mathrm{M}_{k+2\nu}(\rho)$. If $\mathcal{D}(k,\rho)(\phi) = 0$, we find that the first $2\lfloor m \rfloor + 1 \geqslant 2m$ Taylor coefficients of $f \circ \phi$ vanish for any linear functional f on V_ρ. On the other hand, $z \mapsto f \circ \phi(\tau, Nz)$ is a holomorphic elliptic function of index $N^2 m \in \mathbb{Z}$. By assumption, it has at least $N^2(2\lfloor m \rfloor + 2)$ zeros in the range $z = \alpha + \beta\tau$, $\alpha, \beta \in [0,1)$. Consequently, $f \circ \phi = 0$ and, since f was any functional on V_ρ, we have $\phi = 0$. $\square$

Consider the restriction of σ_l to $\mathrm{Aff}_1(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ (see (2.1)) and denote it by σ_l as well. For $m \in \mathbb{Z}$, it was shown in [IK11] that

$$\mathrm{J}_{k,m}(\sigma_l) \cong \bigoplus_{i=0}^{l} \mathrm{J}_{k+i,m}.$$

This isomorphism is induced by differential operators with constant coefficients as stated on [IK11, p. 786]. Hence, it generalizes to arbitrary $m \in \mathbb{Q}$ and to any representation $\rho : \Gamma^{\mathrm{J}} \to \mathrm{GL}(V_\rho)$. We have

$$\mathrm{J}_{k,m}(\sigma_l \otimes \rho)[d] \cong \bigoplus_{i=0}^{l} \mathrm{J}_{k+i,m}(\rho)[d].$$

Combining this with the statement of Theorem 4.3, we obtain the following result.

COROLLARY 4.4. *Let $k \in \mathbb{Z}$ and $m \in \mathbb{Q}$, and $\rho : \Gamma^{\mathrm{J}} \to V_\rho$ be an irreducible unitary representation. Then there is an injective map*

$$\mathrm{J}_{k,m}(\sigma_l \otimes \rho)[d] \hookrightarrow \bigoplus_{i=0}^{l} \begin{cases} \displaystyle\bigoplus_{\nu=0}^{\lfloor m \rfloor} M_{k+i+2\nu}(\rho)[d] & \text{if } (-1)^{k+i}\rho(-I_2) \text{ is trivial,} \\ \displaystyle\bigoplus_{\nu=0}^{\lfloor m \rfloor - 1} M_{k+i+2\nu+1}(\rho)[d] & \text{otherwise.} \end{cases}$$

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We finish this section with a statement on the connection of Siegel modular forms and Jacobi forms.

Proposition 4.5. *Let $\Phi \in \mathrm{M}^{(2)}_{k,l}(\rho)$, $k \in \frac{1}{2}\mathbb{Z}$, $0 \leqslant l \in \mathbb{Z}$ be a Siegel modular form, where ρ is a representation of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$. Then, for all $m \in \mathbb{Q}$, the function*

$$\phi_m(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(\Phi; n, r, m)\, q^n \zeta^r$$

is a Jacobi form of weight k, index m and type $\sigma_l \otimes \rho$.

In particular, we have a Fourier Jacobi expansion

$$\Phi(Z) = \sum_{0 \leqslant m \in \mathbb{Q}} \phi_m(\tau, z)\, q'^{\,m}.$$

Proof. This is a straightforward verification. □

5. Formal Fourier Jacobi expansions

5.1 Definition and basic properties

Given a Siegel modular form $\Phi \in \mathrm{M}^{(2)}_{k,l}(\rho)$, there is a Fourier Jacobi expansion as discussed at the end of the previous section. The notion of formal Fourier Jacobi expansions mimics this.

Definition 5.1. Fix $k \in \frac{1}{2}\mathbb{Z}$, $0 \leqslant l \in \mathbb{Z}$ and a representation ρ of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ with finite index kernel. Let

$$\Phi(Z) = \sum_{0 \leqslant m \in \mathbb{Q}} \phi_m(\tau, z)\, q'^{\,m} \in \prod_{0 \leqslant m \in \mathbb{Q}} \mathrm{J}_{k,\,m}(\sigma_l \otimes \rho)$$

be a formal series of Jacobi forms. We call Φ a *formal Fourier Jacobi expansion* of weight (k, l) and type ρ if

$$c(\phi_m;\, n, r) = (\det{}^k \otimes \sigma_l)^{-1}(S)\, \rho^{-1}(\mathrm{rot}(S))\, c(\phi_n;\, m, r)$$

for all $n, r \in \mathbb{Q}$, where $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Denote the space of such expansions by $\mathrm{FM}^{(2)}_{k,l}(\rho)$.

From Proposition 4.5, we infer that there is an injective map

$$\mathrm{M}^{(2)}_{k,l}(\rho) \hookrightarrow \mathrm{FM}^{(2)}_{k,l}(\rho).$$

Further, $\bigoplus_k \mathrm{FM}^{(2)}_{k,l}(\rho)$ carries a module structure over $\mathrm{M}^{(2)}_\bullet = \bigoplus_{k \in 2\mathbb{Z}} \mathrm{M}^{(2)}_k$, as is deduced immediately from the next statement.

Proposition 5.2. *The graded module of formal Fourier Jacobi expansions*

$$\mathrm{FM}^{(2)}_{\bullet,l}(\rho) = \bigoplus_{k \in 2\mathbb{Z}} \mathrm{FM}^{(2)}_{k,l}(\rho)$$

carries a module structure over $\mathrm{FM}^{(2)}_\bullet$ by means of multiplication of formal series. Given $\sum_m \phi_m(\tau, z)\, q'^{\,m} \in \mathrm{FM}^{(2)}_{k_1}$ and $\sum_m \psi_m(\tau, z)\, q'^{\,m} \in \mathrm{FM}^{(2)}_{k_2,l}(\rho)$, the multiplication is defined by

$$\Big(\sum_m \phi_m(\tau, z)\, q'^{\,m}\Big) \cdot \Big(\sum_m \psi_m(\tau, z)\, q'^{\,m}\Big) = \sum_m \Big(\sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ m_1 + m_2 = m}} \phi_{m_1}(\tau, z) \cdot \psi_{m_2}(\tau, z) \Big)\, q'^{\,m}.$$

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Proof. Provided that the right-hand side of the above definition of multiplication is an element of $\mathrm{FM}^{(2)}_{k_1+k_2,l}(\rho)$, it is clear that $\mathrm{FM}^{(2)}_{\bullet,l}(\rho)$ satisfies the axioms of a module. We therefore only have to verify that the right-hand side is a formal Fourier Jacobi expansion.

From the definition of Jacobi forms, we see that for $\phi_{m_1} \in \mathrm{J}_{k_1,m_1}$ and $\psi_{m_2} \in \mathrm{J}_{k_2,m_2}(\sigma_l \otimes \rho)$, we have $\phi_{m_1}\psi_{m_2} \in \mathrm{J}_{k_1+k_2,m_1+m_2}(\sigma_l \otimes \rho)$. That is, the right-hand side is an element of $\prod_{0\leqslant m\in\mathbb{Q}} \mathrm{J}_{k_1+k_2,m}(\sigma_l \otimes \rho) q'^{m}$, as required. To check the symmetry condition of Definition 5.1, note that we have

$$\begin{aligned}
&\sum_{\substack{m_1,m_2\in\mathbb{Q}\\ m_1+m_2=m}} c(\phi_{m_1}\psi_{m_2};\, n, r)\\
&\quad= \sum_{\substack{m_1,m_2\in\mathbb{Q}\\ m_1+m_2=m}} \sum_{\substack{n_1,n_2,r_1,r_2\in\mathbb{Q}\\ n_1+n_2=n\\ r_1+r_2=r}} c(\phi_{m_1}; n_1, r_1) c(\psi_{m_2}; n_2, r_2)\\
&\quad= \sum_{\substack{m_1,m_2\in\mathbb{Q}\\ m_1+m_2=m\\ n_1,n_2,r_1,r_2\in\mathbb{Q}\\ n_1+n_2=n\\ r_1+r_2=r}} \det{}^{-k_1}(S) c(\phi_{n_1}; m_1, r_1) (\det{}^{k_2} \otimes \sigma_l)^{-1}(S) \rho^{-1}(\mathrm{rot}(S)) c(\psi_{n_2}; m_2, r_2)\\
&\quad= \sum_{\substack{n_1,n_2\in\mathbb{Q}\\ n_1+n_2=n}} (\det{}^{k_1+k_2} \otimes \sigma_l)^{-1}(S) \rho^{-1}(\mathrm{rot}(S)) c(\phi_{n_1}\psi_{n_2}; m, r). \qquad \square
\end{aligned}$$

Proposition 5.3. *Fix $k, k' \in 2\mathbb{Z}$, $0 \leqslant l \in \mathbb{Z}$ and a representation ρ of $\mathrm{Sp}_2(\mathbb{Z})$ with finite index kernel. Let $\Psi \in \mathrm{FM}^{(2)}_{k'}$ and $\Phi \in \mathrm{FM}^{(2)}_{k,l}(\rho)$. If $\Psi\Phi = 0$, then $\Psi = 0$ or $\Phi = 0$.*

Proof. Suppose that $\Psi \neq 0$ and $\Phi \neq 0$. We can choose minimal $m, m' \in \mathbb{Q}$ such that $\phi_m \neq 0$ and $\psi \neq 0$. Clearly, $\psi_{m'}\phi_m \neq 0$, since both $\psi_{m'}$ and ϕ_m are holomorphic functions on $\mathbb{H}^{\mathrm{J}}$. By definition of multiplication, we have $(\Psi\Phi)(Z) = \psi_{m'}(\tau, z)\phi_m(\tau, z) q'^{m+m'} + O(q'^{m+m'+\epsilon})$ for some $\epsilon > 0$. This proves the statement. $\square$

Proposition 5.4. *Given ρ with finite index kernel, there is $0 < \mu_\rho \in \mathbb{Z}$ such that for every $\Phi(Z) \in \mathrm{FM}^{(2)}_{k,l}(\rho)$, we have $\phi_m(\tau, z) = 0$ if $m \notin (1/\mu_\rho)\mathbb{Z}$.*

Proof. Using the symmetry condition, we have to show that there is $\mu \in \mathbb{Z}$ such that

$$c(\phi_m; n, r) = 0$$

if $n \notin (1/\mu)\mathbb{Z}$. Since ρ has finite index kernel, the matrix

$$\rho\left(\begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$$

has μth roots of unity as eigenvalues for some $0 < \mu \in \mathbb{Z}$. Because ϕ_n is a Jacobi form of type ρ, its Fourier expansion is supported on (n, r) with $n \in (1/\mu)\mathbb{Z}$, as desired. $\square$

In analogy with Siegel modular forms, we define a vanishing order for formal Fourier Jacobi expansions:

$$\mathrm{ord}\,\Phi = \inf\{m \in \mathbb{Q} : \phi_m(\tau, z) \neq 0\}. \tag{5.1}$$

Note that $\operatorname{ord}\Phi \in (1/\mu_\rho)\mathbb{Z} \cup \{-\infty\}$ if $\Phi \in \mathrm{FM}^{(2)}_{k,l}(\rho)$. The space of formal Fourier Jacobi expansions of vanishing order at least m is denoted by $\mathrm{FM}^{(2)}_{k,l}(\rho)[m]$.

Lemma 5.5. *Suppose that Φ and Ψ are formal Fourier Jacobi expansions of the same weight and type. If $\phi_m = \psi_m$ for $0 \leqslant m < m'$ for some $0 < m' \in \mathbb{Q}$, then $\phi_{m'} - \psi_{m'} \in \mathrm{J}_{k,m'}(\sigma_l \otimes \rho)[m']$.*

Proof. For any $0 \leqslant n < m'$ and any $r \in \mathbb{Z}$, we have

$$\begin{aligned} c(\phi_{m'} - \psi_{m'}; n, r) &= c(\phi_{m'}; n, r) - c(\psi_{m'}; n, r) \\ &= (\det{}^k \otimes \sigma_l)^{-1}(S)\, \rho^{-1}(\mathrm{rot}(S))(c(\phi_n; m', r) - c(\psi_n; m', r)) = 0. \end{aligned}$$ □

Proposition 5.6. *There is a (noncanonical) map*

$$\mathrm{FM}^{(2)}_{k,l}(\rho) \hookrightarrow \prod_{0 \leqslant m \in (1/\mu_\rho)\mathbb{Z}} \mathrm{FM}^{(2)}_{k,l}(\rho)[m] / \mathrm{FM}^{(2)}_{k,l}(\rho)\left[m - \frac{1}{\mu_\rho}\right] \hookrightarrow \prod_{0 \leqslant m \in (1/\mu_\rho)\mathbb{Z}} \mathrm{J}_{k,m}(\sigma_l \otimes \rho)[m].$$

Proof. The first map is associated to the decreasing filtration by vanishing orders. Its kernel equals $\bigcap_m \mathrm{FM}^{(2)}_{k,l}(\rho)[m] = \{0\}$. The maps

$$\begin{gathered} \mathrm{FM}^{(2)}_{k,l}(\rho)[m] / \mathrm{FM}^{(2)}_{k,l}(\rho)\left[m - \frac{1}{\mu_\rho}\right] \hookrightarrow \mathrm{J}_{k,m}(\sigma_l \otimes \rho)[m], \\ \sum_{m'} \phi_{m'}(\tau, z) \exp(2\pi i\, m'\tau') \longmapsto \phi_m(\tau, z) \end{gathered}$$

are well defined by Lemma 5.5. They are injective by Proposition 5.4. □

5.2 Formal Fourier Jacobi expansions as a free module

The goal of this section is to establish the next theorem.

Theorem 5.7. *Let $0 \leqslant l \in \mathbb{Z}$ and fix a representation of $\mathrm{Sp}_2(\mathbb{Z})$ whose kernel has finite index. Then the module $\mathrm{FM}^{(2)}_{\bullet,l}(\rho)$ is free over $\mathrm{M}^{(2)}_\bullet$.*

Proposition 5.8. *We keep the assumptions of Theorem 5.7 on l and ρ. Further, suppose that k and k' are even integers. Let*

$$\Phi(Z) = \sum_{0 \leqslant m \in \mathbb{Q}} \phi(\tau, z)\, q'^{\,m} \in \mathrm{FM}^{(2)}_{k,l}(\rho) \quad \textit{and} \quad \Psi(Z) = \sum_{0 \leqslant m \in \mathbb{Q}} \psi(\tau, z)\, q'^{\,m} \in \mathrm{M}^{(2)}_{k'}.$$

Suppose that $\Psi\Phi \in \mathrm{M}^{(2)}_{k+k',l}(\rho)$. Then $\sum_{0 \leqslant m \in \mathbb{Q}} \phi(\tau, z)\, q'^{\,m}$ is convergent and $\Phi \in \mathrm{M}^{(2)}_{k,l}(\rho)$.

Proof of Theorem 5.7. In light of Proposition 5.3, it suffices to show that the module $\mathrm{FM}^{(2)}_{\bullet,l}(\rho) / \mathrm{M}^{(2)}_{\bullet,l}(\rho)$ is torsion free. This is an immediate consequence of Proposition 5.8. □

The proof of Proposition 5.8 uses the partial toroidal compactification. Let $\Gamma \subset \mathrm{Sp}_2(\mathbb{Z})$ be a sufficiently small congruence subgroup such that ρ is trivial on Γ and such that the partial toroidal compactification $X^{(1)}_\Gamma$ of $\Gamma\backslash\mathbb{H}^{(2)}$ by two-dimensional strata is defined as a complex variety. A detailed explanation can be found in [BvdGHZ08, ch. 3.11], in particular, p. 206. Note that van der Geer writes $\mathcal{A}^{(1)}_2$ for $X^{(1)}_\Gamma$. Note also that $X^{(1)}_\Gamma$ is canonical and does not depend on the choice of a cone decomposition. Further, all vector bundles of type $\det^k \otimes \sigma_l$ extend to $X^{(1)}_\Gamma$. This is, for example, stated in [BvdGHZ08, ch. 3.11].

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Proposition 5.9. *Every divisor on* $\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}^{(2)}$ *meets the two-dimensional boundary of* $X_\Gamma^{(1)}$.

Proof. Fix a toroidal compactification $\overline{\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}^{(2)}}$ as in [Hul00]. By taking its closure, every divisor on $\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}^{(2)}$ extends to $\overline{\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}^{(2)}}$. Referring to [Mum83], Hulek states that the free part of the Picard group of $\overline{\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}^{(2)}}$ is generated by the boundary class, say Δ, and the class corresponding to weight-one Siegel modular forms.

Fix a divisor D as in the assumption. By passing to a suitable multiple and adding multiples of Δ, we can assume that D is the divisor of some Siegel modular form Φ. We can further assume that the weight k of Φ is a multiple of 10, and that Φ is not a power of Igusa's χ_{10} (whose divisor meets the boundary of $\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}^{(2)}$). Then division of Φ by the $(\mathrm{ord}\,\Phi)$th power of χ_{10} yields a meromorphic Siegel modular forms $\widetilde{\Phi}$ with

$$\lim_{\tau'\to i\infty} \widetilde{\Phi}(Z) = \widetilde{\phi}_0(\tau, z) \not\equiv 0. \tag{5.2}$$

By construction, ϕ_0 is a meromorphic Jacobi form of index 0. It has positive weight. Indeed, if $k \leqslant 10\,\mathrm{ord}\,\Phi$, then Φ is a power of χ_{10} (see, for example, [Man92]). We conclude that ϕ_0 does not vanish identically and has zeros on $\mathbb{H}^{\mathrm{J}}$, which proves the statement. $\square$

Proof of Proposition 5.8. Let

$$\widetilde{\Phi} = (\Psi\Phi)/\Psi,$$

which is a meromorphic Siegel modular form, since $\Psi\Phi$ is a holomorphic Siegel modular form by the assumptions. We have to show that $\widetilde{\Phi}$ is holomorphic. Indeed, then there is a Fourier Jacobi expansion

$$\widetilde{\Phi}(Z) = \sum_m \widetilde{\phi}_m(\tau, z)\, q'^{\,m}$$

attached to it. Since we know that $\Psi\widetilde{\Phi} = \Psi\Phi$, we can use Proposition 5.3 to conclude that $\phi_m = \widetilde{\phi}_m$ for all m.

In order to show that $\widetilde{\Phi}$ is holomorphic, we employ the partial toroidal compactification. For a linear functional $f \in V(\rho)^\vee$ on $V(\rho)$, set

$$(\Psi\Phi)_f = f \circ \Psi\Phi \quad \text{and} \quad \widetilde{\Phi}_f = f \circ \widetilde{\Phi}.$$

Since ρ is trivial on Γ, $(\Psi\Phi)_f$ is a Siegel modular form for Γ and therefore yields a section in the vector bundle $V(\det^{k+k'} \otimes \sigma_l)$ over $X_\Gamma^{(1)}$. Similarly, we can view Ψ and $\widetilde{\Phi}_f$ as a holomorphic section of $V(\det^{k'})$ and a meromorphic section of $V(\det^k \otimes \sigma_l)$, respectively. The polar divisor of $\widetilde{\Phi}_f$ is contained in the zero divisor of Ψ. By Proposition 5.9, it suffices to show that $\widetilde{\Phi}_f$ is holomorphic at every boundary point $x \in X_\Gamma^{(1)}$ at infinity.

To study the regularity of $\widetilde{\Phi}_f$ locally, we use Hironaka's theory [Hir64a, Hir64b] in a way that is similar to [Bru04]. Given a divisor D on $X_\Gamma^{(1)}$, we obtain a smooth blow up $\pi : \widehat{X}_\Gamma^{(1)} \to X_\Gamma^{(1)}$ such that $\pi^{-1}(D)$ is normal crossing and such that we have local holomorphic coordinates. For D, we take the union of the boundary divisor of $X_\Gamma^{(1)}$ and the zero divisor of Ψ. To show that $\widetilde{\Phi}$ is holomorphic at the boundary of $X_\Gamma^{(1)}$, we fix an arbitrary point $x \in \widehat{X}_\Gamma^{(1)}$ that maps to the boundary component of $X_\Gamma^{(1)}$ at infinity. Let U be a neighborhood of x, which we will shrink in the next steps if necessary.

We pull back Ψ, $(\Psi\Phi)_f$ and $\widetilde{\Phi}$ and restrict to U. To ease notation, we set

$$\Psi|_U = (\Psi \circ \pi)|_U, \quad ((\Psi\Phi)_f)|_U = ((\Psi\Phi)_f \circ \pi)|_U \quad \text{and} \quad (\widetilde{\Phi}_f)|_U = (\widetilde{\Phi}_f \circ \pi)|_U.$$

Hironaka's theory yields local coordinates τ, z and q', where the preimage of the boundary at infinity of $X_\Gamma^{(1)}$ is given by $q' = 0$. In order to reduce technical complications, we shrink U if necessary and assume that U is a product set $U^{\mathrm{J}} \times U^{q'}$, where $U^{q'}$ is a neighborhood of $q' = 0$ in the q' plane. We are in the following situation:

$$\Psi|_U, ((\Psi\Phi)_f)|_U \in \mathcal{O}(U), \quad (\widetilde{\Phi}_f)|_U \in \mathcal{O}(U^{\mathrm{J}})[\![q']\!].$$

We further have $\Psi|_U \cdot (\widetilde{\Phi}_f)|_U = ((\Psi\Phi)_f)|_U$. We have to show that $(\widetilde{\Phi})_f|_U$ is convergent and holomorphic.

This will be a consequence of Proposition 5.10. Fix an arbitrary one-dimensional submanifold $U^{\mathrm{J}\prime}$ of U^{J} and set $U' = U^{\mathrm{J}\prime} \times U^{q'}$. It suffices to consider the restriction $(\widetilde{\Phi})_f|_{U'}$ of $(\widetilde{\Phi})_f|_U$ to U'. In the assumptions of Proposition 5.10, we let f equal $(\widetilde{\Phi})_f|_{U'}$. Local coordinates in $U^{\mathrm{J}\prime}$ correspond to the variable x and q' corresponds to y. Since $(\widetilde{\Phi})_f|_{U'}$ is a power series in q', the first assumption of Proposition 5.10 holds. The second one holds, since we have resolved the zero divisor of Ψ, which contains the polar divisor of $(\widetilde{\Phi})_f$. The third assumption of Proposition 5.10 holds by construction of $(\widetilde{\Phi})_f|_{U'}$. This establishes holomorphicity of $\widetilde{\Phi}_f$ and thus completes the proof. $\square$

The remainder of this section is of purely function theoretic nature. The proof of Proposition 5.10 builds up on Lemmas 5.12 and 5.13, which summarize slightly involved computations with polynomials whose coefficients are meromorphic functions.

In the proofs of the next theorem and lemmas, we will repeatedly use the Pochhammer symbol

$$(a)_n = \prod_{i=0}^{n-1} (a - i), \tag{5.3}$$

which is defined for $a \in \mathbb{C}$ and $0 \leqslant n \in \mathbb{Z}$.

PROPOSITION 5.10. *Let $f(x, y)$ be a function that is meromorphic in a neighborhood U of $x = y = 0$ in $\mathbb{C}^2$. Assume the following:*

(i) *$\{(x, y) \in U : y = 0\}$ is not a polar divisor of f;*

(ii) *no irreducible polar divisor of f has a branching point at $x = y = 0$;*

(iii) *for all $0 \leqslant n \in \mathbb{Z}$, the function $\partial_y^n|_{y=0} f(x, y)$ admits a holomorphic continuation to a neighborhood of $x = 0$.*

Then f is holomorphic in $x = y = 0$.

Proof. By the assumptions on f and the theory of meromorphic functions presented, for example, in [Nis01] (following work of Oka), we can express f in a sufficiently small neighborhood of $x = y = 0$ as follows:

$$f(x, y) = \tilde{f}(x, y) \frac{\sum_{i=0}^{d_P} P_i(x) y^i}{\prod_{j=1}^{n_Q} (Q_{j,1}(x) y + Q_{j,0}(x))^{1+m_j}}. \tag{5.4}$$

We have $0 \leqslant n_Q, d_P, m_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant n_Q)$. Functions that appear on the right-hand side are holomorphic in some neighborhood of $x = y = 0$ or $x = 0$, respectively. They satisfy $\tilde{f}(0, 0) \neq 0$, $P_{d_P}(0) \neq 0$, $P_i(0) = 0$ $(0 \leqslant i < d_P)$, $Q_{j,1}(0) \neq 0$ and $Q_{j,0}(0) = 0$. We can assume that for $j \neq j'$, $Q_{j,0}/Q_{j,1} \neq Q_{j',0}/Q_{j',1}$ as functions in a neighborhood of $x = 0$. By dividing both sides by $\tilde{f}$, we can and will assume without loss of generality that $\tilde{f}(x,y)=1$. Write $P(x,y)=\sum_i P_i(x)y^i$ for the numerator of (5.4). We may assume that the numerator and denominator in (5.4) are coprime, which amounts to the assertion that for all $1 \leqslant j \leqslant n_Q$, we have $P(x,\,-Q_{j,0}(x)/Q_{j,1}(x)) \neq 0$ as functions in a neighborhood of $x=0$.

We will show by contradiction that $n_Q=0$, that is, $f(x,y)=P(x,y)$ is holomorphic in $x=y=0$.

For convenience, we suppress the dependence on x. Furthermore, we use exponents $0 \leqslant e_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant n_Q)$. Away from $x=0$, we have

$$\begin{aligned}
\partial_y^n|_{y=0} f(\cdot,y) &= \sum_{i=0}^{d_P} \binom{n}{i} i!P_i\, \partial_y^{n-i}|_{y=0}\Bigg(\prod_{j=1}^{n_Q} (Q_{j,1}\,y+Q_{j,0})^{-1-m_j}\Bigg)\\
&= \sum_{i=0}^{d_P} \binom{n}{i} i!P_i \sum_{\sum_j e_j=n-i} \binom{n-i}{e_1,e_2,\ldots,e_{n_Q}} \prod_{j=1}^{n_Q} (-1)^{e_j}(m_j+e_j)_{e_j}\, Q_{j,1}^{e_j} Q_{j,0}^{-1-m_j-e_j}\\
&= \Bigg(\prod_{j=1}^{n_Q} Q_{j,0}^{-1-m_j}\Bigg) \sum_{i=0}^{d_P} P_i \sum_{\sum_j e_j=n-i} \prod_{j=1}^{n_Q} \binom{m_j+e_j}{m_j} (-Q_{j,1}/Q_{j,0})^{e_j}.
\end{aligned}$$

Assuming $n_Q \neq 0$, apply Lemmas 5.12 and 5.13 to obtain a contradiction. □

Lemma 5.11. *Let $H_j(x)$ $(1 \leqslant j \leqslant n_H)$ be pairwise distinct meromorphic functions in a neighborhood of $x=0$. Then, for all $0<n\in\mathbb{Z}$ and nonnegative exponents $e_j\in\mathbb{Z}$, we have*

$$\sum_{\sum_{j=1}^{n_H} e_j=n} \prod_{j=1}^{n_H} H_j^{e_j} = \sum_{j=1}^{n_H} H_j^n \frac{\prod_{\substack{j'=1\\ j'\neq j}}^{n_H} H_{j'}}{\prod_{\substack{j',j''=1\\ j'\neq j,\, j'\neq j''}}^{n_H} (H_{j'}-H_{j''})}.$$

Proof. Given $j' \neq j''$, we have

$$(H_{j'}-H_{j''}) \sum_{\sum_{j=1}^{n_H} e_j=n} H_j^{e_j} = H_{j'} \sum_{\substack{\sum_j e_j=n\\ e_{j''}=0}} \prod_{j=1}^{n_H} H_j^{e_j} - H_{j''} \sum_{\substack{\sum_j e_j=n\\ e_{j'}=0}} \prod_{j=1}^{n_H} H_j^{e_j}.$$

Now, for fixed j', we have

$$\Bigg(\prod_{\substack{1\leqslant j''\leqslant n_H\\ j''\neq j'}} (H_{j'}-H_{j''})\Bigg) \sum_{\sum_{j=1}^{n_H} e_j=n} \prod_{j=1}^{n_H} H_j^{e_j} = H_{j'}^n \prod_{j''\neq j'} H_{j''} + O(H_{j'}^{n_H}).$$

Since the remainder term $O(H_j^{n_H})$ does not depend on n, this proves the theorem. □

Lemma 5.12. *Let $H_j(x)$ $(1 \leqslant j \leqslant n_H)$ be pairwise distinct, nonzero meromorphic functions in a neighborhood of 0. Fix a holomorphic function $P(x,y)=\sum_{i=0}^{d_P} P_i(x)y^i$, and suppose that $P(x,H_j(x)^{-1}) \neq 0$ as functions of x for all $1\leqslant j\leqslant n_H$. Given nonnegative integers m_j, we have*

$$\sum_{i=0}^{d_P} P_i \sum_{\sum_{j=1}^{n_H} e_j=n-i} \binom{m_j+e_j}{m_j} H_j^{e_j} = \sum_{j=1}^{n_H} H_j^n R_j(H_1,\ldots,H_{n_H};n)$$

with rational functions R_j. Moreover, the $R_j(H_1(x),\ldots,H_{n_H}(x);n)$ $(1\leqslant j\leqslant n_H)$ are nonzero polynomials in n with coefficients that are meromorphic functions in x.

Proof. Write ∂_j for the formal derivative with respect to H_j. Using Lemma 5.11, we find that

$$\sum_{\sum_{j=1}^{n_H} e_j = n-i} \binom{m_j + e_j}{m_j} H_j^{e_j} = \left(\prod_{j=1}^{n_H} \frac{\partial_{H_j}^{m_j}}{m_j!}\right)\left(\sum_{\sum_{j=1}^{n_H} e_j = n-i} \prod_{j=1}^{n_H} H_j^{e_j+m_j}\right)$$
$$= \left(\prod_{j=1}^{n_H} \frac{\partial_{H_j}^{m_j}}{m_j!}\right)\left(\sum_{j=1}^{n_H} H_j^{n-i} \frac{\prod_{\substack{j'=1\\ j'\neq j}}^{n_H} H_{j'}}{\prod_{\substack{j',j''=1\\ j'\neq j, j'\neq j''}}^{n_H} (H_{j'} - H_{j''})}\right).$$

We insert this into the left-hand side of the formula in the statement. Note that the sum over i and the formal derivatives commute, since P_i, as a variable, is independent of H_j. We obtain

$$\left(\prod_{j=1}^{n_H} \frac{\partial_{H_j}^{m_j}}{m_j!}\right)\left(\sum_{j=1}^{n_H} H_j^{n} \frac{P(H_j^{-i})\prod_{\substack{j'=1\\ j'\neq j}}^{n_H} H_{j'}}{\prod_{\substack{j',j''=1\\ j'\neq j, j'\neq j''}}^{n_H} (H_{j'} - H_{j''})}\right).$$

This proves the first assertion of the lemma. The second assertion follows straightforwardly, when carrying out differentiation. $\square$

LEMMA 5.13. *Let $H_j(x)(1 \leqslant j \leqslant n_H)$ be pairwise distinct, nonzero meromorphic functions in a neighborhood of $x = 0$ whose pole order at $x = 0$ is at least one. Further, let $R_j(x, n)(1 \leqslant j \leqslant n_H)$ be polynomials in n whose coefficients are meromorphic in x.*

If, for all $0 < n \in \mathbb{Z}$,

$$\sum_{j=1}^{n_H} H_j^n R_j(x, n)$$

is regular at $x = 0$, then $R_j = 0$ for all $1 \leqslant j \leqslant n_H$.

Proof. Set $J = \{1 \leqslant j \leqslant n_J : R_j \neq 0\}$. The statement is equivalent to $J = \emptyset$. We assume that J is not empty.

Let $o_H = -\min_{j\in J}(\mathrm{ord}_{x=0} H_j)$ be the maximal pole order at $x = 0$ of the H_j, and $J_H = \{j \in J : -\mathrm{ord}_{x=0} H_j = o_H\}$ be the set of indices for which this maximum is attained. Further, let $d_R = \max_{j\in J_H}(\deg_n R_j)$ be the maximal degree of the R_j as a polynomial in n, and let $J_R = \{j \in J_H : \deg_n R_j = d_R\}$. By construction, $J \neq \emptyset$ implies $J_H \neq \emptyset$, which implies $J_R \neq \emptyset$. Write R_{j,d_R} for the d_Rth coefficient of R_j.

Choose some $0 \leqslant o_R \in \mathbb{Z}$ such that the coefficients of $x^{o_R} R_j(x, n)$ are holomorphic in $x = 0$ for all $1 \leqslant j \leqslant n$. We rewrite the regularity assumption:

$$\sum_{j=1}^{n_H} H_j^n R_j(x, n) = n^{d_R} x^{-n o_H - o_R} \sum_{j=1}^{n_H} (x^{o_H} H_j)^n (n^{-d_R} x^{o_R} R_j(x, n))$$

must be regular at $x = 0$. In other words, for all $0 \leqslant l < n_H$, we have

$$\sum_{j=1}^{n_H} (x^{o_H} H_j)^l ((x^{o_H} H_j)^n (x^{o_R} R_{j,d_R}(x) + O((n + l)^{-1}))) \equiv 0 \pmod{x^{n o_H + o_R}}.$$

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The Vandermonde matrix $((x^{o_H} H_j)^l)_{j,l}$ is invertible as a matrix of meromorphic functions. The 0th coefficient of $(x^{o_H} H_j)^n$ is nonzero for $j \in J_H \supseteq J_R$. By choosing n large enough, we therefore see that

$$x^{o_R} R_{j,d_R}(x) \equiv O(n^{-1}) \pmod{x^{\tilde{n}(n)}}$$

with $\tilde{n}(n) \to \infty$ as $n \to \infty$. This shows that $R_{j,d_R} = 0$ for all $j \in J_H$. This contradicts the choice of d_R and hence proves the lemma. □

5.3 Dimension estimates

In this section, we compare the dimensions of $\mathrm{M}^{(2)}_{k,l}(\rho)$ and $\mathrm{FM}^{(2)}_{k,l}(\rho)$ as $k \to \infty$. The idea of the proof of the next theorem is very much inspired by [IPY13], and was initially brought up by Aoki [Aok00].

Theorem 5.14. *Fix $0 \leqslant l \in \mathbb{Z}$ and a representation ρ of $\mathrm{Sp}_2(\mathbb{Z})$ with finite index kernel. We have*

$$\dim \mathrm{FM}^{(2)}_{k,l}(\rho) \leqslant \dim \mathrm{M}^{(2)}_{k,l}(\rho) + O(k^2)$$

as $k \to \infty$, $k \in 2\mathbb{Z}$. The implied constants of the remainder term depend on l and ρ.

Proof of Theorem 5.14. Proposition 5.6 gives us an inclusion

$$\mathrm{FM}^{(2)}_{k,l}(\rho) \hookrightarrow \prod_{0 \leqslant m \in (1/\mu_\rho)\mathbb{Z}} \mathrm{J}_{k,m}(\sigma_l \otimes \rho)[m].$$

We are hence reduced to estimating the dimension of the right-hand side. Corollary 4.4 says that there is a map

$$\mathrm{J}_{k,m}(\sigma_l \otimes \rho)[m] \hookrightarrow \bigoplus_{i=0}^{l} \mathrm{J}_{k+i,m}(\rho)[m].$$

Insert this into the previous equation. Then Lemma 3.3 shows that it suffices to treat the case $l = 0$.

By Proposition 4.2, we need to consider

$$\bigoplus_{0 \leqslant m_0 < 1} \bigoplus_{0 \leqslant m \in m_0 + \mathbb{Z}} \mathrm{J}_{k,m}(\rho(m_0))[m],$$

where $\rho = \bigoplus_{m_0} \rho(m_0)$ and $\rho(m_0)(I_2, (0,0), 1) = \exp(2\pi i\, m_0)$.

We consider the generating function of this series, and adapt the calculations presented in [IPY13, § 4]. The change of order of summation is justified because all coefficients are positive.

$$\begin{aligned}
&\sum_{2|k \in \mathbb{Z}} \sum_{0 \leqslant m_0 < 1} \sum_{m=0}^{\infty} \sum_{\nu=0}^{m} \dim \mathrm{M}_{k+2\nu}(\rho(m_0))[m_0 + m] t^k \\
&\quad = \sum_{0 \leqslant m_0 < 1} \sum_{m=0}^{\infty} \sum_{\nu=0}^{m} \Bigg(\sum_{2|k \in \mathbb{Z}} \dim \mathrm{M}_{k+2\nu-12m}(\rho(m_0))[m_0]\, t^{k+2\nu-12m} \Bigg) t^{12m-2\nu} \\
&\quad \leqslant \frac{\sum_{0 \leqslant m_0 < 1} p_{\mathrm{g}}(\rho(m_0); t)}{(1-t^4)(1-t^6)} \sum_{m=0}^{\infty} \sum_{\nu=0}^{m} t^{12m-2\nu} \\
&\quad = \frac{p_{\mathrm{g}}(\rho; t)}{(1-t^4)(1-t^6)} \sum_{m=0}^{\infty} \sum_{\nu=0}^{m} t^{12m-2\nu}.
\end{aligned}$$

Here, '$\leqslant$' means that every coefficient with respect to t on the left-hand side is less than or equal to the corresponding coefficient on the right-hand side. The double sum equals

$$\sum_{\nu=0}^{\infty}\sum_{m=\nu}^{\infty} t^{12m-2\nu} = \frac{1}{1-t^{12}}\sum_{\nu=0}^{\infty} t^{12\nu-2\nu} = \frac{1}{(1-t^{10})(1-t^{12})}.$$

This proves the statement, since the generators of $\bigoplus_{2|k} \mathrm{M}_k^{(2)}$ have weights $4, 6, 10$ and 12. $\square$

5.4 Rigidity

We combine the results of §§ 5.1 and 5.2 to establish rigidity of formal Fourier Jacobi expansions. The next lemma will help us to cover weights that are not even integers.

Lemma 5.15. *There is a vector-valued Siegel modular form of:*

(i) *odd weight;*
(ii) *half-integral weight $k \notin \mathbb{Z}$*

whose components have no common zero.

Proof. Assuming that we have proved the second statement, let f be a Siegel modular form of half-integral weight whose components have no common zero. Then $f \otimes f$ has odd weight and its components do not have common zeros, either.

The second part can be established by means of theta nulls. They define a map $\Gamma(4,8)\backslash \mathbb{H}_2 \hookrightarrow \mathbb{P}^9$ for a suitable subgroup $\Gamma(4,8) \subseteq \widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ (see, for example, [Igu64]). Let θ be the vector whose components consist of all theta nulls. Then $\theta|_{1/2}\gamma$, where γ runs through a system of representatives of $\Gamma(4,8)\backslash\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$, yields a vector-valued modular form with the desired properties. $\square$

Theorem 5.16. *For every $k \in \frac{1}{2}\mathbb{Z}$, $0 \leqslant l \in \mathbb{Z}$ and every representation ρ of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ with finite index kernel, we have $\mathrm{FM}_{k,l}^{(2)}(\rho) = \mathrm{M}_{k,l}^{(2)}(\rho)$.*

Proof. We first prove the case of $k \in 2\mathbb{Z}$. By the remark after Definition 3.1, we can assume that ρ factors through $\mathrm{Sp}_2(\mathbb{Z})$. By Theorem 5.7, $\mathrm{FM}_{\bullet,l}^{(2)}(\rho)$ is free over $\mathrm{M}_{\bullet}^{(2)}$. Suppose that $\mathrm{FM}_{\bullet,l}^{(2)}(\rho) \neq \mathrm{M}_{\bullet,l}^{(2)}(\rho)$. Then its rank over $\mathrm{M}_{\bullet}^{(2)}$ exceeds $\dim \sigma_l \cdot \dim \rho$. Therefore, we would have

$$\mathrm{FM}_{\bullet,l}^{(2)}(\rho) \geqslant (1 + \dim \sigma_l \cdot \dim \rho) \dim \mathrm{M}_k^{(2)} + O(k^2),$$

contradicting Theorem 5.14.

Now suppose that $k \in \frac{1}{2}\mathbb{Z}$. Choose a Siegel modular form Ψ as in Lemma 5.15 such that $\Phi \otimes \Psi$ has even weight. By the above, it is a Siegel modular form. Write Ψ_i for the components of Ψ. The meromorphic Siegel modular form $(\Phi\Psi_i)/\Psi_i$ is independent of i, since $(\Phi\Psi_i)\Psi_{i'} = (\Phi\Psi_{i'})\Psi_i$ for all i, i'. By our choice, the components of Ψ have no common zero. Therefore, $(\Phi\Psi_i)/\Psi_i$ is holomorphic and possesses a Fourier Jacobi expansion. This expansion equals Φ in light of Proposition 5.3. $\square$

6. Generating functions for special cycles on Shimura varieties

Let $\mathcal{L}$ be an integral lattice of signature $(n, 2)$ with attached quadratic form $q_{\mathcal{L}}$. We adopt the notation used in the introduction. That is, we denote the dual of $\mathcal{L}$ by $\mathcal{L}^{\#}$, and write $\mathrm{disc}^r \mathcal{L} = (\mathcal{L}^{\#}/\mathcal{L})^r$ for powers of the attached discriminant form. Fix a subgroup Γ of the orthogonal group

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$\mathrm{O}(\mathcal{L})$ that fixes the discriminant form $\operatorname{disc}\mathcal{L}$ pointwise. Write $\mathrm{Gr}^-(\mathcal{L}\otimes\mathbb{C})$ for the Grassmannian of two-dimensional negative subspaces of $\mathcal{L}\otimes\mathbb{C}$. There are rational quadratic cycles on the Shimura variety $X_\Gamma = \Gamma\backslash\mathrm{Gr}^-(\mathcal{L}\otimes\mathbb{R})$. Given a tuple of vectors $v=(v_1,\ldots,v_r)$ in $\mathcal{L}\otimes\mathbb{Q}$, let

$$Z(v) = \{W \in \mathrm{Gr}^-(\mathcal{L}\otimes\mathbb{R}) : \operatorname{span}(v)\perp W\}.$$

This cycle is nontrivial if $\operatorname{span}(v)\subseteq\mathcal{L}\otimes\mathbb{Q}$ is positive.

Kudla defined so-called special cycles on X_Γ [Kud97]. Associate to each v a moment matrix $q_{\mathcal{L}}(v)=\frac{1}{2}(\langle v_i,v_j\rangle_{\mathcal{L}})_{1\leqslant i,j\leqslant r}$, where $\langle v,w\rangle_{\mathcal{L}} = q_{\mathcal{L}}(v+w)-q_{\mathcal{L}}(v)-q_{\mathcal{L}}(w)$. Given a semi-positive-definite matrix $0\leqslant T\in\mathrm{Sym}_r(\mathbb{Q})$ and $\lambda\in\operatorname{disc}^r\mathcal{L}$, set

$$\Omega(T,\mu) = \{v\in\mu+\mathcal{L}^r : T = q_{\mathcal{L}}(v)\}.$$

Define

$$Z(T,\mu) = \sum_{v\in\Gamma\backslash\Omega(T,\mu)} Z(v).$$

The symmetries Γ act on $\Omega(T,\mu)$, and there are finitely many orbits. Therefore, $Z(T,\mu)$ descends to a cycle on X_Γ, which we also denote by $Z(T,\mu)$. Writing $\mathrm{rk}(T)$ for the rank of T, we find that $Z(T,\mu)$ is a $\mathrm{rk}(T)$-cycle, whose class in $\mathrm{CH}^{\mathrm{rk}(T)}(X_\Gamma)_{\mathbb{C}}$ is denoted by $\{Z(T,\mu)\}$.

Let $\omega^\vee$ be the anti-canonical bundle on X_Γ, and write $\{\omega^\vee\}$ for its class in $\mathrm{CH}^1(X_\Gamma)_{\mathbb{C}}$. Fix a linear functional on $\mathrm{CH}^r(X_\Gamma)_{\mathbb{C}}$. Define the formal Fourier expansion

$$\Theta_{\Gamma,f}(Z) = \sum_{\substack{\mu\in\operatorname{disc}^r\mathcal{L}\\ 0\leqslant T\in\mathrm{Sym}_r(\mathbb{Q})}} f(\{Z(T,\mu)\}\cdot\{\omega^\vee\}^{r-\mathrm{rk}(T)})\exp(2\pi i\operatorname{trace}(TZ))\mathfrak{e}_\mu. \tag{6.1}$$

Here, $\mathfrak{e}_\mu$ is a canonical basis vector of the representation space $\mathbb{C}[\operatorname{disc}^r\mathcal{L}]$ of the Weil representation of the double cover $\widetilde{\mathrm{Sp}}_r(\mathbb{Z})$ of $\mathrm{Sp}_r(\mathbb{Z})$.

We now restrict to the case $r=2$. In his thesis, Zhang [Zha09] proved that $\Theta_{\Gamma,f}(Z)$ is a formal Fourier Jacobi expansion of weight $1+n/2$. This is essentially [Zha09, Proposition 2.6, p. 22]. Note that he does not make use of Condition 1 of his Theorem 2.5 while proving this proposition.

Remark 6.1. The type of $\Theta_{\Gamma,f}$ stated in [Zha09] is slightly incorrect. On page 20, Zhang affirms that, in his notation,

$$F(T,\underline{\lambda}) = \sqrt{\det(A)}^{n'+n}\sqrt{\det(A)}^{n'-n}F(A^{\mathrm{trace}}TA,\underline{\lambda}A).$$

He argues that this holds if n' is even. But, actually, $F(T,\underline{\lambda})$ is basis independent. Consequently, the required invariance does not hold if $4\nmid n'$, while $n'=2$ in our application. This can be fixed easily, by considering the subrepresentation of the Weil representation $\rho_{L,r}$ (in Zhang's notation), which comes with the matching sign.

Corollary 6.2. *For any subgroup $\Gamma\subseteq O(L)$ as above and every linear functional f on the Chow group $\mathrm{CH}^2(X_\Gamma)_{\mathbb{C}}$, the function $\Theta_{\Gamma,f}$ is a vector-valued Siegel modular form of weight $1+n/2$.*

Proof. This follows when combining Zhang's results and Theorem 5.16. □

Using this statement, we can reprove [Zha09, Theorem 3.1].

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COROLLARY 6.3. *The space*

$$\operatorname{span}(\{Z(T,\mu)\} : 0 < T \in \operatorname{Sym}_2(\mathbb{Q}), \mu \in \operatorname{disc}^r \mathcal{L}) \subseteq \mathrm{CH}^2(X_\Gamma)_\mathbb{C}$$

is finite dimensional.

Proof. This follows from $\dim \mathrm{M}_k^{(2)}(\rho) < \infty$, which holds for all $k \in \frac{1}{2}\mathbb{Z}$ and any representation ρ of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ with finite index kernel. □

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Appendix A. Computing Siegel modular forms

In this section, we assume that the base field for all modular forms is $\mathbb{Q}$. All statements concerning dimensions and Fourier expansions that we will make use of hold for any sufficiently large field $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. If readers wish, they can adjust Algorithm A.5 accordingly. For all representations in this section, we assume that the kernel is a congruence subgroup.

We deal with truncated Fourier expansions of Jacobi forms, Siegel modular forms and formal Fourier Jacobi expansions. We start by defining appropriate index sets. Given $0 < B \in \mathbb{Z}$, set

$$\begin{aligned} I^{(2)}(B) &= \{(n,r,m) \in \mathbb{Q}^3 : 0 \leqslant m, n < B, 4nm - r^2 \geqslant 0\}, \\ I^{\mathrm{J}}(m;B) &= \{(n,r) \in \mathbb{Q}^2 : 0 \leqslant n < B, 4nm - r^2 \geqslant 0\}. \end{aligned}$$

By abuse of notation, we denote the Fourier expansion map always by the same symbol $\mathcal{FE}_B$. Given any representation ρ of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ or Γ^J with representation space V_ρ, define

$$\begin{aligned} \mathcal{FE}_B : \mathrm{M}_k^{(1)}(\rho) &\to V_\rho^B, \quad \Phi \longmapsto (c(\Phi;n))_{0\leqslant n<B}, \\ \mathcal{FE}_B : \mathrm{M}_{k,l}^{(2)}(\rho) &\to (V_{\sigma_l} \otimes V_\rho)^{I^{(2)}(B)}, \quad \Phi \longmapsto (c(\Phi;n,r,m))_{(n,r,m)}, \\ \mathcal{FE}_B : \mathrm{J}_{k,m}(\sigma_l \otimes \rho) &\to (V_{\sigma_l} \otimes V_\rho)^{I^\mathrm{J}(m;B)}, \quad \phi \longmapsto (c(\phi;n,r))_{(n,r)}, \\ \mathcal{FE}_B : \mathrm{FM}_{k,l}^{(2)}(\rho) &\to \bigoplus_{0\leqslant m<B} (V_{\sigma_l} \otimes V_\rho)^{I^\mathrm{J}(m;B)}, \quad (\phi_m)_m \longmapsto ((c(\phi_m;n,r))_{(n,r)})_{0\leqslant m<B}. \end{aligned}$$

There are truncation maps from $V^{I^\mathrm{J}(m;B')}$ to $V^{I^\mathrm{J}(m;B)}$ if $B' \geqslant B$. When comparing Fourier expansions that have different 'precision', we will freely apply them.

PROPOSITION A.1. *Given $0 \leqslant k \in \frac{1}{2}\mathbb{Z}$ and a representation ρ of $\widetilde{\mathrm{Sp}}_1(\mathbb{Z})$, the map $\mathcal{FE}_B : \mathrm{M}_k^{(1)}(\rho) \to V_\rho^B$ is injective for $B > k/12 + 1$.*

Proof. Suppose that there is a modular form $\phi \in \mathrm{M}_k^{(1)}(\rho)$ whose first $l := \lfloor k/12 \rfloor + 1$ Fourier coefficients vanish. Then $\Delta^{-l}\phi$, where Δ is the discriminant form, is a nonzero, holomorphic modular form of negative weight, which cannot be. □

PROPOSITION A.2. *Given $0 \leqslant k \in \frac{1}{2}\mathbb{Z}$, $0 < m \in \mathbb{Q}$ and a representation ρ of $\widetilde{\Gamma}^\mathrm{J}$ with representation space V_ρ, the map $\mathcal{FE}_B : \mathrm{J}_{k,m}(\sigma_l \otimes \rho) \to (V_{\sigma_l} \otimes V_\rho)^{I^\mathrm{J}(m;B)}$ is injective for $B > (k + l + 2\lfloor m \rfloor)/12 + 1$.*

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Proof. The bijection

$$\mathrm{J}_{k,m}(\sigma_l \otimes \rho) \cong \bigoplus_{i=0}^{l} \mathrm{J}_{k+i,m}(\rho)$$

is induced by differential operators with constant coefficients. Hence, it is compatible with taking Fourier expansions. It is thus sufficient to treat the case $l = 0$.

We have

$$\mathcal{FE}_B(\mathrm{J}_{k,m}(\rho)) \hookrightarrow \begin{cases} \displaystyle\bigoplus_{\nu=0}^{\lfloor m \rfloor} \mathcal{FE}_B(\mathrm{M}^{(1)}_{k+2\nu}(\rho)) & \text{if } (-1)^k \rho(-I_2) \text{ is trivial,} \\ \displaystyle\bigoplus_{\nu=0}^{\lfloor m \rfloor - 1} \mathcal{FE}_B(\mathrm{M}^{(1)}_{k+2\nu+1}(\rho)) & \text{otherwise.} \end{cases}$$

Elements of $\mathrm{M}^{(1)}_{k+i}(\rho)$, $0 \leqslant i \leqslant 2\lfloor m \rfloor$, are uniquely determined by their first $(k + 2\lfloor m \rfloor)/12 + 1$ Fourier coefficients. This proves the proposition. □

Proposition A.3. *Let $k \in \frac{1}{2}\mathbb{Z}$, $l \in \mathbb{Z}$ and $0 < m \in \mathbb{Q}$. Further, let ρ be a representation of $\widetilde{\Gamma}^{\mathrm{J}}$ with representation space V_ρ. Given $0 < B \in \mathbb{Z}$, fix families*

$$(\phi_m)_{0 \leqslant m < B}, (\psi_m)_{0 \leqslant m < B} \in \bigoplus_{0 \leqslant m < B} \mathrm{J}_{k,m}(\sigma_l \otimes \rho).$$

Suppose that they satisfy $\mathcal{FE}_{(k+l)/10+1}(\phi_m) = \mathcal{FE}_{(k+l)/10+1}(\psi_m)$ for all $0 \leqslant m \leqslant (k + l)/10$ and

$$\begin{aligned} c(\phi_m; n, r) &= (\det{}^k \otimes \sigma_l)^{-1}(S)\, \rho^{-1}(\mathrm{rot}(S)) c(\phi_n; m, r), \\ c(\psi_m; n, r) &= (\det{}^k \otimes \sigma_l)^{-1}(S)\, \rho^{-1}(\mathrm{rot}(S)) c(\psi_n; m, r) \end{aligned}$$

for all $0 < n, m < B$ and all $r \in \mathbb{Z}$. Then we have $\phi_m = \psi_m$ for all $0 \leqslant m < B$.

Proof. By Proposition A.2, ϕ_m and ψ_m are determined by $c(\phi_m; n, r)$ and $c(\psi_m; n, r)$ with $n \leqslant (k + l + 2\lfloor m \rfloor)/12$. For $m \leqslant (k + l)/10$, we have $(k + l + 2m)/12 < (k + l)/10 + 1$ and hence $\phi_m = \psi_m$.

The rest of the proof is analogous to the proof of Lemma 5.5. It suffices to show that ϕ_m, $m > (k + l)/10$, is uniquely determined by all $\phi_{m'}$, $m' < m$. Employing Proposition A.2, this follows from the inequality $(k + l + 2m)/12 < m$, which holds exactly if $m > (k + l)/10$. □

Given $k \in \frac{1}{2}\mathbb{Z}$, $l \in \mathbb{Z}$, a representation ρ of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ with finite index kernel and $0 < B \in \mathbb{Z}$, define

$$\begin{aligned} \mathrm{FM}^{(2)}_{k,l}(\rho)_B = \Big\{ (\phi_m)_{0 \leqslant m < B} \in \bigoplus_{0 \leqslant m < B} \mathcal{FE}_B(\mathrm{J}_{k,m}(\sigma_l \otimes \rho)) :& \\ c(\phi_m; n, r) = (\det{}^k \otimes \sigma_l)^{-1}(S)\, \rho^{-1}(\mathrm{rot}(S))\, c(\phi_n; m, r)& \\ \text{for all } m, n < B \text{ and all } r \in \mathbb{Z} \Big\}.& \end{aligned}$$

PROPOSITION A.4. *Given $k \in \frac{1}{2} \in \mathbb{Z}$, $l \in \mathbb{Z}$, a representation ρ of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ with finite index kernel and $(k+l)/10 < B \leqslant B' \in \mathbb{Z}$, the following inclusion holds:*

$$\mathrm{FM}^{(2)}_{k,l}(\rho)_{B'} \subseteq \mathrm{FM}^{(2)}_{k,l}(\rho)_{B},$$

where we implicitly truncate elements of the space on the right-hand side.

Proof. This follows from Proposition A.3: given the first $(k+l)/10$ Fourier Jacobi coefficients of any element in $\mathrm{FM}^{(2)}_{k,l}(\rho)_B$ or $\mathrm{FM}^{(2)}_{k,l}(\rho)_{B'}$, all others are uniquely defined. □

ALGORITHM A.5. Let $0 \leqslant k \in \frac{1}{2}\mathbb{Z}$, $0 \leqslant l \in \mathbb{Z}$ and let ρ be a representation of $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ with finite index kernel. Given $(k+l)/10 < B \in \mathbb{Z}$, the following algorithm computes the space $\mathcal{FE}_B(\mathrm{M}^{(2)}_{k,l}(\rho))$.

(1) Compute $\mathcal{FE}_B(\mathrm{J}_{k,m}(\sigma_l \otimes \rho))$ for all $0 \leqslant m < B$.

(2) Compute $\mathrm{FM}^{(2)}_{k,l}(\rho)_B$.

(3) If $\dim \mathrm{FM}^{(2)}_{k,l}(\rho)_B = \dim \mathrm{M}^{(2)}_{k,l}(\rho)$, then we are done. Otherwise, increase B and go back to Step (1).

After processing these steps, there is a one-to-one correspondence of elements $\Phi \in \mathcal{FE}_B(\mathrm{M}^{(2)}_{k,l}(\rho))$ and elements $(\phi_m) \in \mathrm{FM}^{(2)}_{k,l}(\rho)_B$ via $c(\Phi; n, r, m) = c(\phi_m; n, r)$.

Remarks A.6. All steps except for the last one can be implemented based on the current state of knowledge.

(1) Step (1) requires use of results in [IPY13] and [Rau12]. In the former paper an explicit bijection of $\mathrm{J}_{k,m}(\sigma_l \otimes \rho)$ and $\bigoplus_{i=0}^{l} \mathrm{J}_{k+i,m}(\rho)$ was given. In order to compute $\mathcal{FE}_B(\mathrm{J}_{k,m}(\rho))$ using the latter work, note that $\mathrm{J}_{k,m}(\rho) \cong \mathrm{M}^{(1)}_{k-1/2}(\check{\rho}_m \otimes \rho)$ ($\check{\rho}_m$ is the dual of the Weil representation for the lattice $(2m)$ – see [Rau12]) via theta decomposition.

(2) Step (2) can be done by means of basic linear algebra.

(3) Step (3) requires computations of $\dim \mathrm{M}^{(2)}_{k,l}(\rho)$. If ρ is trivial and $k \geqslant 4$ ($l = 0$) or $k \geqslant 5$ ($l > 0$), this has been done by Tsushima [Tsu83]. To the author's knowledge, the case of nontrivial ρ has not yet been treated.

Proof. The algorithm terminates, because we have

$$\mathrm{FM}^{(2)}_{k,l}(\rho) = \bigcap_{0<B\in\mathbb{Z}} \mathrm{FM}^{(2)}_{k,l}(\rho)_B,$$

and $\dim \mathrm{FM}^{(2)}_{k,l}(\rho)_B$ is monotonically decreasing for sufficiently large B by Proposition A.4.

Correctness of Algorithm A.5 follows, because we have

$$\mathcal{FE}_B(\mathrm{M}^{(2)}_{k,l}(\rho)) \subseteq \mathrm{FM}^{(2)}_{k,l}(\rho)_B$$

for all $0 < B \in \mathbb{Z}$. If the dimension check in Step (3) succeeds, then equality holds. □

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