7
The Cauchy problem in General Relativity

In this chapter we shall give an outline of the Cauchy problem in General Relativity. We shall show that, given certain data on a spacelike three-surface $\mathscr{S}$, there is a unique maximal future Cauchy development $D^+(\mathscr{S})$ and that the metric on a subset $\mathscr{U}$ of $D^+(\mathscr{S})$ depends only on the initial data on $J^-(\mathscr{U}) \cap \mathscr{S}$. We shall also show that this dependence is continuous if $\mathscr{U}$ has a compact closure in $D^+(\mathscr{S})$. This discussion is included here because of its intrinsic interest, because it uses some of the results of the previous chapter, and because it demonstrates that the Einstein field equations do indeed satisfy postulate (*a*) of § 3.2 that signals can only be sent between points that can be joined by a non-spacelike curve. However it is not really needed for the remaining three chapters, and so could be skipped by the reader more interested in singularities.

In § 7.1, we discuss the various difficulties and give a precise formulation of the problem. In § 7.2 we introduce a global background metric $\hat{\mathbf{g}}$ to generalize the relation which holds between the Ricci tensor and the metric in each coordinate patch to a single relation which holds over the whole manifold. We impose four gauge conditions on the covariant derivatives of the physical metric $\mathbf{g}$ with respect to the background metric $\hat{\mathbf{g}}$. These remove the four degrees of freedom to make diffeomorphisms of a solution of Einstein's equations, and lead to the second order hyperbolic reduced Einstein equations for $\mathbf{g}$ in the background metric $\hat{\mathbf{g}}$. Because of the conservation equations, these gauge conditions hold at all times if they and their first derivatives hold initially.

In § 7.3 we show that the essential part of the initial data for $\mathbf{g}$ on the three-dimensional manifold $\mathscr{S}$ can be expressed as two three-dimensional tensor fields h^{ab}, χ^{ab} on $\mathscr{S}$. The three-dimensional manifold $\mathscr{S}$ is then imbedded in a four-dimensional manifold $\mathscr{M}$ and a metric $\mathbf{g}$ is defined on $\mathscr{S}$ such that h^{ab} and χ^{ab} become respectively the first and second fundamental forms of $\mathscr{S}$ in $\mathbf{g}$. This can be done in such a way that the gauge conditions hold on $\mathscr{S}$. In § 7.4 we establish some

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basic inequalities for second order hyperbolic equations. These relate integrals of squared derivatives of solutions of such equations to their initial values. These inequalities are used to prove the existence and uniqueness of solutions of second order hyperbolic equations. In § 7.5 the existence and uniqueness of solutions of the reduced empty space Einstein equations is proved for small perturbations of an empty space solution. The local existence and uniqueness of empty space solutions for arbitrary initial data is then proved by dividing the initial surface up into small regions which are nearly flat, and then joining the resulting solutions together. In § 7.6 we show there is a unique maximal empty space solution for given initial data and that in a certain sense this solution depends continuously on the initial data. Finally in §7.7 we indicate how these results may be extended to solutions with matter.

7.1 The nature of the problem

The Cauchy problem for the gravitational field differs in several important respects from that for other physical fields.

(1) The Einstein equations are *non-linear*. Actually in this respect they are not so different from other fields, for while the electromagnetic field, the scalar field, etc., *by themselves* obey linear equations in a given space–time, they form a non-linear system when their mutual interactions are taken into account. The distinctive feature of the gravitational field is that it is *self-interacting*: it is non-linear even in the absence of other fields. This is because it defines the space–time over which it propagates. To obtain a solution of the non-linear equations one employs an iterative method on approximate linear equations whose solutions are shown to converge in a certain neighbourhood of the initial surface.

(2) Two metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ on a manifold $\mathscr{M}$ are physically equivalent if there is a diffeomorphism $\phi: \mathscr{M} \to \mathscr{M}$ which takes $\mathbf{g}_1$ into $\mathbf{g}_2$ ($\phi_* \mathbf{g}_1 = \mathbf{g}_2$), and clearly $\mathbf{g}_1$ satisfies the field equations if and only if $\mathbf{g}_2$ does. Thus the solutions of the field equations can be unique only up to a diffeomorphism. In order to obtain a definite member of the equivalence class of metrics which represents a space–time, one introduces a fixed 'background' metric and imposes four 'gauge conditions' on the covariant derivatives of the physical metric with respect to the background metric. These conditions remove the four degrees of freedom to make diffeomorphisms and lead to a unique solution for the metric components. They are analogous to the Lorentz condition

which is imposed to remove the gauge freedom for the electromagnetic field.

(3) Since the metric defines the space–time structure, one does not know in advance what the domain of dependence of the initial surface is and hence what the region is on which the solution is to be determined. One is simply given a three-dimensional manifold $\mathscr{S}$ with certain initial data $\boldsymbol{\omega}$ on it, and is required to find a four-dimensional manifold $\mathscr{M}$, an imbedding $\theta: \mathscr{S} \to \mathscr{M}$ and a metric $\mathbf{g}$ on $\mathscr{M}$ which satisfies the Einstein equations, agrees with the initial values on $\theta(\mathscr{S})$ and is such that $\theta(\mathscr{S})$ is a Cauchy surface for $\mathscr{M}$. We shall say that $(\mathscr{M}, \theta, \mathbf{g})$, or simply $\mathscr{M}$, is a *development* of $(\mathscr{S}, \boldsymbol{\omega})$. Another development $(\mathscr{M}', \theta', \mathbf{g}')$ of $(\mathscr{S}, \boldsymbol{\omega})$ will be called an *extension* of $\mathscr{M}$ if there is a diffeomorphism α of $\mathscr{M}$ into $\mathscr{M}'$ which leaves the image of $\mathscr{S}$ pointwise fixed and takes $\mathbf{g}'$ into $\mathbf{g}$ (i.e. $\theta^{-1}\alpha^{-1}\theta' = \mathrm{id}$ on $\mathscr{S}$, and $\alpha_* \mathbf{g}' = \mathbf{g}$). We shall show that provided the initial data $\boldsymbol{\omega}$ satisfies certain *constraint equations* on $\mathscr{S}$, there will exist developments of $(\mathscr{S}, \boldsymbol{\omega})$ and further, there will be a development which is maximal in the sense that it is an extension of any development of $(\mathscr{S}, \boldsymbol{\omega})$. Note that by formulating the Cauchy problem in these terms we have included the freedom to make diffeomorphisms, since any development is an extension of any diffeomorphism of itself which leaves the image of $\mathscr{S}$ pointwise fixed.

7·2 The reduced Einstein equations

In chapter 2, the Ricci tensor was obtained in terms of coordinate partial derivatives of the components of the metric tensor. For the purposes of this chapter it will be convenient to obtain an expression that applies to the whole manifold $\mathscr{M}$ and not just to each coordinate neighbourhood separately. To this end we introduce a *background metric* $\hat{\mathbf{g}}$ as well as the physical metric $\mathbf{g}$. With two metrics one has to be careful to maintain the distinction between covariant and contravariant indices. (To avoid confusion, we shall suspend the usual conventions for raising and lowering indices.) The covariant and contravariant forms of $\mathbf{g}$ and $\hat{\mathbf{g}}$ are related by

$$g^{ab}g_{bc} = \delta^a{}_c = \hat{g}^{ab}\hat{g}_{bc}. \tag{7.1}$$

It will be convenient to take the contravariant form g^{ab} of the metric to be more fundamental and the covariant form g_{ab} as derived from it by (7.1). Using the alternating tensor $\hat{\eta}_{abcd}$ defined by the background metric, this relation can be expressed explicitly as

$$g_{ab} = \frac{1}{3!} g^{cd} g^{ef} g^{ij} (\det \mathbf{g})\, \hat{\eta}_{acei} \hat{\eta}_{bdfj}, \tag{7.2}$$

where $$(\det \mathbf{g})^{-1} \equiv \frac{1}{4!} g^{ab} g^{cd} g^{ef} g^{ij} \hat{\eta}_{acei} \hat{\eta}_{bdfj}$$

is the determinant of the components of g^{ab} in a basis which is orthonormal with respect to the metric $\hat{\mathbf{g}}$.

The difference between the connection $\mathbf{\Gamma}$ defined by $\mathbf{g}$ and the connection $\hat{\mathbf{\Gamma}}$ defined by $\hat{\mathbf{g}}$ is a tensor, and can be expressed in terms of the covariant derivative of $\mathbf{g}$ with respect to $\hat{\mathbf{\Gamma}}$ (cf § 3.3):

$$\begin{aligned} \delta \Gamma^a{}_{bc} &\equiv \Gamma^a{}_{bc} - \hat{\Gamma}^a{}_{bc} \\ &= \tfrac{1}{2} g^{ij}{}_{|k} (g_{bi} g_{cj} g^{ak} - g_{bi} \delta^k{}_c \delta^a{}_j - g_{ci} \delta^k{}_b \delta^a{}_j), \end{aligned} \tag{7.3}$$

where we have used a stroke to denote covariant differentiation with respect to $\hat{\mathbf{\Gamma}}$ and the symbol δ to denote the difference between quantities defined from $\mathbf{g}$ and $\hat{\mathbf{g}}$. Then from (2.20),

$$\delta R_{ab} = \delta \Gamma^d{}_{ab|d} - \delta \Gamma^d{}_{ad|b} + \delta \Gamma^d{}_{ab} \delta \Gamma^e{}_{de} - \delta \Gamma^d{}_{ae} \delta \Gamma^e{}_{bd}. \tag{7.4}$$

Thus

$$\begin{aligned} \delta(R^{ab} - \tfrac{1}{2} g^{ab} R) &= g^{ai} g^{bj} \delta R_{ij} + 2 \delta g^{i(a} g^{b)j} \hat{R}_{ij} - \delta g^{ai} \delta g^{bj} \hat{R}_{ij} \\ &\quad - \tfrac{1}{2} \delta g^{ab} \hat{R} - \tfrac{1}{2} g^{ab} (\delta g^{ij} \hat{R}_{ij} + g^{ij} \delta R_{ij}) \\ &= \tfrac{1}{2} g^{ij} \delta g^{ab}{}_{|ij} - g^{i(a} \psi^{b)}{}_{|i} + \tfrac{1}{2} g^{ab} (\psi^i{}_{|i} - g_{cd} g^{ij} \delta g^{cd}{}_{|ij}) \\ &\quad + (\text{terms in } \delta g^{cd}{}_{|i} \text{ and } \delta g^{ef}), \end{aligned} \tag{7.5}$$

$$\psi^b \equiv g^{bc}{}_{|c} - \tfrac{1}{2} g^{bc} g_{de} g^{de}{}_{|c} = (\det \mathbf{g})^{-1} ((\det \mathbf{g}) g^{bc})_{|c} = (\det \mathbf{g})^{-1} \phi^{bc}{}_{|c} \tag{7.6}$$

and $$\phi^{bc} \equiv (\det \mathbf{g})\, \delta g^{bc}.$$

The plan is now as follows. We choose some suitable background metric $\hat{\mathbf{g}}$ and express the Einstein equations in the form

$$R^{ab} - \tfrac{1}{2} R g^{ab} = \delta(R^{ab} - \tfrac{1}{2} R g^{ab}) + \hat{R}^{ab} - \tfrac{1}{2} \hat{g}^{ab} \hat{R} = 8\pi T^{ab}. \tag{7.7}$$

One regards this as a second order non-linear set of differential equations to determine $\mathbf{g}$ in terms of the values of it and its first derivatives on some initial surface. Of course to complete the system one has to specify the equations governing the physical fields which make up the energy–momentum tensor T^{ab}. However even when this is done one does not have a system of equations which uniquely determines the time development in terms of the initial values and first derivatives. The reason for this is, as was mentioned above, that a solution of the Einstein equations can be unique only up to a diffeomorphism. In order to obtain a definite solution one removes this freedom to make diffeomorphisms by imposing four *gauge conditions* on the covariant derivatives of $\mathfrak{g}$ with respect to the background metric $\hat{\mathfrak{g}}$. We shall use the so-called 'harmonic' conditions

$$\psi^b = \phi^{bc}{}_{|c} = 0$$

which are analogous to the Lorentz gauge conditions $A^i{}_{;i} = 0$ in electrodynamics. With this condition one obtains the *reduced Einstein equations*

$$g^{ij}\phi^{ab}{}_{|ij} + (\text{terms in } \phi^{cd}{}_{|e} \text{ and } \phi^{ab}) = 16\pi T^{ab} - 2\hat{R}^{ab} + \hat{g}^{ab}\hat{R}. \quad (7.8)$$

We shall denote the left-hand side of (7.8) by $E^{ab}{}_{cd}(\phi^{cd})$, where $E^{ab}{}_{cd}$ is the *Einstein operator*. For suitable forms of the energy–momentum tensor T^{ab} these are second order hyperbolic equations for which we shall demonstrate the existence and uniqueness of solutions in § 7.5. We still have to check that the harmonic conditions are consistent with the Einstein equations. That is to say: we derived (7.8) from the Einstein equations by assuming that $\phi^{bc}{}_{|c}$ was zero. We now have to verify that the solution that (7.8) gives rise to does indeed have this property. To do this, differentiate (7.8) and contract. This gives an equation of the form

$$g^{ij}\psi^b{}_{|ij} + B_c{}^{bi}\,\psi^c{}_{|i} + C_c{}^b\,\psi^c = 16\pi T^{ab}{}_{;a}, \quad (7.9)$$

where a semi-colon denotes differentiation with respect to g, and the tensors $B_c{}^{bi}$ and $C_c{}^b$ depend on $\hat{g}^{ab}$, $\hat{R}^a{}_{bcd}$, g^{ab} and $g^{ab}{}_{|c}$. Equations (7.9) may be regarded as second order linear hyperbolic equations for ψ^b. Since the right-hand side vanishes, one can use the uniqueness theorem for such equations (proposition 7.4.5) to show that ψ^b will vanish everywhere if it and its first derivatives are zero on the initial surface. We shall see in the next section that this can be arranged by a suitable diffeomorphism.

We still have to show that the unique solution obtained by imposing the harmonic gauge condition is related by a diffeomorphism to any other solution of the Einstein equations with the same initial data. This will be done in § 7.4 by making a special choice of the background metric.

7.3 The initial data

As (7.8) is a second order hyperbolic system it seems that to determine the solution one should prescribe the values of g^{ab} and $g^{ab}{}_{|c}u^c$ on the initial surface $\theta(\mathscr{S})$, where u^c is some vector field which is not tangent to $\theta(\mathscr{S})$. However not all these twenty components are significant or independent: some can be given arbitrary initial values without changing the solution by more than a diffeomorphism, and others have to obey certain consistency conditions.

Consider a diffeomorphism $\mu: \mathscr{M} \to \mathscr{M}$ which leaves $\theta(\mathscr{S})$ pointwise fixed. This will induce a map μ_* which takes g^{ab} at $p \in \theta(\mathscr{S})$ into a new tensor $\mu_* g^{ab}$ at p. If $n_a \in T^*{}_p$ is orthogonal to $\theta(\mathscr{S})$ (i.e. $n_a V^a = 0$ for any $V^a \in T_p$ tangent to $\theta(\mathscr{S})$) and normalized so that $n_a \hat{g}^{ab} n_b = -1$ then, by suitable choice of μ, $n_a \mu_* g^{ab}$ can be made equal to any vector at p which is not tangent to $\theta(\mathscr{S})$. Thus the components $n_a g^{ab}$ are not significant. On the other hand as μ leaves $\theta(\mathscr{S})$ pointwise fixed, the induced metric $h_{ab} = \theta^* g_{ab}$ on $\mathscr{S}$ will remain unchanged. It is therefore only this part of $\mathbf{g}$ which lies in $\theta(\mathscr{S})$ which need be given to determine the solution. The other components $n_a g^{ab}$ can be prescribed arbitrarily without changing the solution by more than a diffeomorphism. Another way of seeing this is to recall that we formulated the Cauchy problem in terms of certain data on a disembodied three-manifold $\mathscr{S}$ and then looked for an imbedding into some four-manifold $\mathscr{M}$. Now on $\mathscr{S}$ itself one cannot define a four-dimensional tensor field like $\mathbf{g}$ but only a three-dimensional metric $\mathbf{h}$, which we shall take to be positive definite. The contravariant and covariant forms of $\mathbf{h}$ are related by

$$h^{ab} h_{bc} = \delta^a{}_c, \tag{7.10}$$

where now $\delta^a{}_c$ is a three-dimensional tensor in $\mathscr{S}$. The imbedding θ will carry h_{ab} into a contravariant tensor field $\theta_* h^{ab}$ on $\theta(\mathscr{S})$ which has the property

$$n_a \theta_* h^{ab} = 0. \tag{7.11}$$

As $n_a g^{ab}$ is arbitrary, one may now define $\mathbf{g}$ on $\theta(\mathscr{S})$ by

$$g^{ab} = \theta_* h^{ab} - u^a u^b, \tag{7.12}$$

where u^a is any vector field on $\theta(\mathscr{S})$ which is nowhere zero or tangent to $\theta(\mathscr{S})$. Defining g_{ab} by (7.1), one has:

$$h_{ab} = \theta^* g_{ab}, \quad n_a g^{ab} = -n_a u^a u^b, \quad g_{ab} u^a u^b = -1. \tag{7.13}$$

Thus h_{ab} is the metric induced on $\mathscr{S}$ by $\mathbf{g}$ and u^a is the unit vector orthogonal to $\theta(\mathscr{S})$ in the metric $\mathbf{g}$.

The situation with the first derivatives $g^{ab}{}_{|c}u^c$ is similar: $n_a g^{ab}{}_{|c}u^c$ can be given any value by suitable diffeomorphisms. However there is now an additional complication in that $g^{ab}{}_{|c}$ depends not only on $\mathbf{g}$ but also on the background metric $\hat{\mathbf{g}}$ on $\mathscr{M}$. In order to give a description of the significant part of the first derivative of $\mathbf{g}$ in terms only of tensor fields defined on $\mathscr{S}$, we proceed as follows. We prescribe a symmetric contravariant tensor field χ^{ab} on $\mathscr{S}$. Under the imbedding χ^{ab} is mapped into a tensor field $\theta_*\chi^{ab}$ on $\theta(\mathscr{S})$. We require that this is equal to the second fundamental form (see § 2.7) of the submanifold $\theta(\mathscr{S})$ in the metric $\mathbf{g}$. This gives

$$\begin{aligned}\theta_*\chi^{ab} &= \theta_* h^{ac}\theta_* h^{bd}(u^e g_{ec})_{;d} \\ &= \theta_* h^{ac}\theta_* h^{bd}((u^e g_{ec})_{|d} - \delta\Gamma^f{}_{cd}u^e g_{ef}).\end{aligned} \tag{7.14}$$

Using (7.3), one has

$$\theta_*\chi^{ab} = \tfrac{1}{2}\theta_* h^{ac}\theta_* h^{bd}(-g_{ci}g_{dj}g^{ij}{}_{|k}u^k + g_{bi}u^i{}_{|c} + g_{ci}u^i{}_{|b}). \tag{7.15}$$

This may be inverted to give $g^{ab}{}_{|c}u^c$ in terms of $\theta_*\chi^{ab}$:

$$\tfrac{1}{2}g^{ab}{}_{|c}u^c = -\theta_*\chi^{ab} + \theta_* h^{ac}\theta_* h^{bd}g_{i(c}u^i{}_{|d)} + u^{(a}W^{b)}, \tag{7.16}$$

where W^b is some vector field on $\theta(\mathscr{S})$. It can be given any required value by a suitable diffeomorphism μ.

The tensor fields h^{ab} and χ^{ab} cannot be prescribed completely independently on $\mathscr{S}$. For multiplying the Einstein equations (7.7) by n_a, one obtains four equations which do not contain $g^{ab}{}_{|cd}u^c u^d$, the second derivatives of $\mathbf{g}$ out of $\mathscr{S}$. Thus there must be four relations between g^{ab}, $g^{ab}{}_{|c}u^c$ and $n_a T^{ab}$. Using (2.36) and (2.35), they can be expressed as equations in the three-manifold $\mathscr{S}$:

$$\chi^{cd}{}_{\|d}h_{ce} - \chi^{cd}{}_{\|e}h_{cd} = 8\pi\theta^*(T_{de}u^d), \tag{7.17}$$

$$\tfrac{1}{2}(R' + (\chi^{dc}h_{dc})^2 - \chi^{ab}\chi^{cd}h_{ac}h_{bd}) = 8\pi\theta^*(T_{de}u^d u^e), \tag{7.18}$$

where a double stroke $\|$ denotes covariant differentiation in $\mathscr{S}$ with respect to the metric $\mathbf{h}$, and R' is the curvature scalar of $\mathbf{h}$.

The data $\boldsymbol{\omega}$ on $\mathscr{S}$ that is required to determine the solution therefore consists of the initial data for the matter fields (in the case of a scalar field ϕ for example, this would consist of two functions on $\mathscr{S}$ representing the value of ϕ and its normal derivative) and two tensor fields h^{ab} and χ^{ab} on $\mathscr{S}$ which obey the *constraint equations* (7.17–18). These contraint equations are elliptic equations on the surface $\mathscr{S}$ which impose four constraints on the twelve independent components of (h^{ab}, χ^{ab}). In such situations, one can show one can prescribe eight of these components independently and then solve the constraint equations to find the other four, see e.g. Bruhat (1962). We shall call a pair $(\mathscr{S}, \boldsymbol{\omega})$ satisfying these conditions, an *initial data set*. We then imbed $\mathscr{S}$ in some suitable four-manifold $\mathscr{M}$ with metric $\mathbf{g}$ and define g^{ab} on $\theta(\mathscr{S})$ by (7.12) for some suitable choice of u^a. We shall take u^a to be $g^{ab}n_b$. Thus it will be the unit vector orthogonal to $\theta(\mathscr{S})$ in both the metric $\mathbf{g}$ and $\hat{\mathbf{g}}$. We shall also exploit our freedom of choice of W^a in the definition of $g^{ab}{}_{|c}u^c$ by (7.16) to make ψ^b zero on $\theta(\mathscr{S})$. This requires

$$W^b = -g^{bc}{}_{|d}\, g_{ce}\, \theta_* h^{ed} + \tfrac{1}{2} g_{cd}\, g^{cd}{}_{|e}\, \theta_* h^{eb} + u^b(g_{cd}\,\theta_*\chi^{cd} - g_{ic}\, u^i{}_{|d}\, \theta_* h^{cd}). \qquad (7.19)$$

(Note that all the derivatives in (7.19) are tangent to $\theta(\mathscr{S})$ as is required by the fact that the fields involved have been defined only on $\theta(\mathscr{S})$.) To ensure that ψ^b vanishes everywhere one also needs $\psi^b{}_{|c}u^c$ to be zero on $\theta(\mathscr{S})$. However this now follows from the constraint equations providing the reduced Einstein equations (7.8) hold on $\theta(\mathscr{S})$. One may therefore proceed to solve (7.8) as a second order non-linear hyperbolic system on the manifold $\mathscr{M}$ with metric $\hat{\mathbf{g}}$.

(Note that there are 10 such equations for the ϕ's; in proving the existence of solutions of these 10 equations we do not split them into a set of constraint equations and a set of evolution equations, and so the question as to whether the constraint equations are conserved does not arise.)

7.4 Second order hyperbolic equations

In this section we shall reproduce some results on second order hyperbolic equations given in Dionne (1962). They will be generalized to apply to a whole manifold, not just one coordinate neighbourhood. These results will be used in the following sections to prove the existence and uniqueness of developments for an initial data set $(\mathscr{S}, \boldsymbol{\omega})$.

We first introduce a number of definitions. We use Latin letters to denote multiple contravariant or covariant indices; thus a tensor of type (r, s) will be written as $K^I{}_J$, and we denote by $|I| = r$ the number of indices that the multiple index I represents. We introduce a positive definite metric e_{ab} on $\mathscr{M}$ and define

$$e_{IJ} = \underbrace{e_{ab}\, e_{cd} \ldots e_{pq}}_{r \text{ times}}, \quad e^{IJ} = \underbrace{e^{ab} e^{cd} \ldots e^{pq}}_{r \text{ times}},$$

where $|I| = |J| = r$. We then define the magnitude $|K^I{}_J|$ (or simply, $|\mathbf{K}|$) as $(K^I{}_J K^L{}_M e_{IL} e^{JM})^{\frac{1}{2}}$ where repeated multiple indices imply contraction over all the indices they represent. We define $|D^m K^I{}_J|$ (or simply, $|D^m\mathbf{K}|$) to be $|K^I{}_{J|L}|$ where $|L| = m$ and as before, $|$ indicates covariant differentiation with respect to $\hat{\mathbf{g}}$.

Let $\mathscr{N}$ be an imbedded submanifold of $\mathscr{M}$ with compact closure in $\mathscr{M}$. Then $\|K^I{}_J, \mathscr{N}\|_m$ is defined to be

$$\left\{\sum_{p=0}^{m} \int_{\mathscr{N}} (|D^p K^I{}_J|)^2 \, d\sigma\right\}^{\frac{1}{2}},$$

where $d\sigma$ is the volume element on $\mathscr{N}$ induced by $\mathbf{e}$. We also define $\|\mathbf{K}, \mathscr{N}\tilde{\|}_m$ to be the same expression where the derivatives are taken only in directions tangent to $\mathscr{N}$. Clearly, $\|\mathbf{K}, \mathscr{N}\|_m \geqslant \|\mathbf{K}, \mathscr{N}\tilde{\|}_m$.

The *Sobolev spaces* $W^m(r, s, \mathscr{N})$ (or simply $W^m(\mathscr{N})$) are then defined to be the vector spaces of tensor fields $K^I{}_J$ of type (r, s) whose values and derivatives (in the sense of distributions) are defined almost everywhere on $\mathscr{N}$ (i.e. except, possibly, on a set of measure zero; for the rest of this section 'almost everywhere' is to be understood almost everywhere) and for which $\|K^I{}_J, \mathscr{N}\tilde{\|}_m$ is finite. With the norms $\| \quad , \mathscr{N}\tilde{\|}_m$ the Sobolev spaces are Banach spaces in which the C^m tensor fields of type (r, s) form dense subsets. If $\mathbf{e}'$ is another continuous positive definite metric on $\mathscr{M}$ then there will be positive constants C_1 and C_2 such that

$$C_1 |K^I{}_J| \leqslant |K^I{}_J|' \leqslant C_2 |K^I{}_J| \quad \text{on} \quad \mathscr{N},$$

and

$$C_1 \|K^I{}_J, \mathscr{N}\tilde{\|}_m \leqslant \|K^I{}_J, \mathscr{N}\tilde{\|}_m' \leqslant C_2 \|K^I{}_J, \mathscr{N}\tilde{\|}_m.$$

Thus $\| \quad , \mathscr{N}\tilde{\|}_m'$ will be an equivalent norm. Similarly another C^m background metric $\hat{\mathbf{g}}'$ will give an equivalent norm. In fact it follows from two lemmas given below that if $\hat{\mathbf{g}}'' \in W^m(\mathscr{N})$ and $2m$ is greater than the dimension of $\mathscr{N}$, then the norm obtained using the covariant derivatives defined by $\hat{\mathbf{g}}''$ is again equivalent.

We now quote three fundamental results on Sobolev spaces. The proofs can be derived from results given in Sobolev (1963). They require a mild restriction on the shape of $\mathscr{N}$. A sufficient condition will be that for each point p of the boundary $\partial\mathscr{N}$ it should be possible to imbed an n-dimensional half cone in $\bar{\mathscr{N}}$ with vertex at p, where n is the dimension of $\mathscr{N}$. In particular this condition will be satisfied if the boundary $\partial\mathscr{N}$ is smooth.

Lemma 7.4.1

There is a positive constant P_1 (depending on $\mathscr{N}$, $\mathbf{e}$ and $\hat{\mathbf{g}}$) such that for any field $K^I{}_J \in W^m(\mathscr{N})$ with $2m > n$, where n is the dimension of $\mathscr{N}$, $|\mathbf{K}| \leqslant P_1 \|\mathbf{K}, \widetilde{\mathscr{N}}\|_m$ on $\mathscr{N}$.

From this and the fact that the vector space of all continuous fields $K^I{}_J$ on $\mathscr{N}$ is a Banach space with norm $\sup_{\mathscr{N}} |\mathbf{K}|$, it follows that if $K^I{}_J \in W^m(\mathscr{N})$ where $2m > n$, then $K^I{}_J$ is continuous on $\mathscr{N}$. Similarly if $K^I{}_J \in W^{m+p}(\mathscr{N})$, then $K^I{}_J$ is C^p on $\mathscr{N}$.

Lemma 7.4.2

There is a positive constant P_2 (depending on $\mathscr{N}$, $\mathbf{e}$ and $\hat{\mathbf{g}}$) such that for any fields $K^I{}_J$, $L^P{}_Q \in W^m(\mathscr{N})$ with $4m \geqslant n$,

$$\|K^I{}_J L^P{}_Q, \mathscr{N}\|_0 \leqslant P_2 \|\mathbf{K}, \mathscr{N}\|_m \|\mathbf{L}, \mathscr{N})\|_m.$$

From this and the previous lemma it follows that if $n \leqslant 4$ and $2m > n$, then for any two fields $K^I{}_J$, $L^P{}_Q \in W^m(\mathscr{N})$, the product $K^I{}_J L^P{}_Q$ is also in $W^m(\mathscr{N})$.

Lemma 7.4.3

If $\mathscr{N}'$ is an $(n-1)$-dimensional submanifold smoothly imbedded in $\mathscr{N}$, there is a positive constant P_3 (depending on $\mathscr{N}$, $\mathscr{N}'$, $\mathbf{e}$ and $\hat{\mathbf{g}}$) such that for any field $K^I{}_J \in W^{m+1}(\mathscr{N})$,

$$\|\mathbf{K}, \mathscr{N}'\|_m \leqslant P_3 \|\mathbf{K}, \mathscr{N}\|_{m+1}.$$

We shall prove the existence and uniqueness of developments for $(\mathscr{S}, \boldsymbol{\omega})$ when $h^{ab} \in W^{4+a}(\mathscr{S})$ and $\chi^{ab} \in W^{3+a}(\mathscr{S})$ where a is any non-negative integer. (If $\mathscr{S}$ is non-compact, we mean by $h^{ab} \in W^m(\mathscr{S})$ that $h^{ab} \in W^m(\mathscr{N})$ for any open subset $\mathscr{N}$ of $\mathscr{S}$ with compact closure.) A sufficient condition for this is that h^{ab} be C^{4+a} and χ^{ab} be C^{3+a} on $\mathscr{S}$; by lemma 7.4.1, a necessary condition is that h^{ab} be C^{2+a} and χ^{ab} be C^{1+a}. The solution obtained for g^{ab} will belong to $W^{4+a}(\mathscr{H})$ for each smooth spacelike surface $\mathscr{H}$ and so the $(2+a)$th derivatives will be bounded, i.e. g^{ab} will be $C^{(2+a)-}$ on $\mathscr{M}$.

These differentiability conditions can be weakened to cases such as shock waves where the solution departs from W^4 behaviour on well-behaved hypersurfaces; see Choquet–Bruhat (1968), Papapetrou and Hamoui (1967), Israel (1966), and Penrose (1972*a*). However no proof is known for cases in which such departures occur generally. The W^4 condition for the existence and uniqueness of developments is an improvement on previous work (Choquet-Bruhat (1968)) but it is somewhat stronger than one would like since the Einstein equations can be defined in a distributional sense if the metric is continuous and its generalized derivatives are locally square integrable (i.e. if $\mathbf{g}$ is C^0 and W^1). On the other hand any W^p conditions for p less than 4 would

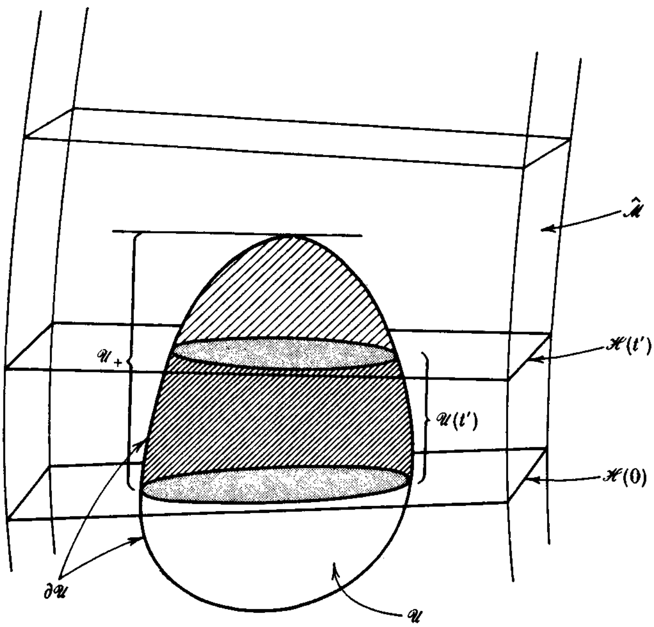

FIGURE 48. $\mathscr{U}$ is an open set with compact closure in the manifold $\hat{\mathscr{M}} = \mathscr{H} \times R^1$. $\mathscr{U}_+$ is the region of $\mathscr{U}$ for which $t \geqslant 0$ and $\mathscr{U}(t')$ is the region of $\mathscr{U}$ between $t = 0$ and $t = t' > 0$.

not guarantee the uniqueness of geodesics, or, for p less than 3, their existence. Our own view is that these differences of differentiability conditions are not important since as explained in § 3.1, the model for space–time may as well be taken to be C^∞.

In order to prove the existence and uniqueness of developments we now establish some fundamental inequalities (lemmas 7.4.4 and 7.4.6) for second order hyperbolic equations, in a manner similar to that of the conservation theorem in § 4.3.

Consider a manifold $\hat{\mathscr{M}}$ of the form $\mathscr{H} \times R^1$ where $\mathscr{H}$ is a three-dimensional manifold. Let $\mathscr{U}$ be an open set of $\hat{\mathscr{M}}$ with compact closure which has boundary $\partial\mathscr{U}$ and which intersects $\mathscr{H}(0)$, where $\mathscr{H}(t)$ denotes the surface $\mathscr{H}\times\{t\}$, $t\in R^1$. Let $\mathscr{U}_+$ and $\mathscr{U}(t')$ denote the parts of $\mathscr{U}$ for which $t \geqslant 0$ and $t' \geqslant t \geqslant 0$ respectively (figure 48). On $\mathscr{U}_+$ let $\hat{\mathbf{g}}$ be a C^{2-} background metric and let $\mathbf{e}$ be a C^{1-} positive definite metric. We shall consider tensor fields $K^I{}_J$ which obey second order hyperbolic equations of the form

$$L(K) \equiv A^{ab}K^I{}_{J|ab} + B^{aPI}{}_{QJ}K^Q{}_{P|a} + C^{PI}{}_{QJ}K^Q{}_P = F^I{}_J, \quad (7.20)$$

where $\mathbf{A}$ is a Lorentz metric on $\mathscr{U}_+$ (i.e. a symmetric tensor field of signature $+2$), $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{F}$ are tensor fields of type indicated by their indices, and $|$ denotes covariant differentiation with respect to the metric $\hat{\mathbf{g}}$.

Lemma 7.4.4

If (1) $\partial\mathscr{U} \cap \overline{\mathscr{U}}_+$ is achronal with respect to $\mathbf{A}$,

(2) there exists some $Q_1 > 0$ such that on $\overline{\mathscr{U}}_+$

$$A^{ab}t_{|a}t_{|b} \leqslant -Q_1$$

and

$$A^{ab}W_aW_b \geqslant Q_1 e^{ab}W_aW_b$$

for any form $\mathbf{W}$ which satisfies $A^{ab}t_{|a}W_b = 0$,

(3) there exists some Q_2 such that on $\overline{\mathscr{U}}_+$

$$|\mathbf{A}| \leqslant Q_2, \quad |\mathbf{DA}| \leqslant Q_2, \quad |\mathbf{B}| \leqslant Q_2, \quad |\mathbf{C}| \leqslant Q_2,$$

then there exists some positive constant P_4 (depending on $\mathscr{U}$, $\mathbf{e}$, $\hat{\mathbf{g}}$, Q_1 and Q_2) such that for all solutions $K^I{}_J$ of (7.20),

$$\|\mathbf{K}, \mathscr{H}(t)\cap\mathscr{U}_+\|_1 \leqslant P_4\{\|\mathbf{K}, \mathscr{H}(0)\cap\mathscr{U}_+\|_1 + \|\mathbf{F}, \mathscr{U}(t)\|_0\}.$$

One forms the 'energy tensor' S^{ab} for the field $K^I{}_J$ in analogy to the energy–momentum tensor of a scalar field of unit mass (§ 3.2):

$$S^{ab} = \{(A^{ac}A^{bd} - \tfrac{1}{2}A^{ab}A^{cd})K^I{}_{J|c}K^P{}_{Q|d} - \tfrac{1}{2}A^{ab}K^I{}_JK^P{}_Q\}e^{JQ}e_{IP}. \quad (7.21)$$

The tensor S^{ab} obeys the dominant energy condition (§ 4.3) with respect to the metric $\mathbf{A}$ (i.e. if W_a is timelike with respect to $\mathbf{A}$ then $S^{ab}W_aW_b \geqslant 0$ and $S^{ab}W_a$ is non-spacelike with respect to $\mathbf{A}$). Moreover by conditions (2) and (3) there will be positive constants Q_3 and Q_4 such that

$$Q_3(|\mathbf{K}|^2 + |\mathbf{DK}|^2) \leqslant S^{ab}t_{|a}t_{|b} \leqslant Q_4(|\mathbf{K}|^2 + |\mathbf{DK}|^2). \quad (7.22)$$

We now apply lemma 4.3.1 to S^{ab}, taking $\mathscr{U}_+$ as the compact region $\mathscr{F}$

and using the volume element $\mathrm{d}\hat{v}$ and covariant differentiation defined by the metric $\hat{\mathbf{g}}$:

$$\int_{\mathscr{H}(t)\cap\overline{\mathscr{U}}_+} S^{ab}t_{|a}\,\mathrm{d}\hat{\sigma}_b \leqslant \int_{\mathscr{H}(0)\cap\overline{\mathscr{U}}_+} S^{ab}t_{|a}\,\mathrm{d}\hat{\sigma}_b + \int_0^t \left\{\int_{\mathscr{H}(t')\cap\overline{\mathscr{U}}_+} (PS^{ab}t_{|a} + S^{ab}{}_{|a})\,\mathrm{d}\hat{\sigma}_b\right\}\mathrm{d}t' \quad (7.23)$$

where P is a positive constant independent of S^{ab}. (The sign has been changed in the first term on the right-hand side since the surface element $\mathrm{d}\hat{\sigma}_b$ of the surface $\mathscr{H}(t)$ is taken to have the same orientation as $t_{|b}$, i.e. $\mathrm{d}\hat{\sigma}_b = t_{|b}\,\mathrm{d}\tilde{\sigma}$ where $\mathrm{d}\tilde{\sigma}$ is a positive definite measure on $\mathscr{H}(t)$.) Since $\mathbf{e}$ and $\hat{\mathbf{g}}$ are continuous there will be positive constants Q_5 and Q_6 such that on $\overline{\mathscr{U}}_+$

$$Q_5\,\mathrm{d}\sigma \leqslant \mathrm{d}\tilde{\sigma} \leqslant Q_6\,\mathrm{d}\sigma, \quad (7.24)$$

where $\mathrm{d}\sigma$ is the area element on $\mathscr{H}(t)$ induced by $\mathbf{e}$. Thus by (7.22) and (7.23) there is some Q_7 such that

$$\|\mathbf{K}, \mathscr{H}(t)\cap\mathscr{U}_+\|_1{}^2 \leqslant Q_7\left\{\|\mathbf{K}, \mathscr{H}(0)\cap\mathscr{U}_+\|_1{}^2 + \int_0^t \|\mathbf{K}, \mathscr{H}(t')\cap\mathscr{U}_+\|_1{}^2\,\mathrm{d}t' + \int_0^t (S^{ab}{}_{|b}t_{|a}\,\mathrm{d}\sigma)\,\mathrm{d}t'\right\}. \quad (7.25)$$

By (7.20),

$$S^{ab}{}_{|b} = A^{ac}K^I{}_{J|c}F^P{}_Q\,e^{JQ}e_{IP} + (\text{terms quadratic in } K^I{}_J \text{ and } K^P{}_{Q|c} \text{ with coefficients involving } A^{cd},\ A^{cd}{}_{|e},\ \hat{R}^c{}_{def},\ B^{cPI}{}_{QJ} \text{ and } C^{PI}{}_{QJ}). \quad (7.26)$$

Since the coefficients are all bounded on $\mathscr{U}_+$, there is some Q_8 such that

$$S^{ab}{}_{|b}t_{|a} \leqslant Q_8\{|\mathbf{F}|^2 + |\mathbf{K}|^2 + |\mathbf{DK}|^2\}. \quad (7.27)$$

Thus there is some Q_9 such that, from (7.25) and (7.27),

$$\|\mathbf{K}, \mathscr{H}(t)\cap\mathscr{U}_+\|_1{}^2 \leqslant Q_9\left\{\|\mathbf{K}, \mathscr{H}(0)\cap\mathscr{U}_+\|_1{}^2 + \int_0^t \|\mathbf{K}, \mathscr{H}(t')\cap\mathscr{U}_+\|_1{}^2\,\mathrm{d}t' + \|\mathbf{F}, \mathscr{U}(t)\|_0{}^2\right\}.$$

This is of the form

$$\mathrm{d}x/\mathrm{d}t \leqslant Q_9\{x+y\}, \quad (7.28)$$

where

$$x(t) = \int_0^t \|\mathbf{K}, \mathscr{H}(t')\cap\mathscr{U}_+\|_1{}^2\,\mathrm{d}t'.$$

Therefore

$$x \leqslant e^{Q_9 t}\int_0^t e^{-Q_9 t'}y(t')\,\mathrm{d}t'. \quad (7.29)$$

Since y is a monotonically increasing function of t and since t is bounded on $\bar{\mathscr{U}}_+$, there is some Q_{10} such that

$$x \leqslant Q_{10} y.$$

Thus $\|\mathbf{K}, \mathscr{H}(t) \cap \mathscr{U}_+\|_1 \leqslant P_4\{\|\mathbf{K}, \mathscr{H}(0) \cap \mathscr{U}_+\|_1 + \|\mathbf{F}, \mathscr{U}(t)\|_0\}$, where

$$P_4 = (Q_9 + Q_{10})^{\frac{1}{2}}. \qquad \square$$

With this inequality one can immediately prove the uniqueness of solutions of second order hyperbolic equations which are linear, i.e. for which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{F}$ do not depend on $\mathbf{K}$. For suppose $K^{1I}{}_J$ and $K^{2I}{}_J$ were solutions of the equation $L(\mathbf{K}) = \mathbf{F}$ which had the same initial values and first derivatives on $\mathscr{H}(0) \cap \mathscr{U}$. Then one can apply the above result to the equation $L(\mathbf{K}^1 - \mathbf{K}^2) = 0$ and obtain

$$\|\mathbf{K}^1 - \mathbf{K}^2, \mathscr{H}(t) \cap \mathscr{U}_+\|_1 = 0.$$

Therefore $\mathbf{K}^1 = \mathbf{K}^2$ on $\bar{\mathscr{U}}_+$. One has thus

Proposition 7.4.5

Let $\mathbf{A}$ be a C^{1-} Lorentz metric on $\hat{\mathscr{M}}$ and let $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{F}$ be locally bounded. Let $\mathscr{H} \subset \hat{\mathscr{M}}$ be a three-surface which is spacelike and acausal with respect to $\mathbf{A}$. Then if $\mathscr{V}$ is a set in $D^+(\mathscr{H}, \mathbf{A})$, the solution on $\mathscr{V}$ of the linear equation (7.20) is uniquely determined by its values and the values of its first derivatives on $\mathscr{H} \cap J^-(\mathscr{V}, \mathbf{A})$.

By proposition 6.6.7, $D^+(\mathscr{H}, \mathbf{A})$ is of the form $\mathscr{H} \times R^1$. If $q \in \mathscr{V}$, then by proposition 6.6.6, $J^-(q) \cap J^+(\mathscr{H})$ is compact and so may be taken for $\bar{\mathscr{U}}_+$. $\square$

Thus a physical field obeying a linear equation of the form (7.20) will satisfy the causality postulate (a) of §3.2 provided the null cone of $\mathbf{A}$ coincides with or lies within the null cone of the space–time metric $\mathbf{g}$.

In order to prove the existence of solutions of the equations (7.20) we shall need inequalities for higher order derivatives of $\mathbf{K}$. We shall now take the background metric $\hat{\mathbf{g}}$ to be at least C^{5+a} where a is a non-negative integer and we shall take $\mathscr{U}$ to be such that $\mathscr{H}(0) \cap \bar{\mathscr{U}}$ has a smooth boundary and such that there is a diffeomorphism

$$\lambda\colon (\mathscr{H}(0) \cap \bar{\mathscr{U}}) \times [0, t_1] \to \bar{\mathscr{U}}_+$$

which has the property that for each $t \in [0, t_1]$,

$$\lambda\{(\mathscr{H}(0) \cap \bar{\mathscr{U}}), t\} = \mathscr{H}(t) \cap \bar{\mathscr{U}}_+.$$

We do this so that there shall be upper bounds $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$ to the constants P_1, P_2 and P_3 in lemmas 7.4.1–7.4.3 for the surface $\mathscr{H}(t) \cap \mathscr{U}_+$.

Lemma 7.4.6

If conditions (1) and (2) of lemma 7.4.4 hold and if

(4) there is some Q_3 such that

$$\|\mathbf{A}, \mathscr{U}_+\|_{4+a} < Q_3, \quad \|\mathbf{B}, \mathscr{U}_+\|_{3+a} < Q_3, \quad \|\mathbf{C}, \mathscr{U}_+\|_{3+a} < Q_3$$

(by lemma 7.4.1, this implies condition (3)), then there exist positive constants $P_{5,a}$ (depending on $\mathscr{U}$, $\mathbf{e}$, $\hat{\mathbf{g}}$, a, Q_1 and Q_3) such that

$$\|\mathbf{K}, \mathscr{H}(t) \cap \mathscr{U}_+\|_{4+a} \leqslant P_{5,a}\{\|\mathbf{K}, \mathscr{H}(0) \cap \mathscr{U}_+\|_{4+a} + \|\mathbf{F}, \mathscr{U}(t)\|_{3+a}\}. \quad (7.30)$$

From lemma 7.4.4 one has an inequality for $\|\mathbf{K}, \mathscr{H}(t) \cap \mathscr{U}_+\|_1$. To obtain an inequality for $\|\mathbf{K}, \mathscr{H}(t) \cap \mathscr{U}_+\|_2$ one forms the 'energy' tensor S^{ab} for the first derivatives $K^I{}_{J|c}$ and proceeds as before. The divergence $S^{ab}{}_{|b}$ can now be evaluated by differentiating equations (7.20):

$$\begin{aligned} S^{ab}{}_{|b} = A^{ad}K^I{}_{J|cd}F^P{}_{Q|e}\,e^{ec}e^{JQ}e_{IP} + (&\text{terms quadratic in } K^I{}_J, \\ &K^I{}_{J|c} \text{ and } K^I{}_{J|cd} \text{ with coefficients involving } A^{cd}, \\ &A^{cd}{}_{|e}, \hat{R}^c{}_{def}, \hat{R}^c{}_{def|g}, B^{cPI}{}_{QJ}, B^{cPI}{}_{QJ|d}, C^{PI}{}_{QJ} \\ &\text{and } C^{PI}{}_{QJ|d}). \end{aligned} \quad (7.31)$$

With the possible exceptions of $B^{cPI}{}_{QJ|d}$ and $C^{PI}{}_{QJ|d}$, these coefficients are all bounded on $\overline{\mathscr{U}}_+$ in the case $a = 0$. When integrated over the surface $\mathscr{H}(t') \cap \mathscr{U}_+$, the term in (7.31) involving $B^{cPI}{}_{QJ|d}$ is

$$-\int_{\mathscr{H}(t') \cap \mathscr{U}_+} A^{ab}K^I{}_{J|cb}B^{dPR}{}_{QS|e}K^S{}_{R|d}\,e^{ce}e^{QJ}e_{PI}\,\mathrm{d}\hat{\sigma}_a. \quad (7.32)$$

There is some Q_4 such that for all t', (7.32) is less than or equal to

$$Q_4\int_{\mathscr{H}(t') \cap \mathscr{U}_+} |\mathrm{D}\mathbf{B}|\,|\mathrm{D}\mathbf{K}|\,|\mathrm{D}^2\mathbf{K}|\,\mathrm{d}\sigma$$
$$\leqslant \tfrac{1}{2}Q_4\int_{\mathscr{H}(t') \cap \mathscr{U}_+} (|\mathrm{D}^2\mathbf{K}|^2 + |\mathrm{D}\mathbf{B}|^2\,|\mathrm{D}\mathbf{K}|^2)\,\mathrm{d}\sigma. \quad (7.33)$$

By lemma 7.4.2,

$$\int_{\mathscr{H}(t') \cap \mathscr{U}_+} |\mathrm{D}\mathbf{B}|^2\,|\mathrm{D}\mathbf{K}|^2\,\mathrm{d}\sigma \leqslant \tilde{P}_2{}^2\|\mathbf{B}, \mathscr{H}(t') \cap \mathscr{U}_+\|_2{}^2\,\|\mathbf{K}, \mathscr{H}(t') \cap \mathscr{U}_+\|_2{}^2,$$

where, by condition (4) and lemma 7.4.3, $\|\mathbf{B}, \mathscr{H}(t') \cap \mathscr{U}_+\|_2 < \tilde{P}_3Q_3$. The term involving $C^{PI}{}_{QJ|d}$ can be bounded similarly. Thus by lemma 4.3.1 there is some constant Q_5 such that

$$\int_{\mathscr{H}(t) \cap \mathscr{U}_+} (|\mathrm{D}^2\mathbf{K}| + |\mathrm{D}\mathbf{K}|^2)\,\mathrm{d}\sigma \leqslant Q_5\left\{\int_{\mathscr{H}(0) \cap \mathscr{U}_+} (|\mathrm{D}^2\mathbf{K}|^2 + |\mathrm{D}\mathbf{K}|^2)\,\mathrm{d}\sigma\right.$$
$$\left. + \int_0^t \|\mathbf{K}, \mathscr{H}(t') \cap \mathscr{U}_+\|_2{}^2\,\mathrm{d}t' + \int_{\mathscr{U}(t)} |\mathrm{D}\mathbf{F}|^2\,\mathrm{d}\sigma\right\}. \quad (7.34)$$

By lemma 7.4.4,

$$\int_{\mathscr{H}(t)\cap\mathscr{U}_+}|\mathbf{K}|^2\,\mathrm{d}\sigma \leqslant \|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_1{}^2$$
$$\leqslant 2P_4{}^2\{\|\mathbf{K},\mathscr{H}(0)\cap\mathscr{U}\|_1{}^2+\|\mathbf{F},\mathscr{U}(t)\|_0{}^2\}. \quad (7.35)$$

Adding this to (7.34), one obtains

$$\|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_2{}^2 \leqslant Q_6\Big\{\|\mathbf{K},\mathscr{H}(0)\cap\mathscr{U}\|_2{}^2$$
$$+\int_0^t\|\mathbf{K},\mathscr{H}(t')\cap\mathscr{U}_+\|_2{}^2\,\mathrm{d}t'+\|\mathbf{F},\mathscr{U}(t)\|_1{}^2\Big\}, \quad (7.36)$$

where $Q_6 = Q_5+2P_4$. By a similar argument to that in lemma 7.4.4, there is some constant Q_7 such that

$$\|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_2 \leqslant Q_7\{\|\mathbf{K},\mathscr{H}(0)\cap\mathscr{U}\|_2+\|\mathbf{F},\mathscr{U}(t)\|_1\}. \quad (7.37)$$

From lemma 7.4.1 it now follows that on $\mathscr{U}_+$,

$$|\mathbf{K}| \leqslant \tilde{P}_1Q_7\{\|\mathbf{K},\mathscr{H}(0)\cap\mathscr{U}\|_2+\|\mathbf{F},\mathscr{U}(t)\|_0\}. \quad (7.38)$$

Using this one may proceed in a similar way to establish an inequality for $\|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_3$. The divergence of the 'energy' tensor now gives a term of the form

$$Q_8\int_{\mathscr{H}(t')\cap\mathscr{U}_+}(|\mathrm{D}^3\mathbf{K}|^2+|\mathrm{D}^2\mathbf{B}|^2\,|\mathrm{D}\mathbf{K}|^2)\,\mathrm{d}\sigma. \quad (7.39)$$

By lemma 7.4.2 the second term above is bounded by

$$Q_8\tilde{P}_2{}^2\|\mathbf{B},\mathscr{H}(t')\cap\mathscr{U}_+\|_3{}^2\,\|\mathbf{K},\mathscr{H}(t')\cap\mathscr{U}_+\|_2{}^2,$$

where by condition (4), $\|\mathbf{B},\mathscr{H}(t)\cap\mathscr{U}_+\|_3$ is defined for almost all values of t' and is square integrable with respect to t'. Thus one can obtain an inequality for $\|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_3$ in the same manner as for $\|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_2$. The procedure for higher order derivatives is similar. □

Corollary

There exist constants $P_{6,a}$ and $P_{7,a}$ such that

$$\|\mathbf{K},\mathscr{H}(t)\cap\mathscr{U}_+\|_{4+a} \leqslant P_{6,a}\{\|\mathbf{K},\mathscr{H}(0)\cap\tilde{\mathscr{U}}\|_{4+a}$$
$$+\|K^I{}_{J|a}u^a,\mathscr{H}(0)\cap\tilde{\mathscr{U}}\|_{3+a}+\|\mathbf{F},\tilde{\mathscr{U}}_+\|_{3+a}\},$$

and $$\|\mathbf{K},\mathscr{U}_+\|_{4+a} \leqslant P_{7,a}\{\text{ditto}\},$$

where u^a is some C^{3+a} vector field on $\mathscr{H}(0)$ which is nowhere tangent to $\mathscr{H}(0)$.

By (7.20), the second and higher derivatives of $\mathbf{K}$ out of the surface $\mathscr{H}(0)$ may be expressed in terms of $\mathbf{F}$ and its derivatives out of $\mathscr{H}(0)$, $K^I{}_{J|a}u^a$ and derivatives of $\mathbf{K}$ in the surface $\mathscr{H}(0)$. By lemma 7.4.3,

$$\left.\begin{aligned} &\|\mathbf{A}, \mathscr{H}(0) \cap \mathscr{U}\|_{3+a} < \tilde{P}_3 Q_3, \\ &\|\mathbf{B}, \mathscr{H}(0) \cap \mathscr{U}\|_{2+a} < \tilde{P}_3 Q_3, \\ &\|\mathbf{C}, \mathscr{H}(0) \cap \mathscr{U}\|_{2+a} < \tilde{P}_3 Q_3, \\ &\|\mathbf{F}, \mathscr{H}(0) \cap \mathscr{U}\|_{2+a} < \tilde{P}_3 \|\mathbf{F}, \mathscr{U}_+\|_{3+a}. \end{aligned}\right\} \tag{7.40}$$

Thus there will be some constant Q_4 such that

$$\begin{aligned} \|\mathbf{K}, \mathscr{H}(0) \cap \mathscr{U}\|_{4+a} \leqslant Q_4 \{ &\|\mathbf{K}, \mathscr{H}(0) \cap \tilde{\mathscr{U}}\|_{4+a} \\ &+ \|K^I{}_{J|a}u^a, \mathscr{H}(0) \cap \tilde{\mathscr{U}}\|_{3+a} + \|\mathbf{F}, \tilde{\mathscr{U}}_+\|_{3+a}\}. \end{aligned} \tag{7.41}$$

The second result follows immediately, since t is bounded on $\mathscr{U}_+$. □

We can now proceed to prove the existence of solutions of linear equations of the form (7.20). We first suppose that the components of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{F}$, $\mathbf{u}$ and $\hat{\mathbf{g}}$ are analytic functions of the local coordinates x^1, x^2, x^3 and x^4 ($x^4 = t$) on a coordinate neighbourhood $\mathscr{V}$ and take the initial data $K^I{}_J = {}_0K^I{}_J$ and $K^I{}_{J|a}u^a = {}_1K^I{}_J$ to be analytic functions of the coordinates x^1, x^2 and x^3 on $\mathscr{H}(0) \cap \mathscr{V}$. Then from (7.20) one can calculate the partial derivatives $\partial^2(K^I{}_J)/\partial t^2$, $\partial^3(K^I{}_J)/\partial t^2\,\partial x^i$, $\partial^3(K^I{}_J)/\partial t^3$, etc. of the components of $\mathbf{K}$ out of the surface $\mathscr{H}(0)$ in terms of derivatives of ${}_0\mathbf{K}$ and ${}_1\mathbf{K}$ in $\mathscr{H}(0)$. One can then express $K^I{}_J$ as a formal power series in x^1, x^2, x^3 and t about the origin of coordinates p. By the Cauchy–Kowaleski theorem (Courant and Hilbert (1962), p. 39) this series will converge in some ball $\mathscr{V}(r)$ of coordinate radius r to give a solution of (7.20) with the given initial conditions. One now selects an analytic atlas from the C^∞ atlas of $\mathscr{M}$, covers $\mathscr{H}(0) \cap \bar{\mathscr{U}}$ with coordinate neighbourhoods of the form $\mathscr{V}(r)$ from this atlas, and in each coordinate neighbourhood constructs a solution as above. One thus obtains a solution on a region $\mathscr{U}(t_2)$ for some $t_2 > 0$. One then repeats the process using $\mathscr{H}(t_2)$. By the Cauchy–Kowaleski theorem, the ratio of successive intervals of t for which the power series converges is independent of the initial data and so the solution can be extended to the whole of $\mathscr{U}_+$ in a finite number of steps. This proves the existence of solutions of linear equations of the form (7.20) when the coefficients, the source term and the initial data are all analytic. We shall now remove the requirement of analyticity.

Proposition 7.4.7

If conditions (1), (2) and (4) hold and if

(5) $\mathbf{F} \in W^{3+a}(\mathscr{U}_+)$,

(6) ${}_0\mathbf{K} \in W^{4+a}(\mathscr{H}(0) \cap \overline{\mathscr{U}})$, ${}_1\mathbf{K} \in W^{3+a}(\mathscr{H}(0) \cap \overline{\mathscr{U}})$,

then there exists a unique solution $\mathbf{K} \in W^{4+a}(\mathscr{U}_+)$ of the linear equation (7.20) such that on $\mathscr{H}(0)$, $K^I{}_J = {}_0K^I{}_J$ and $K^I{}_{J|a}u^a = {}_1K^I{}_J$.

We prove this result by approximating the coefficients and initial data by analytic fields and showing that the analytic solutions obtained converge to a field which is a solution of the given equations with the given initial conditions. Let $\mathbf{A}_n$ $(n = 1, 2, 3, \ldots)$ be a sequence of analytic fields on $\overline{\mathscr{U}}_+$ which converge strongly to $\mathbf{A}$ in $W^{4+a}(\mathscr{U}_+)$. ($\mathbf{A}_n$ is said to converge strongly to $\mathbf{A}$ in W^m if $\|\mathbf{A}_n - \mathbf{A}\|_m$ converges to zero.) Let $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{F}_n$ be analytic fields on $\overline{\mathscr{U}}_+$ which converge strongly to $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{F}$ respectively in $W^{3+a}(\mathscr{U}_+)$, and let ${}_0\mathbf{K}_n$ and ${}_1\mathbf{K}_n$ be analytic fields on $\mathscr{H}(0) \cap \overline{\mathscr{U}}$ which converge strongly to ${}_0\mathbf{K}$ and ${}_1\mathbf{K}$ in $W^{4+a}(\mathscr{H}(0) \cap \mathscr{U})$ and $W^{3+a}(\mathscr{H}(0) \cap \mathscr{U})$ respectively. For each value of n there will be an analytic solution $\mathbf{K}_n$ to (7.20) with the initial values $K_n{}^I{}_J = {}_0K_n{}^I{}_J$, $K_n{}^I{}_{J|a}u^a = {}_1K_n{}^I{}_J$. By the corollary to lemma 7.4.6, $\|\mathbf{K}_n, \mathscr{U}_+\|_{4+a}$ will be bounded as $n \to \infty$. Therefore by a theorem of Riesz (1955) there will be a field $\mathbf{K} \in W^{4+a}(\mathscr{U}_+)$ and a subsequence $\mathbf{K}_{n'}$ of the $\mathbf{K}_n$ such that for each b, $0 \leqslant b \leqslant 4 + a$, $D^b\mathbf{K}_{n'}$ converges weakly to $D^b\mathbf{K}$. (A sequence of fields $I_n{}^I{}_J$ on $\mathscr{N}$ is said to converge weakly to $I^I{}_J$ if for each C^∞ field $J^I{}_J$,

$$\int_{\mathscr{N}} I_n{}^I{}_J J^J{}_I \, d\sigma \to \int_{\mathscr{N}} I^I{}_J J^J{}_I \, d\sigma.\Big)$$

Since $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$ converge strongly to $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $W^3(\mathscr{U}_+)$, $\sup|\mathbf{A} - \mathbf{A}_n|$, $\sup|\mathbf{B} - \mathbf{B}_n|$ and $\sup|\mathbf{C} - \mathbf{C}_n|$ will converge to zero. Thus $L_{n'}(\mathbf{K}_{n'})$ will converge weakly to $L(\mathbf{K})$. But $L_{n'}(\mathbf{K}_{n'})$ is equal to $\mathbf{F}_{n'}$ which converges strongly to $\mathbf{F}$. Therefore $L(\mathbf{K}) = \mathbf{F}$. On $\mathscr{H}(0) \cap \overline{\mathscr{U}}$ $K_{n'}{}^I{}_J$ and $K_{n'}{}^I{}_{J|a}u^a$ will converge weakly to $K^I{}_J$ and $K^I{}_{J|a}u^a$ which must therefore be equal to ${}_0K^I{}_J$ and ${}_1K^I{}_J$ respectively. Thus $\mathbf{K}$ is a solution of the given equation with the given initial conditions. By proposition 7.4.5 it is unique. Since each $\mathbf{K}_n$ satisfies the inequality in lemma 7.4.6, $\mathbf{K}$ will satisfy it also. □

7.5 The existence and uniqueness of developments for the empty space Einstein equations

We shall now apply the results of the previous section to the Cauchy problem in General Relativity. We shall first deal with the Einstein equations for empty space ($T^{ab} = 0$), and shall discuss the effect of matter in § 7.7.

The reduced Einstein equations

$$E^{ab}{}_{cd}(\phi^{cd}) = 8\pi T^{ab} - (\hat{R}^{ab} - \tfrac{1}{2}\hat{R}\hat{g}^{ab}) \tag{7.42}$$

are *quasi-linear* second order hyperbolic equations. That is, they have the form (7.20) where the coefficients **A**, **B** and **C** are functions of **K** and **DK** (actually, in this case $A^{ab} = g^{ab}$ is a function of ϕ^{ab} and not of $\phi^{ab}{}_{|c}$). To prove the existence of solutions of these equations we proceed as follows. We take some suitable trial field ϕ'^{ab} and use this to determine the values of the coefficients **A**, **B** and **C** in the operator E. Using these values we then solve (7.42) as a *linear* equation with the prescribed initial data and obtain a new field ϕ''^{ab}. We thus have a map α which takes $\boldsymbol{\phi}'$ into $\boldsymbol{\phi}''$, and we show that under suitable conditions this map has a fixed point (i.e. there is some ϕ such that $\alpha(\boldsymbol{\phi}) = \boldsymbol{\phi}$). This fixed point will be the desired solution of the quasi-linear equation.

We shall take the background metric $\hat{\mathbf{g}}$ to be a solution of the empty space Einstein equations and choose the surfaces $\mathscr{H}(t) \cap \overline{\mathscr{U}}_+$ and $\partial\mathscr{U} \cap \overline{\mathscr{U}}_+$ to be spacelike in $\hat{\mathbf{g}}$. Then by lemma 7.4.1 there will be some positive constants $\tilde{Q}_a$ such that if for some value of $a \geqslant 0$

$$\|\boldsymbol{\phi}', \mathscr{U}_+\|_{4+a} < \tilde{Q}_a, \tag{7.43}$$

then the coefficients $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ determined by $\boldsymbol{\phi}'$ satisfy conditions (1), (2) and (4) of lemma 7.4.6 for given values of Q_1 and Q_3. From (7.41) one then has

$$\|\boldsymbol{\phi}'', \mathscr{U}_+\|_{4+a} \leqslant P_{7,a}\{\|{}_0\boldsymbol{\phi}, \mathscr{H}(0) \cap \tilde{\overline{\mathscr{U}}}\|_{4+a} + \|{}_1\boldsymbol{\phi}, \mathscr{H}(0) \cap \tilde{\overline{\mathscr{U}}}\|_{3+a}\}.$$

Thus the map α: $W^{4+a}(\mathscr{U}_+) \to W^{4+a}(\mathscr{U}_+)$ will take the closed ball $W(r)$ of radius r ($r < \tilde{Q}_a$) in $W^{4+a}(\mathscr{U}_+)$ into itself provided that

$$\|{}_0\boldsymbol{\phi}, \mathscr{H}(0) \cap \tilde{\overline{\mathscr{U}}}\|_{4+a} \leqslant \tfrac{1}{2} r P_{7,a}{}^{-1}$$

and

$$\|{}_1\boldsymbol{\phi}, \mathscr{H}(0) \cap \tilde{\overline{\mathscr{U}}}\|_{3+a} \leqslant \tfrac{1}{2} r P_{7,a}{}^{-1}. \tag{7.44}$$

We shall show that α has a fixed point if (7.44) holds and if r is sufficiently small.

Suppose $\boldsymbol{\phi}_1'$ and $\boldsymbol{\phi}_2'$ are in $W(r)$. The fields $\boldsymbol{\phi}_1'' = \alpha(\boldsymbol{\phi}_1')$ and $\boldsymbol{\phi}_2'' = \alpha(\boldsymbol{\phi}_2')$ satisfy $E_1'(\boldsymbol{\phi}_1'') = 0$, $E_2'(\boldsymbol{\phi}_2'') = 0$ where E_1' is the Einstein operator with coefficients $\mathbf{A}_1'$, $\mathbf{B}_1'$ and $\mathbf{C}_1'$ determined by $\boldsymbol{\phi}_1'$. Thus

$$E_1'(\boldsymbol{\phi}_1'' - \boldsymbol{\phi}_2'') = -(E_1' - E_2')(\boldsymbol{\phi}_2''). \tag{7.45}$$

Since the coefficients $\mathbf{A}_1'$, $\mathbf{B}_1'$ and $\mathbf{C}_1'$ depend differentiably on $\boldsymbol{\phi}_1'$ and $\mathrm{D}\boldsymbol{\phi}_1'$ for $\boldsymbol{\phi}_1'$ in $W(r)$, there will be some constant Q_4 such that on $\bar{\mathscr{U}}_+$

$$\left.\begin{aligned} |\mathbf{A}'_1 - \mathbf{A}'_2| &\leqslant Q_4|\boldsymbol{\phi}'_1 - \boldsymbol{\phi}'_2|, \\ |\mathbf{B}'_1 - \mathbf{B}'_2| &\leqslant Q_4(|\boldsymbol{\phi}'_1 - \boldsymbol{\phi}'_2| + |\mathrm{D}\boldsymbol{\phi}'_1 - \mathrm{D}\boldsymbol{\phi}'_2|), \\ |\mathbf{C}'_1 - \mathbf{C}'_2| &\leqslant Q_4(|\boldsymbol{\phi}'_1 - \boldsymbol{\phi}'_2| + |\mathrm{D}\boldsymbol{\phi}'_1 - \mathrm{D}\boldsymbol{\phi}'_2|). \end{aligned}\right\} \tag{7.46}$$

Therefore by lemmas 7.4.1 and 7.4.6,

$$|(E'_1 - E'_2)(\boldsymbol{\phi}''_2)| \leqslant 3rQ_4\tilde{P}_1 P_{7,a}^{-1} P_{6,a}(|\boldsymbol{\phi}'_1 - \boldsymbol{\phi}'_2| + |\mathrm{D}\boldsymbol{\phi}'_1 - \mathrm{D}\boldsymbol{\phi}'_2|).$$

We now apply lemma 7.4.4 to (7.45) to obtain the result

$$\|\boldsymbol{\phi}''_1 - \boldsymbol{\phi}''_2, \mathscr{U}_+\|_1 \leqslant rQ_5\|\boldsymbol{\phi}'_1 - \boldsymbol{\phi}'_2, \mathscr{U}_+\|_1, \tag{7.47}$$

where Q_5 is some constant independent of r. Thus for sufficiently small r, the map α will be contracting in the $\| \; \|_1$ norm (i.e. $\|\alpha(\boldsymbol{\phi}_1) - \alpha(\boldsymbol{\phi}_2)\|_1 < \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_1$) and the sequence $\alpha^n(\boldsymbol{\phi}'_1)$ will converge strongly in $W^1(\mathscr{U}_+)$ to some field $\boldsymbol{\phi}$. But by the theorem of Riesz some subsequence of the $\alpha^n(\boldsymbol{\phi}'_1)$ will converge weakly to some field $\tilde{\boldsymbol{\phi}} \in W(r)$. Thus $\boldsymbol{\phi}$ must equal $\tilde{\boldsymbol{\phi}}$ and so be in $W(r)$. Therefore $\alpha(\boldsymbol{\phi})$ will be defined. Now

$$\|\alpha(\boldsymbol{\phi}) - \alpha^{n+1}(\boldsymbol{\phi}'_1), \mathscr{U}_+\|_1 \leqslant rQ_5\|\boldsymbol{\phi} - \alpha^n(\boldsymbol{\phi}'_1), \mathscr{U}_+\|_1.$$

As $n \to \infty$, the right-hand side tends to zero. This implies that $\|\alpha(\boldsymbol{\phi}) - \boldsymbol{\phi}, \mathscr{U}_+\|_1 = 0$ and so that $\alpha(\boldsymbol{\phi}) = \boldsymbol{\phi}$. Since the map α is contracting the fixed point is unique in $W(r)$. We have therefore proved:

Proposition 7.5.1

If $\hat{\mathbf{g}}$ is a solution of the empty space Einstein equations, the reduced empty space Einstein equations have a solution $\boldsymbol{\phi} \in W^{4+a}(\mathscr{U}_+)$ if $\|_0\boldsymbol{\phi}, \mathscr{H}(0) \cap \bar{\mathscr{U}}\|_{4+a}$ and $\|_1\boldsymbol{\phi}, \mathscr{H}(0) \cap \bar{\mathscr{U}}\|_{3+a}$ are sufficiently small. $\|\boldsymbol{\phi}, \mathscr{H}(0) \cap \bar{\mathscr{U}}_+\|_{4+a}$ will be bounded and so $\boldsymbol{\phi}$ will be at least $C^{(2+a)-}$. □

This solution will be locally unique even among solutions which are not in $W^4(\mathscr{U}_+)$.

Proposition 7.5.2

Let $\tilde{\boldsymbol{\phi}}$ be a C^{1-} solution of the reduced empty space Einstein equations with the same initial data on an open set $\mathscr{V} \subset \mathscr{H}(0) \cap \mathscr{U}$. Then $\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ on a neighbourhood of $\mathscr{V}$ in $\mathscr{U}_+$.

Since $\tilde{\boldsymbol{\phi}}$ is continuous one can find a neighbourhood $\mathscr{U}'$ of $\mathscr{V}$ in $\mathscr{U}$ such that the conditions of lemma 7.4.4 hold for **A**, **B** and **C**. As before one has

$$\tilde{E}(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}) = -(\tilde{E}-E)(\boldsymbol{\phi}). \tag{7.48}$$

Similarly there will be some Q_6 such that

$$\|(\tilde{E}-E)(\boldsymbol{\phi}), \mathscr{H}(t) \cap \mathscr{U}'_+\|_0 \leqslant Q_6 \|\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \mathscr{H}(t) \cap \mathscr{U}'_+\|_1.$$

Applying lemma 7.4.4 to (7.48) one obtains an inequality of the form

$$\mathrm{d}x/\mathrm{d}t \leqslant Q_7 x,$$

where
$$x = \int_0^t \|\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \mathscr{H}(t') \cap \mathscr{U}'_+\|_1 \,\mathrm{d}t'.$$

Therefore $\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ on $\overline{\mathscr{U}}'_+$. □

Proposition 7.5.1 shows that if one makes a sufficiently small perturbation in the initial data of an empty space solution of the Einstein equations one obtains a solution in a region $\mathscr{U}_+$. What one wants however is to prove the existence of developments for any initial data h^{ab} and χ^{ab} which satisfy the constraint equations on a three-manifold $\mathscr{S}$. To do this we proceed as follows. We take $\mathscr{M}$ to be R^4, **e** to be the Euclidean metric and $\hat{\mathbf{g}}$ to be the flat, Minkowski metric (this is a solution of the empty space Einstein equations). In the usual Minkowski coordinates x^1, x^2, x^3 and x^4 ($x^4 = t$) we take $\mathscr{U}$ to be such that $\partial\mathscr{U} \cap \overline{\mathscr{U}}_+$ is spacelike and $\mathscr{H}(0) \cap \overline{\mathscr{U}}$ consists of the points for which $(x^1)^2+(x^2)^2+(x^3)^2 \leqslant 1$, $x^4 = 0$. The idea now is that any metric appears nearly flat if looked at on a fine enough scale. Therefore if one maps a sufficiently small region of $\mathscr{S}$ onto $\mathscr{H}(0) \cap \overline{\mathscr{U}}$, one can use proposition 7.5.1 and obtain a solution on $\mathscr{U}_+$. We then repeat this for other portions of $\mathscr{S}$ and join up the resulting solutions to form a manifold $\mathscr{M}$ with metric $\mathbf{g}$ which is a development of $(\mathscr{S}, \boldsymbol{\omega})$.

Let $\mathscr{V}_1$ be a coordinate neighbourhood in $\mathscr{S}$ with coordinates y^1, y^2 and y^3 such that at p, the origin of the coordinates, the coordinate components of h^{ab} equal δ^{ab}. Let $\mathscr{V}_1(f_1)$ be the open ball of coordinate radius f_1 about p. Define an imbedding θ_1: $\mathscr{V}_1(f_1) \to \mathscr{U}$ by $x^i = f_1{}^{-1}y^i$ ($i = 1, 2, 3$), $x^4 = 0$. By the usual law of transformation of a basis, the components of $\theta_* h^{ab}$ and $\theta_* \chi^{ab}$ with respect to the coordinates $\{x\}$ are f_1^{-2} times the components of h^{ab} and χ^{ab} with respect to the coordinates $\{y\}$. We define new fields h'^{ab} and χ'^{ab} on $\mathscr{V}_1$ by $h'^{ab} = f_1^2 h^{ab}$ and $\chi'^{ab} = f_1^3 \chi^{ab}$. Then since $\mathbf{h}$ is continuous (in fact C^{2+a}) on $\mathscr{S}$ one can make $g'^{ab} - \hat{g}^{ab}$ and $g'^{ab}{}_{|c} u^c$ arbitrarily small on $\mathscr{H}(0) \cap \mathscr{U}$ by taking f_1 sufficiently small, where g'^{ab} and $g'^{ab}{}_{|c} u^c$ are defined from h'^{ab} and χ'^{ab} in the manner of §7.3. The derivatives of g'^{ab} and $g'^{ab}{}_{|c} u^c$ in the surface $\mathscr{H}(0)$ will also become smaller as f_1 is made smaller. Thus $\|{}_0\boldsymbol{\phi}', \mathscr{H}(0) \cap \tilde{\overline{\mathscr{U}}}\|_{4+a}$ and $\|{}_1\boldsymbol{\phi}', \mathscr{H}(0) \cap \tilde{\overline{\mathscr{U}}}\|_{3+a}$ can be made small enough that proposition 7.5.1 can be applied and a solution for $\boldsymbol{\phi}'$ obtained on $\mathscr{U}_+$. Then $g_1^{ab} = f_1^{-2} g'^{ab}$ will be a solution of the reduced Einstein equations with the initial data determined by h^{ab} and χ^{ab}. Similarly one can obtain a solution on $\mathscr{U}_-$, the part of $\mathscr{U}$ on which $t \leqslant 0$.

One can now cover $\mathscr{S}$ by coordinate neighbourhoods $\mathscr{V}_\alpha(f_\alpha)$ of the form $\mathscr{V}_1(f_1)$, map them by imbeddings θ_α to neighbourhoods $\mathscr{U}_\alpha$ of the form $\mathscr{U}$ and obtain solutions $g_\alpha{}^{ab}$ on $\mathscr{U}_\alpha$. The problem now is to identify suitable points in the overlaps to make the collection of the $\mathscr{U}_\alpha$ into a manifold with a metric $\mathbf{g}$. To do this we make use of the harmonic gauge condition

$$\phi^{bc}{}_{|c} = g^{bc}{}_{|c} - \tfrac{1}{2} g^{bc} g_{de} g^{dc}{}_{|c} = 0. \tag{7.49}$$

By the definition (7.3) of $\delta\Gamma^a{}_{bc}$, this is equivalent to $g^{de}\,\delta\Gamma^b{}_{de} = 0$. Therefore for any function z,

$$z_{;ab} g^{ab} = z_{|ab} g^{ab} - \delta\Gamma^c{}_{ab} z_{|c} g^{ab} = z_{|ab} g^{ab}. \tag{7.50}$$

If the background metric is the Minkowski metric and z is one of the Minkowski coordinates x^1, x^2, x^3 and x^4, the right-hand side of (7.50) will vanish. Suppose now one has an arbitrary W^{4+a} Lorentz metric $\mathbf{g}$ on a manifold $\mathscr{M}$. In some neighbourhood $\mathscr{Y} \subset \mathscr{M}$ one can find four solutions z^1, z^2, z^3 and z^4 of the linear equation

$$z_{;ab} g^{ab} = 0 \tag{7.51}$$

which are such that their gradients are linearly independent at each point of $\mathscr{Y}$. We may then define a diffeomorphism $\mu: \mathscr{Y} \to \hat{\mathscr{M}}$ by $x^a = z^a$ $(a = 1, 2, 3, 4)$. This diffeomorphism will have the property that the metric $\mu_* g^{ab}$ on $\hat{\mathscr{M}}$ will satisfy the harmonic gauge condition with respect to the Minkowski metric $\hat{\mathbf{g}}$ on $\hat{\mathscr{M}}$. Thus if the metric $\mathbf{g}$ is a solution of the Einstein equations on $\mathscr{M}$, the metric $\mu_* \mathbf{g}$ will be a solution of the reduced Einstein equations on $\hat{\mathscr{M}}$ with the background metric $\hat{\mathbf{g}}$.

The procedure to identify points in the overlap between two neighbourhoods $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$ is therefore to solve (7.51) on $\mathscr{U}_\alpha$ for the coordinates $x_\beta{}^1$, $x_\beta{}^2$, $x_\beta{}^3$ and $x_\beta{}^4$ using the initial values for $x_\beta{}^a$ and $x_\beta{}^a{}_{|b}u^b$ determined by the overlap of the coordinate neighbourhoods $\mathscr{V}_\alpha$ and $\mathscr{V}_\beta$ on $\mathscr{S}$. In fact $x_\beta{}^i{}_{|a}u^a = 0$ $(i = 1, 2, 3)$ and $x_\beta{}^4{}_{|a}u^a = 1$ where $u^a = \partial/\partial x_\alpha{}^a$ is the unit vector in $\mathscr{U}_\alpha$ orthogonal to $\mathscr{H}(0)$ in the metric $\hat{\mathbf{g}}$. Thus $x_\beta{}^4 = x_\alpha{}^4$ though $x_\beta{}^i$ will not in general be equal to $x_\alpha{}^i$. By proposition 7.4.7. the coordinates $x_\beta{}^a$ will be $C^{(2+a)-}$ functions on $\mathscr{U}_\alpha$. (In proposition 7.4.7 the background metric with respect to which the covariant derivatives are taken has to be $C^{(5+a)-}$. Thus it cannot be applied directly to (7.51), since the covariant derivatives are taken with respect to $\mathbf{g}$, which is only W^{4+a}. However one can introduce a C^{5+a} background metric $\tilde{\mathbf{g}}$ and express (7.51) in the form

$$z_{\|ab}g^{ab} + z_{\|a}B^a = 0,$$

where $\|$ indicates covariant differentiation with respect to $\tilde{\mathbf{g}}$. Proposition 7.4.7 can then be applied to this equation.)

Since the gradients of $x_\beta{}^a$ are linearly independent on $\mathscr{H}(0) \cap \mathscr{U}_\alpha$, they will be linearly independent on some neighbourhood $\mathscr{U}''_\alpha$ of $\mathscr{H}(0)$ in $\mathscr{U}_\alpha$. The metric $\mu_* g^{ab}_\alpha$ will be at least C^{1-} on $\mu(\mathscr{U}''_\alpha)$ in $\mathscr{U}_\beta$. Since it will obey the reduced empty space Einstein equations on $\mathscr{U}_\beta$ in the background metric $\hat{\mathbf{g}}$ and since it has the same initial data on $\theta_\beta(\mathscr{V}_\alpha \cap \mathscr{V}_\beta)$, it must coincide with $\mathbf{g}_\beta$ on some neighbourhood $\mathscr{U}_\beta{}'$ of $\theta_\beta(\mathscr{V}_\alpha \cap \mathscr{V}_\beta)$ in $\mathscr{U}_\beta$. This shows that one may join together $\mathscr{U}''_\alpha$ and $\mathscr{U}'_\beta$ to obtain a development of the region $\mathscr{V}_\alpha \cup \mathscr{V}_\beta$ of $\mathscr{S}$. Taking the covering $\{\mathscr{V}_\alpha\}$ of $\mathscr{S}$ to be locally finite, one may proceed in a similar fashion to join together the subsets of the other neighbourhoods $\{\mathscr{U}_\alpha\}$ to obtain a development of $\mathscr{S}$, i.e. a manifold $\mathscr{M}$ with a metric $\mathbf{g}$ and an imbedding $\theta: \mathscr{S} \to \mathscr{M}$ such that $\mathbf{g}$ satisfies the empty space Einstein equations and agrees with the prescribed initial data $\boldsymbol{\omega}$ on $\theta(\mathscr{S})$, which is a Cauchy surface for $\mathscr{M}$. If $(\mathscr{M}', \mathbf{g}')$ is another development of $(\mathscr{S}, \boldsymbol{\omega})$ one can by a similar procedure establish a diffeomorphism μ between some neighbourhood of $\theta'(\mathscr{S}')$ in $\mathscr{M}'$ and some neighbourhood of $\theta(\mathscr{S})$ in $\mathscr{M}$ such that $\mu_* g'^{ab} = g^{ab}$. We have therefore proved:

The local Cauchy development theorem

If $h^{ab} \in W^{4+a}(\mathscr{S})$ and $\chi^{ab} \in W^{3+a}(\mathscr{S})$ satisfy the empty space constraint equations there exist developments $(\mathscr{M}, \mathbf{g})$ for the empty space Einstein equations such that $\mathbf{g} \in W^{4+a}(\mathscr{M})$ and $\mathbf{g} \in W^{4+a}(\mathscr{H})$ for any smooth spacelike surface $\mathscr{H}$. These developments are locally unique in that if $(\mathcal{M}', \mathbf{g}')$ is another W^{4+a} development of $(\mathcal{S}, \boldsymbol{\omega})$ then $(\mathcal{M}, \mathbf{g})$ and $(\mathcal{M}', \mathbf{g}')$ are both extensions of some common development of $(\mathcal{S}, \boldsymbol{\omega})$.

That $\mathbf{g} \in W^{4+a}(\mathcal{H})$ follows from lemma 7.4.6 since the surfaces of constant t can be chosen arbitrarily. □

7.6 The maximal development and stability

We have shown that if the initial data satisfied the empty space constraint equations one can find a development, i.e. one can construct a solution some distance into the future and past of the initial surface. In general, this development can be extended further into the future and past to give a larger development of $(\mathcal{S}, \boldsymbol{\omega})$. However we shall show by an argument similar to that of Choquet-Bruhat and Geroch (1969) that there is a unique (up to a diffeomorphism) development $(\mathcal{M}, \mathbf{g})$ of $(\mathcal{S}, \boldsymbol{\omega})$ which is an extension of any other development of $(\mathcal{S}, \boldsymbol{\omega})$.

Recall that $(\mathcal{M}_1, \mathbf{g}_1)$ is an extension of $(\mathcal{M}_2, \mathbf{g}_2)$ if there is an imbedding $\mu: \mathcal{M}_2 \to \mathcal{M}_1$ such that $\mu_* \mathbf{g}_2 = \mathbf{g}_1$, and such that $\theta_1^{-1}\mu\theta_2$ is the identity map on $\mathcal{S}$. Given a point $q \in \mathcal{S}$, and a distance s one can uniquely determine points $p_1 \in \mathcal{M}_1$ and $p_2 \in \mathcal{M}_2$ by going a distance s along the geodesics orthogonal to $\theta_1(\mathcal{S})$ and $\theta_2(\mathcal{S})$ through $\theta_1(q)$ and $\theta_2(q)$ respectively. Since $\mu(p_2)$ must equal p_1, the imbedding μ must be unique. One can therefore partially order the set of all developments of $(\mathcal{S}, \boldsymbol{\omega})$, writing $(\mathcal{M}_2, \mathbf{g}_2) \leqslant (\mathcal{M}_1, \mathbf{g}_1)$ if $(\mathcal{M}_1, \mathbf{g}_1)$ is an extension of $(\mathcal{M}_2, \mathbf{g}_2)$. If now $\{(\mathcal{M}_\alpha, \mathbf{g}_\alpha)\}$ is a totally ordered set (a set $\mathcal{A}$ is said to be totally ordered if for every pair a, b of distinct elements of $\mathcal{A}$, either $a \leqslant b$ or $b \leqslant a$) of developments of $(\mathcal{S}, \boldsymbol{\omega})$, one can form the manifold $\mathcal{M}'$ as the union of all the $\mathcal{M}_\alpha$ where for $(\mathcal{M}_\alpha, \mathbf{g}_\alpha) \leqslant (\mathcal{M}_\beta, \mathbf{g}_\beta)$ each $p_\alpha \in \mathcal{M}_\alpha$ is identified with $\mu_{\alpha\beta}(p_\alpha) \in \mathcal{M}_\beta$, where $\mu_{\alpha\beta}: \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the imbedding. The manifold $\mathcal{M}'$ will have an induced metric $\mathbf{g}'$ equal to $\mu_{\alpha *}\mathbf{g}_\alpha$ on each $\mu_\alpha(\mathcal{M}_\alpha)$ where $\mu_\alpha: \mathcal{M}_\alpha \to \mathcal{M}'$ is the natural imbedding. Clearly $(\mathcal{M}', \mathbf{g}')$ will also be a development of $(\mathcal{S}, \boldsymbol{\omega})$; therefore every totally ordered set has an upper bound, and so by Zorn's lemma (see, for example, Kelley (1965), p. 33) there is a maximal development $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ of $(\mathcal{S}, \boldsymbol{\omega})$ whose only extension is itself.

We shall now show that $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ is an extension of every development of $(\mathcal{S}, \boldsymbol{\omega})$. Suppose $(\mathcal{M}', \mathbf{g}')$ is another development of $(\mathcal{S}, \boldsymbol{\omega})$. By the local Cauchy theorem, there exist developments of $(\mathcal{S}, \boldsymbol{\omega})$ of which $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ and $(\mathcal{M}', \mathbf{g}')$ are both extensions. The set of all such common developments is likewise partially ordered and so again by Zorn's lemma there will be a maximal development $(\mathscr{M}'', \mathbf{g}'')$ with the imbeddings $\tilde{\mu}: \mathscr{M}'' \to \tilde{\mathscr{M}}$ and $\mu': \mathscr{M}'' \to \mathscr{M}'$, etc. Let $\mathscr{M}^+$ be the union of $\tilde{\mathscr{M}}$, $\mathscr{M}'$ and $\mathscr{M}''$, where each $p'' \in \mathscr{M}''$ is identified with $\tilde{\mu}(p'') \in \tilde{\mathscr{M}}$ and $\mu'(p'') \in \mathscr{M}'$. If one can show that the manifold $\mathscr{M}^+$ is Hausdorff, the pair $(\mathscr{M}^+, \mathbf{g}^+)$ will be a development of $(\mathscr{S}, \boldsymbol{\omega})$. It will be an extension of both $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ and $(\mathscr{M}', \mathbf{g}')$. However the only extension of $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ is $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ itself, and so $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ must equal $(\mathscr{M}^+, \mathbf{g}^+)$ and be an extension of $(\mathscr{M}', \mathbf{g}')$.

Suppose that $\mathscr{M}^+$ were not Hausdorff. Then there exist points $\tilde{p} \in (\tilde{\mu}(\mathscr{M}''))^{\dot{}} \subset \tilde{\mathscr{M}}$ and $p' \in (\mu'(\mathscr{M}''))^{\dot{}} \subset \mathscr{M}'$ such that every neighbourhood $\mathscr{U}$ of $\tilde{p}$ has the property that $\overline{\mu'(\tilde{\mu}^{-1}(\mathscr{U}))}$ contains p'. Now since $(\mathscr{M}'', \mathbf{g}'')$ is a development, it will be globally hyperbolic as will its image $\tilde{\mu}(\mathscr{M}'')$ in $\tilde{\mathscr{M}}$. Therefore the boundary of $\tilde{\mu}(\mathscr{M}'')$ in $\tilde{\mathscr{M}}$ must be achronal. Let γ be a timelike curve in $\tilde{\mathscr{M}}$ with future endpoint at $\tilde{p}$. Then p' must be a limit point in $\mathscr{M}'$ of the curve $\mu'\tilde{\mu}^{-1}(\gamma)$. In fact it must be a future endpoint, since strong causality holds in $(\mathscr{M}', \mathbf{g}')$. Thus the point p' is unique, given $\tilde{p}$. Further, by continuity vectors at p' can be uniquely associated with vectors at $\tilde{p}$. Thus one can find normal coordinate neighbourhoods $\tilde{\mathscr{U}}$ of $\tilde{p}$ in $\tilde{\mathscr{M}}$ and $\mathscr{U}'$ of p' in $\mathscr{M}'$ such that under the map $\mu'\tilde{\mu}^{-1}$ points of $\tilde{\mathscr{U}} \cap \tilde{\mu}(\mathscr{M}'')$ are mapped into points of $\mathscr{U}' \cap \mu'(\mathscr{M}'')$ with the same coordinate values. This shows that the set $\mathscr{F}$ of all 'non-Hausdorff' points of $(\tilde{\mu}(\mathscr{M}''))^{\dot{}}$ is open in $(\tilde{\mu}(\mathscr{M}''))^{\dot{}}$. We shall suppose that $\mathscr{F}$ is non-empty, and so obtain a contradiction.

If $\tilde{\lambda}$ is a past-directed null geodesic in $\tilde{\mathscr{M}}$ through $\tilde{p} \in \mathscr{F}$, then since one can associate directions at p with directions at p', one can construct a past-directed null geodesic λ' through p' in $\mathscr{M}'$ in the corresponding direction. To each point of $\tilde{\lambda} \cap (\tilde{\mu}(\mathscr{M}''))^{\dot{}}$ there will correspond a point of $\lambda' \cap (\mu'(\mathscr{M}''))^{\dot{}}$ and so every point of $\tilde{\lambda} \cap (\tilde{\mu}(\mathscr{M}''))^{\dot{}}$ will be in $\mathscr{F}$. Since $\tilde{\theta}(\mathscr{S})$ is a Cauchy surface for $\tilde{\mathscr{M}}$, $\tilde{\lambda}$ must leave $(\tilde{\mu}(\mathscr{M}''))^{\dot{}}$ at some point $\tilde{q}$. There will be some point $\tilde{r} \in \mathscr{F}$ in a neighbourhood of $\tilde{q}$ such that there is a spacelike surface $\tilde{\mathscr{H}}$ through $\tilde{r}$ which has the property that $(\tilde{\mathscr{H}} - \tilde{r}) \subset \tilde{\mu}(\mathscr{M}'')$. There will be a corresponding spacelike surface $\mathscr{H}' = (\mu'\tilde{\mu}^{-1}(\tilde{\mathscr{H}} - \tilde{r})) \cup r'$ in $\mathscr{M}'$ through the corresponding point r'. The surfaces $\tilde{\mathscr{H}}$ and $\mathscr{H}'$ may be regarded as images of a three-dimensional manifold $\mathscr{H}$ under imbeddings $\tilde{\psi}: \mathscr{H} \to \tilde{\mathscr{M}}$ and $\psi': \mathscr{H} \to \mathscr{M}'$ such that $\tilde{\psi}^{-1}\tilde{\mu}\mu'^{-1}\psi'$ is the identity map on $\mathscr{H} - \tilde{\psi}^{-1}(\tilde{p})$. The induced metrics $\tilde{\psi}_*(\tilde{\mathbf{g}})$ and $\psi'_*(\mathbf{g}')$ on $\mathscr{H}$ will agree since $\tilde{\mathscr{H}} - \tilde{p}$ and $\mathscr{H}' - p'$ are isometric. By the local Cauchy theorem, they will be in $W^{4+a}(\mathscr{H})$. Similarly the second fundamental forms will agree and be in $W^{3+a}(\mathscr{H})$. Neighbourhoods of $\tilde{\mathscr{H}}$ in $\tilde{\mathscr{M}}$ and $\mathscr{H}'$ in $\mathscr{M}'$ would be W^{4+a} developments of $\mathscr{H}$. By the local Cauchy theorem they must be extensions of the same common development $(\mathscr{M}^*, \mathbf{g}^*)$. Joining $(\mathscr{M}^*, \mathbf{g}^*)$ to $(\mathscr{M}'', \mathbf{g}'')$ one would obtain a larger development of $(\mathscr{S}, \boldsymbol{\omega})$, of which $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ and $(\mathscr{M}', \mathbf{g}')$ would be extensions. This is impossible, since $(\mathscr{M}'', \mathbf{g}'')$ was the largest such common development. This shows that $\mathscr{M}^+$ must be Hausdorff, and so that $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ must be an extension of $(\mathscr{M}', \mathbf{g}')$.

We have therefore proved:

The global Cauchy development theorem

If $h^{ab} \in W^{4+a}(\mathscr{S})$ and $\chi^{ab} \in W^{3+a}(\mathscr{S})$ satisfy the empty space constraint equations, there exists a maximal development $(\mathscr{M}, \mathbf{g})$ of the empty space Einstein equations with $\mathbf{g} \in W^{4+a}(\mathscr{M})$ and $\mathbf{g} \in W^{4+a}(\mathscr{H})$ for any smooth spacelike surface $\mathscr{H}$. This development is an extension of any other such development.

We have so far only proved that this development is maximal among W^{4+a} developments. If a is greater than zero, there will also be $W^{4+a-1}, W^{4+a-2}, \ldots, W^4$ developments which are extensions of the W^{4+a} development. However, Choquet-Bruhat (1971) has pointed out that these developments must all coincide with the W^4 development. This is because one can differentiate the reduced Einstein equations and then regard them as *linear* equations on the W^4 development, for the first derivatives of g^{ab}. Then using proposition 7.4.7 one can show that g^{ab} is W^5 on the W^4 development, if the initial data is W^5. By continuing in this way, one can show that if the initial data is C^∞, there will be a C^∞ development which will in fact coincide with the W^4 development.

We have proved the existence and uniqueness of maximal developments only for W^4 or higher metrics. In fact, it is possible to prove the existence of developments for W^3 initial data, but we have not been able to prove the uniqueness in this case. It may be possible to extend the W^4 maximal development either so that the metric does not remain in W^4, or so that $\theta(\mathscr{S})$ does not remain a Cauchy surface. In the latter case, a Cauchy horizon occurs; examples of this were given in chapter 6. On the other hand it may be that some sort of singularity occurs, in which case the development cannot be extended with a metric which is sufficiently differentiable to be interpreted physically. In fact, theorem 4 of the next chapter will show that if $\mathscr{S}$ is compact and $\chi^{ab}h_{ab}$ is negative everywhere on $\mathscr{S}$, then the development cannot be extended to be geodesically complete with a C^{2-} metric, i.e. with locally bounded curvature.

We have shown there is a map from the space of pairs of tensors (h^{ab}, χ^{ab}) on $\mathscr{S}$ which satisfy the constraint equations to the space of equivalence classes of metrics $\mathbf{g}$ on a manifold $\mathscr{M}$, which, by proposition 6.6.8, is diffeomorphic to $\mathscr{S} \times R^1$. If two pairs (h^{ab}, χ^{ab}) and (h'^{ab}, χ'^{ab}) are equivalent under a diffeomorphism $\lambda: \mathscr{S} \to \mathscr{S}$ (i.e. $\lambda_* h^{ab} = h'^{ab}$ and $\lambda_* \chi^{ab} = \chi'^{ab}$) they will produce equivalent metrics $\mathbf{g}$. We thus have a map from equivalence classes of pairs (h^{ab}, χ^{ab}) to equivalence classes of metrics $\mathbf{g}$. Now h^{ab} and χ^{ab} together have twelve independent components. The constraint equations impose four relations between these, and the equivalence under diffeomorphisms may be regarded as removing a further three arbitrary functions, leaving five independent functions. One of these functions may be regarded as specifying the position of $\theta(\mathscr{S})$ within the development $(\mathscr{M}, \mathbf{g})$. Therefore maximal developments of the empty space Einstein equations are specified by four functions of three variables.

One would like to show that the map from equivalence classes of (h^{ab}, χ^{ab}) to equivalence classes of $\mathbf{g}$ is continuous in some sense. The appropriate topology on the equivalence classes for this is the *W^r compact-open topology* (cf. §6.4). Let $\hat{\mathbf{g}}$ be a C^r Lorentz metric on $\mathscr{M}$ and $\mathscr{U}$ be an open set with compact closure. Let V be an open set in $W^r(\mathscr{U})$ and let $O(\mathscr{U}, V)$ be the set of all Lorentz metrics on $\mathscr{M}$ whose restrictions to $\mathscr{U}$ lie in V. The open sets of the W^r compact open topology on the space $\mathscr{L}_r(\mathscr{M})$ of all W^r Lorentz metrics on $\mathscr{M}$ are defined to be the unions and finite intersections of sets of the form $O(U, V)$. The topology of the space $\mathscr{L}_r{}^*(\mathscr{M})$ of equivalence classes of W^r metrics on $\mathscr{M}$ is then that induced by the projection

$$\pi: \mathscr{L}_r(\mathscr{M}) \to \mathscr{L}_r{}^*(\mathscr{M})$$

which assigns a metric to its equivalence class (i.e. the open sets of $\mathscr{L}_r{}^*(\mathscr{M})$ are of the form $\pi(Q)$ where Q is open in $\mathscr{L}_r(\mathscr{M})$). Similarly the W^r compact open topology on the space $\Omega_r(\mathscr{S})$ of all pairs (h^{ab}, χ^{ab}) which satisfy the constraint equations is defined by sets of the form $O(\mathscr{U}, V, V')$ consisting of the pairs for which $h^{ab} \in V$ and $\chi^{ab} \in V'$ where V and V' are open sets in $W^r(\mathscr{S})$ and $W^{r-1}(\mathscr{S})$ respectively. The C^∞ metrics on $\mathscr{M}$ form a subspace $\mathscr{L}_\infty(\mathscr{M})$ of the space $\mathscr{L}$ $(\mathscr{M})$ of all Lorentz metrics on $\mathscr{M}$. Since a C^∞ metric is W^r for any r, one has the W^r topology on $\mathscr{L}_\infty(\mathscr{M})$. One can then define the C^∞ or W^∞ topology on $\mathscr{L}_\infty(\mathscr{M})$ as that given by all the open sets in the W^r topologies on $\mathscr{L}_\infty(\mathscr{M})$ for every r. The C^∞ topology on $\mathscr{L}_\infty{}^*(\mathscr{M})$ and on $\Omega_\infty(\mathscr{S})$ are defined similarly.

One would like to show that the map Δ_r from the space $\Omega_r{}^*(\mathscr{S})$ of equivalence classes of pairs (h^{ab}, χ^{ab}) to the space $\mathscr{L}_r{}^*(\mathscr{M})$ of equivalence classes of metrics is continuous with the W^r compact open topology on both spaces. In other words, suppose one has initial data $h^{ab} \in W^r(\mathscr{S})$ and $\chi^{ab} \in W^{r-1}(\mathscr{S})$ which gives rise to a solution $\mathbf{g} \in W^r(\mathscr{M})$ on $\mathscr{M}$. Then if $\mathscr{V}$ is a region of $\mathscr{M}$ with compact closure, and $\epsilon > 0$, one would like to show there was some region $\mathscr{Y}$ of $\mathscr{S}$ with compact closure and some $\delta > 0$ such that $\|\mathbf{g}' - \mathbf{g}, \mathscr{V}\|_r < \epsilon$ for all initial data (h'^{ab}, χ'^{ab}) such that $\widetilde{\|\mathbf{h}' - \mathbf{h}, \mathscr{Y}\|}_r < \frac{1}{2}\delta$ and $\widetilde{\|\boldsymbol{\chi}' - \boldsymbol{\chi}, \mathscr{Y}\|}_{r-1} < \frac{1}{2}\delta$. This result may be true, but we have been unable to prove it. What we can prove is that this result holds if the metric is $C^{(r+1)-}$. This follows immediately from proposition 7.5.1, taking $\mathbf{g}$ to be the background metric and $\mathscr{U}$ to be some suitable neighbourhood of $J^-(\overline{\mathscr{V}}) \cap J^+(\theta(\mathscr{S}))$. In fact if one examines lemma 7.4.6, one sees that the condition on the background metric can be weakened from $C^{(r+1)-}$ to $W^{(r+1)}$, but not to W^r, since the $(r-1)$th derivatives of the Riemann tensor of the background metric appear. (By the background metric being W^{r+1} we mean that it is W^{r+1} with respect to a further C^{r+1} background metric.) Thus the map $\Delta_r\colon \Omega_r{}^*(\mathscr{S}) \to \mathscr{L}_r{}^*(\mathscr{M})$ from the equivalence classes of initial data to the equivalence classes of metrics will be continuous in the W^r compact open topology at every W^{r+1} metric. Although the W^{r+1} metrics form a dense set in the W^r metrics, there is a possibility that the map might not be continuous at a W^r metric which was not also a W^{r+1} metric. However $\infty + 1 = \infty$ and so the map $\Delta_\infty\colon \Omega^*{}_\infty(\mathscr{S}) \to \mathscr{L}^*{}_\infty(\mathscr{M})$ will be continuous in the C^∞ topology on both spaces.

One can express this result as:

The Cauchy stability theorem

Let $(\mathscr{M}, g)$ be the W^{5+a} $(0 \leqslant a \leqslant \infty)$ maximal development of initial data $\mathbf{h} \in W^{5+a}(\mathscr{S})$ and $\boldsymbol{\chi} \in W^{4+a}(\mathscr{S})$, and let $\mathscr{V}$ be a region of $J^+(\theta(\mathscr{S}))$ with compact closure. Let Z be a neighbourhood of $\mathbf{g}$ in $\mathscr{L}_{5+a}(\mathscr{V})$ and $\mathscr{U}$ be an open neighbourhood in $\theta(\mathscr{S})$ of $J^-(\overline{\mathscr{V}}) \cap \theta(\mathscr{S})$ with compact closure. Then there is some neighbourhood Y of $(\mathbf{h}, \boldsymbol{\chi})$ in $\Omega_{5+a}(\mathscr{U})$ such that for all initial data $(\mathbf{h}', \boldsymbol{\chi}') \in Y$ satisfying the constraint equations, there is a diffeomorphism $\mu\colon \mathscr{M}' \to \mathscr{M}$ with the properties

(1) $\theta^{-1}\mu\theta'$ is the identity on $\theta^{-1}(\mathscr{U})$,

(2) $\mu_* \mathbf{g}' \in Z$,
where $(\mathscr{M}', \mathbf{g}')$ is the maximal development of $(\mathbf{h}', \boldsymbol{\chi}')$. □

Roughly speaking what this theorem says is that if the perturbation of initial data on the Cauchy surface $\theta(\mathscr{S})$ is small on $J^-(\overline{\mathscr{V}}) \cap \theta(\mathscr{S})$, then one gets a new solution which is near the old solution in $\mathscr{V}$. In fact the perturbation of the initial data has to be small on a slightly larger region of the Cauchy surface than $J^-(\overline{\mathscr{V}}) \cap \theta(\mathscr{S})$, since the null cones will be slightly different in the new solution and so $\mathscr{V}$ may not lie in the Cauchy development of $J^-(\overline{\mathscr{V}}) \cap \theta(\mathscr{S})$.

7.7 The Einstein equations with matter

For simplicity we have so far considered the Einstein equations only for empty space. However similar results hold when matter is present providing that the equations governing the matter fields $\Psi_{(i)}{}^I{}_J$ obey certain physically reasonable conditions. The idea is to solve the matter equations with the prescribed initial conditions in a given space–time metric $\mathbf{g}'$. One then solves the reduced Einstein equations (7.42) as *linear* equations with the coefficients determined by $\mathbf{g}'$ and with the source term T'^{ab} determined by $\mathbf{g}'$ and by the solution for the matter fields. One thus obtains a new metric $\mathbf{g}''$ and repeats the procedure with $\mathbf{g}''$ in place of $\mathbf{g}'$. To show that this converges to a solution of the combined Einstein and matter equations one has to impose certain conditions on the matter equations. We shall require:

(*a*) if $\{_0\boldsymbol{\Psi}_{(i)}\} \in W^{4+a}(\mathscr{H})$ and $\{_1\boldsymbol{\Psi}_{(i)}\} \in W^{3+a}(\mathscr{H})$ are the initial data on an achronal spacelike surface $\mathscr{H}$ in a W^{4+a} metric $\mathbf{g}$, there exists a unique solution of the matter equations in a neighbourhood of $\mathscr{H}$ in $D^+(\mathscr{H})$ with $\{\boldsymbol{\Psi}_{(i)}\} \in W^{4+a}(\mathscr{H}')$ for any smooth spacelike surface $\mathscr{H}'$, and

$$\boldsymbol{\Psi}_{(i)} = {}_0\boldsymbol{\Psi}_{(i)}, \quad \Psi_{(i)}{}^I{}_{J|a} u^a = {}_1\Psi_{(i)}{}^I{}_J \quad \text{on } \mathscr{H};$$

(*b*) if $\{\boldsymbol{\Psi}_{(i)}\}$ is a W^{5+a} solution in the W^{5+a} metric $\mathbf{g}$ on the set $\mathscr{U}_+$, then there exist positive constants $\tilde{Q}_1$ and $\tilde{Q}_2$ such that

$$\sum_{(i)} \|\boldsymbol{\Psi}'_{(i)} - \boldsymbol{\Psi}_{(i)}, \mathscr{U}_+\|_{4+a} \leqslant \tilde{Q}_2\{\|\mathbf{g}' - \mathbf{g}, \mathscr{U}_+\|_{4+a}$$
$$+ \sum_{(i)} \|{}_0\boldsymbol{\Psi}'_{(i)} - {}_0\boldsymbol{\Psi}_{(i)}, \mathscr{H}(0) \cap \tilde{\mathscr{U}}\|_{4+a} + \sum_{(i)} \|{}_1\boldsymbol{\Psi}'_{(i)} - {}_1\boldsymbol{\Psi}_{(i)}, \mathscr{H}(0) \cap \tilde{\mathscr{U}}\|_{3+a}\}$$

for any W^{4+a} solution $\{\boldsymbol{\Psi}'_{(i)}\}$ in the metric $\mathbf{g}'$ such that

$$\|\mathbf{g}' - \mathbf{g}, \mathscr{U}_+\|_{4+a} < \tilde{Q}_1$$

and

$$\sum_{(i)} \{\|{}_0\boldsymbol{\Psi}'_{(i)} - {}_0\boldsymbol{\Psi}_{(i)}, \mathscr{H}(0) \cap \tilde{\mathscr{U}}\|_{4+a} + \|{}_1\boldsymbol{\Psi}'_{(i)} - {}_1\boldsymbol{\Psi}_{(i)}, \mathscr{H}(0) \cap \tilde{\mathscr{U}}\|_{3+a}\} < \tilde{Q}_1;$$

(c) the energy–momentum tensor T_{ab} is polynomial in

$$\Psi_{(i)}{}^{I}{}_{J},\quad \Psi_{(i)}{}^{I}{}_{J;a} \quad \text{and} \quad g^{ab}.$$

Condition (a) is the local Cauchy theorem for the matter field in a given space–time metric. Condition (b) is the Cauchy stability theorem for the matter field under a variation of the initial conditions and under a variation of the space–time metric $\mathbf{g}$. If the matter equations are quasi-linear second order hyperbolic equations, these conditions may be established in a similar manner to that for the reduced Einstein equations, providing that the null cones of the matter equations coincide with or lie within the null cone of the space–time metric $\mathbf{g}$. In the case of the scalar field or the electromagnetic potential which obey linear equations, these conditions follow from proposition 7.4.7. One can also deal with a scalar field coupled to the electromagnetic potential; one fixes the metric and the electromagnetic potential, solves the scalar field as a linear equation in that metric and potential, and then solves the electromagnetic field in the given metric with the scalar field as the source. Iterating this procedure one can show that one converges on a set of the form $\mathscr{U}_+$ to a solution of the coupled scalar and electromagnetic equations in the given metric, providing that the initial data are sufficiently small. One then shows, by rescaling the metric and the fields, that for $\mathscr{U}_+$ sufficiently small (as measured by the space–time metric $\mathbf{g}$) one can obtain a solution for any suitable initial data. The same procedure will work for any finite number of coupled quasi-linear second order hyperbolic equations, where the coupling does not involve derivatives higher than the first.

The equations of a perfect fluid are not second order hyperbolic, but form a quasi-linear first order *system*. (For the definition of a first order hyperbolic system, see Courant and Hilbert (1962), p. 577.) Similar results can be obtained for such systems providing that the ray cone coincides with or lies within the null cone of the space–time with metric $\mathbf{g}$. The requirement that the matter equations should be second order hyperbolic equations or first order hyperbolic systems with their cones coinciding with or lying within that of the space–time metric $\mathbf{g}$, may be thought of as a more rigorous form of the local causality postulate of chapter 3.

With the conditions (a), (b) and (c) one can establish propositions 7.5.1 and 7.5.2 for the combined reduced Einstein's equations and the matter equations; from these, the local and global Cauchy development theorems and the Cauchy stability theorem follow.