ON $n+1$ INTERSECTING HYPERSPHERES IN n-SPACE †

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Introduction

The interesting results arising from the study of 'Four intersecting spheres' [9] in a solid made the author think of an analogous picture in higher spaces too and thus the present paper arose.

First we take up a 4-space and develop the relative ideas therein, suggesting the respective results in an n-space, which are then deduced mostly by the method of induction. In fact, the dlcovery of an S-configuration [6] and its dual are due to this study

I. Four dimensional space

1. *Preliminaries*

(a) Let the 5 pairs of points (λ) of intersection of 5 mutually intersecting hyperspheres (A), (B), (C), (D), (E) taken four at at time be tabulated as follows:

The pair of points (λ)	*common to*
P, P'	$(B), (C), (D), (E)$
Q, Q'	$(A), (C), (D), (E)$
R, R'	$(A), (B), (D), (E)$
T, T'	$(A), (B), (C), (E)$
F, F'	$(A), (B), (C), (D)$.

(b) The radical solids h_a, h_b, h_c, h_d, h_e of (A), (B), (C), (D), (E) with their common orthogonal hypersphere (H) form a simplex S', referred to as the *radical simplex* of the 5 hyperspheres whose centres A, B, C, D, E form another simplex S, referred to as their *central simplex*. The vertices $A' = (h_b, h_c, h_d, h_e)$, $B' = (h_a, h_c, h_d, h_e)$, $C' = (h_a, h_b, h_d, h_e)$, $D' =$

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(h_a, h_b, h_c, h_e), $E' = (h_a, h_b, h_c, h_d)$ of S' lie respectively on their radical axes $p = PP'$, $q = QQ'$, $r = RR'$, $t = TT'$, $f = FF'$. It is readily seen that S and S' are polar reciprocal for (H). Hence ([1], [10]) we have

THEOREM 1. *The joins of the corresponding vertices of the radical and central simplexes of* 5 *intersecting hyperspheres in a* 4*-space form an associated set of* 5 *lines such that planes meeting four of them meet the fifth too.*

2. *Perspective simplexes*

(a) Taking one point from each of the 5 pairs of points (λ), we obtain 32 simplexes, referred to as the *simplexes of intersection* of the given 5 hyperspheres, grouped into 16 pairs of *complementary simplexes*: One is $ijklm$, $i'j'k'l'm'$; five are of the type $i'jklm$, $ij'k'l'm'$ and ten like $i'j'klm$, $ijk'l'm'$ where $i, j, k, l, m = P, Q, R, T, F$.

(b) The 5 radical axes p, q, r, t, f (§ 1b) concur at the orthogonal centre H of the 5 hyperspheres. Hence the 16 pairs of complementary simplexes are in perspective from H such that a pair of their corresponding edges meet at one of the 20 points $L_{ij} = (ij, i'j')$, $M_{ij} = (i'j, ij')$ which lie by tens in their 16 solids of perspectivity, thus forming 16 Desargues' 10_3 configurations, by sixes in the 40 planes of perspectivity of the 40 pairs of corresponding tetrahedra (five of the type $ijkl$, $i'j'k'l'$; twenty like $i'jkl$, $ij'k'l'$ and fifteen $i'j'kl$, $ijk'l'$), and by threes on the 40 lines of perspectivity of the 40 pairs of corresponding triangles (ten of the type ijk, $i'j'k'$ and thirty like $i'jk$, $ij'k'$), lying by fours in 40 planes forming 40 quadrilaterlas, one in each plane.

3. *S-configuration*

(a) The 4 points Q, Q', R, R' determine a complete quadrangle inscribed in the circle common to the 3 hyperspheres (A), (D), (E) (see § 1a) having the points H, L_{QR}, M_{QR} as vertices of its diagonal triangle. Hence L_{QR}, M_{QR} lie on the polar of H for this circle as a pair of conjugate points for it [9]. Similarly, the pairs of points L_{FT}, M_{FT}; L_{RF}, M_{RF}; L_{RT}, M_{RT}; L_{QF}, M_{QF}; L_{QT}, M_{QT} are conjuagate and lie respectively on the polars of H for the circles of (A) common to the pair of hyperspheres (B), (C); (B), (D),; (B), (E); (C), (D); (C), (E). Thus the 6 pairs of points L_{ij}, M_{ij} $(i, j \neq P)$ lie in the polar of H for (A), which coincides with the radical solid h_a (§ 1b) of the orthogonal hyperspheres (H), (A), and are conjugate for the sphere-section of (A) by h_a and therefore for (H) too. The other four solids of the radical simplex S' behave similarly; their edges then lie along the 10 joins of the 10 pairs of points L_{ij}, M_{ij} conjugate for (H).

(b) For example, $L_{PQ}M_{PQ} = (h_c, h_d, h_e) = A'B'$, $L_{QR}M_{QR} = (h_a, h_d, h_e) = B'C'$, $L_{RP}M_{RP} = (h_b, h_d, h_e) = C'A'$ and the 3 pair of points L_{PQ}, M_{PQ}; L_{QR}, M_{QR}; L_{RP}, M_{RP} form the 3 pairs of opposite vertices of the quadrilateral formed by the 4 lines $L_{PQ}L_{QR}L_{RP}$, $L_{PQ}M_{QR}M_{RP}$, $L_{QR}M_{RP}M_{PQ}$, $L_{RP}M_{PQ}M_{QR}$ having the plane face $A'B'C' = (h_d, h_e)$ of S' as its diagonal triangle, and are separated harmonically by A', B'; B', C'; C', A' respectively. The other nine plane faces of S' behave similarly. Consequently, the 6 pairs of points L_{ij}, M_{ij} in every solid face of S' form a desmic system [9] with this face as its diagonal tetrahedrom.

(c) The 10 points L_{ij} obviously lie in the solid of perspectivity of the pair of simplexes $PQRTF$, $P'Q'R'T'F'$ and the 10 points M_{ij} are their harmonic conjugates for the corresponding pairs of vertices of S'. Thus the 10 pairs of points L_{ij}, M_{ij} form the pairs of opposite vertices of an *S-configuration* $(S-C)$

$$20\ (.,\ 6,\ 12,\ 8)\ 40\ (3,\ .,\ 4,\ 4)\ 40\ (6,\ 4,\ .,\ 2)\ 16\ (10,\ 10,\ 5,\ .),$$

with S' as its diagonal simplex [6], referred to as the $(S-C)$ *of intersection* of the given 5 hyperspheres. Hence we have

THEOREM 2. *Five interscecting hyperspheres in a 4-space give rise to 16 pairs of complementary simplexes of intersection, each pair perspective from their orthogonal centre H. The 16 solids of perspectivity form their $(S-C)$ of intersection with their radical simplex S' as its diagonal simplex and the 10 pairs of opposite vertices of $(S-C)$ are conjugate for the common orthogonal hypersphere (H) and the circumhypersphere (S') of S'.*

4. *Hyperspheres of intersection*

(a) The circumhyperspheres of the simplexes of intersection are referred to as the *hyperspheres of intersection,* and those circumscribing a pair of complementary ones are referred to as *complementary hyperspheres of intersection* for the given 5 hyperspheres.

The pairs of points (λ) are easily observed to be inverse [7] for the hypersphere (H) (§1). They are therefore antihomologous [9] for any pair of complementary hyperspheres of intersection which are then inverse for (H) and the corresponding edges of their correspondingly inscribed simplexes of intersection are their *antihomologous chords* meeting in their radical solid. Hence we have

THEOREM 3. *The common orthogonal hypersphere (H) of 5 intersecting hyperspheres in a 4-space is a common hypersphere of antisimilitude of the 16 pairs of complementary hyperspheres of intersection and the 16 radical solids of the latter coincide with the 16 solids of perspectivity of the corresponding pairs of complementary simplexes of intersection; the 16 joins of their centres therefore concur at their common centre of similitude H and are normal to their respective radical solids.*

(b) Since the 2 hyperspheres of antisimilitude of any two hyperspheres are coaxal with them and orthogonal to each other, the centre of either is one of their centres of similitude and lies at the pole of their common radical solid for the other ([5]; [7]). Hence from the preceding two theorems follows

THEOREM 4. *The* 16 *second centres of similitude of the* 16 *pairs of complementary hyperspheres of intersection of* 5 *intersecting hyperspheres in a* 4-*space, besides their one common centre of similitude H, lie at the poles of the* 16 *solids of the* $(S-C)$ *of intersection for their common hypersphere of antisimilitude* (H) *and therefore form the dual configuration* $(R.S-C)$, *as the polar reciprocal of the* $(S-C)$ *for* (H), *with the central simplex S* (§ 1*b*) *as its diagonal simplex* [6]. *Thus they form a closed or associated set of* 16 *points such that quadrics, for which S is selfpolar, passing through one of them, pass through all of them* [8].

(c) L_{ij}, M_{ij} being a pair of opposite vertices of the $(S-C)$ and conjugate for the hypersphere (H) (§ 3), we now deduce from the preceding theorem.

THEOREM 5. *The* $(S-C)$ *of intersection of* 5 *intersecting hyperspheres in a* 4-*space is inscribed in its polar reciprocal configuration* $(R.S-C)$ *for their common orthogonal hypersphere* (H) *in such a way that every one of the* 20 *solids of the* $(R.S-C)$ *passes through one vertex of the* $(S-C)$.

5. *Associated* 5 *hyperspheres*

(a) A' is obviously the orthogonal centre of the 5 hyperspheres (H), (B), (C), (D), (E) (§ 1b), and therefore we can construct their common orthogonal hypersphere (A') with centre at A'. Similarly, we have 4 more such hyperspheres (B'), (C'), (D'), (E'). Thus (A'), (B'), (C'), (D'), (E'), referred to as the associated 5 hyperspheres, are mutually related to (A), (B), (C), (D), (E), in such a way that the central simplex of one set is the radical simplex of the other.

(b) L_{PQ}, M_{PQ} being conjugate points for (H), the hypersphere $(L_{PQ} M_{PQ})$ with $L_{PQ}M_{PQ}$ as its diameter is orthogonal to (H) ([5]; [9]). The centre of $(L_{PQ}M_{PQ})$ lying on the line $L_{PQ}M_{PQ} = A'B'$ (§ 3b), the 3 hyperspheres (A'), (B'), $(L_{PQ}M_{PQ})$, with centres collinear and orthogonal to the same hypersphere (H), are coaxal ([5]; [9]). Moreover L_{PQ}, M_{PQ} separate A', B' harmonically (§ 3b). Hence $(L_{PQ}M_{PQ})$ is the hypersphere of similitude [7] and L_{PQ}, M_{PQ} are the centres of similitude of (A'), (B'). All the 10 hyperspheres $(L_{ij}M_{ij})$ behave similarly. Hence we have

THEOREM 6. *The pairs of opposite vertices of the* $(S-C)$ *of intersection of* 5 *intersecting hyperspheres in a* 4-*space are the* 10 *pairs of centres of similitude of the associated* 5 *hyperspheres taken two by two such that its solids are their* 16 *solids of similitude.*

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This proves the theorem: *the* 20 *centres of similitude of* 5 *hyperspheres taken two by two lie by tens in* 16 *solids which may be termed their solids of similitude* [6].

(c) We may also observe that the hyperspheres $(L_{ij}M_{ij})$ are orthogonal to the common orthogonal hypersphere (H) of the associated 5 hyperspheres as well as to the circumhypersphere (S') of their central simplex S' (by Theorem 2). Hence they form a *coaxal net* with their centres in the *Newtonian solid* of the associated hyperspheres [6]. Thus we have the following

THEOREM 7. *The join of the orthogonal centre of* 5 *intersecting hyperspheres in a* 4*-space to the circumcentre of their radical simplex is normal to the Newtonian solid of their associated* 5 *hyperspheres.*

(d) L_{PQ}, M_{PQ} are the centees of similitude of the pair of hyperspheres (A'), (B') and lie on $A'B' = (h_c, h_d, h_e)$. Hence the 2 hyperspheres (L_{PQ}), (M_{PQ}) with centres at L_{PQ}, M_{PQ} and coaxal with (A'), (B') form the pair of their hyperspheres of antisimilitude orthogonal to the hypersphere (H) and therefore to the 3 hyperspheres (C), (D), (E). For L_{PQ}, M_{PQ} lie in the radical solids h_c, h_d, h_e of (H) with (C), (D), (E). Similar results hold for all the 10 pairs of hyperspheres (L_{ij}), (M_{ij}) similarly constructed.

Again every point L_{ij} or M_{ij} lies in 8 solids, of the $(S-C)$ of intersection (§ 3c), which are the radical solids of 8 pairs of complementary hyperspheres of intersection, each pair coaxal with their common hypersphere of antisimilitude (H) (Theorem 3). Thus follows

THEOREM 8. *Every one of the* 20 *hyperspheres of antisimilitude of the associated* 5 *hyperspheres of* 5 *intersecting hyperspheres in a* 4*-space is orthogonal to* 8 *pairs of complementary hyperspheres of intersection and to three of the given hyperspheres. Or, each pair of complementary hyperspheres is orthogonal to* 10 *hyperspheres of antisimilitude with centres in the radical solid of the complementary hyperspheres considered, and every one of the given* 5 *hyperspheres is orthogonal to* 12 *hyperspheres of antisimilitude with centres in the radical solid of their comon orthogonal hypersphere* (H) *and the hypersphere under consideration.*

6. *Isodynamic simplexes*

(a) It may happen that the central and radical simplexes S, S' of the given 5 hyperspheres are in perspective with their solid of perspectivity u the same as that of a pair of complementary simplexes of intersection, say $I = PQRTF$, $I' = P'Q'R'T'F'$. The 10 pairs of their corresponding edges then meet respectively in the 10 points L_{ij} in u. E.g., $(AB, A'B') = L_{PQ} = (PQ, P'Q')$. Now the 3 hyperspheres (L_{PQ}), (A), (B) with collinear centres and orthogonal to the same hypersphere (H) are coaxal. Again, since (L_{PQ}) is orthogonal to the other three given hyperspheres (C), (D), (E)

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and to 8 pairs of complementary hyperspheres of intersection (§ 5d), one pair being (I), (I') circumscribing I, I', it follows that the 2 pairs of their common points P, Q; P', Q' (§ 1a) collinear with L_{PQ} form 2 pairs of inverse points for (L_{PQ}). Thus (A) passing through Q, Q' inverts w.r.t. (L_{PQ}) into (B) passing through P, P'. Or, (L_{PQ}) is a hypersphere of antisimilitude for (A), (B) and passes through the 2 triads of their common points R, T, F,; R', T', F' (§ 1a) which lie respectively on (I), (I'). Hence the joins of L_{PQ} to R, T, F touch (I) at R, T, F, and to R', T', F' touch (I') at R', T', F' respectively. Or, (L_{PQ}) cuts the planes of the triangles PQR, PQT, PQF, $P'Q'R'$, $P'Q'T'$, $P'Q'F'$ in their *Apollonian circles* [2], with common centre at L_{PQ}, as its great circles.

All the 10 hyperspheres (L_{ij}) to S, S', I, I' are similarly related and thus the *tangential simplexes* of I, I' formed of the tangent solids of (I), (I') at the vertices of I, I' are perspective to I, I' with u as their common solid of perspectivity. Or, I, I' are both *isodynamic* with u as their common *Lemoine* solid ([12]—[15]), the (L_{ij}) are their common 10 *Neuberg hyperspheres*, and their *Lemoine points* U, U' lie at the poles of u for (I), (I') ([1], p. 149, Ex. 22; [6], [8], [10]) and therefore on their common *Brocard diameter* joining their circumcentres. In fact, (I), (I') are the *polar quadrics* ([15], [16], (18]) of U, U' for I, I' in analogy with the polar conic of a point for a triangle ([17], [19]). For the tangential simplexes of I, I' are *anticevian* to I, I' for U, U'. Hence we have

THEOREM 9. *If the central and radical simplexes of* 5 *intersecting hyperspheres in a* 4-*space be in perspective with the solid of perspectivity u the same as that of a pair of complementary simplexes of intersection I, I', then: u is one of their* 16 *solids of similitude; I, I' are both isodynamic with common Lemoine solid coinciding with u; their* 10 *hyperspheres of antisimilitude with centres in u coincide with those of their associated hyperspheres and form the common* 10 *Neuberg hyperspheres of I, I'; the Lemoine points of I, I' lie on their common Brocard diameter meeting their* 10 *altitudes and normal to u* [14].

(b) It can be proved that I' is the transform of I in the *homology* $(H, u, -c)$ with centre of homology at the orthogonal centre H of the given hyperspheres, the solid of homology being the radical solid u of the hyperspheres (H), (I) and the constant cross ratio of homology being equal to the negative ratio $-c$ of the power of H for (I) to the square of the radius of (H) ([13]). Obviously then (I) transform into (I') and the tangential simplex of I into that of I' in this homology. Thus we have

THEOREM 10. *If a simplex of intersection I of* 5 *intersecting hyperspheres in a* 4-*space be isodynamic, the same is also true of its complementary one I'; the Lemoine solids of I, I' are then coaxal with their solid of perspectivity u*

and may coincide with u; in this case the central and radical simplexes of the given hyperspheres are in perspective with the same solid of perspectivity as u.

The later part of the proposition follows by retracing the steps of the preceding section and recalling that u is the radical solid of the corresponding pair of complementary hyperspheres of intersection.

(c) Now as an immediate consequence we have

THEOREM 11. *If* 2 *non-complementary simplexes of intersection of* 5 *intersecting hyperspheres in a* 4*-space be isodynamic in such a way that their Lemoine solids coincide respectively with those of their complementary ones, the same is true of every pair of complementary simplexes of intersection. Consequently, the given hyperspheres are mutually orthogonal such that: the* 16 *Brocard diameters of the* 16 *pairs of complementary simplexes of intersection concur at their common orthogonal centre H; the* 16 *solids of the $(S-C)$ of intersection coincide with their* 16 *solids of similitude (Theorem* 6*); the radical solid of the circumhypersphere of their central simplex and their sixth common orthogonal hypersphere (H) coincides with their Newtonian solid* (§ 5c)*; each pair of complementary hyperspheres of intersection is orthogonal to ten of their* 20 *hyperspheres of antisimilitude with centres in the radical solid of the complementary hyperspheres considered, and every one of them to* 12 *hyperspheres of antisimilitude with centres in the radical solid of (H) and the hypersphere considered (Theorem* 8*).*

For S, S' (§ 6a) are now doubly perspective in the same order, by Theorem 10, but such a situation cannot hold unless they coincide, which leads to the necessary consequences. In fact, H, and the vertices of S form an *orthocentric* set of 6 points such that the simplex formed by any five of them is orthogonal, with the sixth point as its orthocentre ([7]).

7. *Umbilical projection*

Let us consider 5 primes in a 5-space meeting in a point H in such a way that their lines of intersection meet a given 4-quadric W in the 5 pairs of points (λ) (§ 1a). The polar prime (H) of H for W meets the given primes in 5 solids that form 4-simplex S', reciprocal to the one S, formed by their poles w.r.t. W, for its section by (H). Taking one point from each of the 5 pairs of points (λ), we obtain 16 pairs of perspective 4-simplexes (§ 2a) whose solids of perspectivity are those common to their respective primes and lie in (H), giving rise to an $(S-C)$ therein (§ 3c). Each of these 16 pairs of primes, therefore, as well as their sections with W, are *harmonically inverse* [8] w.r.t. $(H, (H))$, and the pair of their poles for W are collinear with H. The second centre of *harmonic inversion* [4] for the sections of W by a pair of such primes is the pole of their common solid for its section by (H) [5]. The polar primes of the vertices of S' for W pass through H and form the *associated* 5 primes.

On projection from a point V of W into a given prime p we get all the results, obtained above, at once. For by this projection, prime sections of W project into 3-quadrics, all passing through a fixed 2-quadric section w of W by the solid common to p and the tangent prime (V) of W at V [1].

Taking w to be the ***absolute polarity*** [6] or the 2-sphere at infinity through which all the 3-spheres in p pass [3], we get these 3-spheres as the projections of different prime sections of W from V. We may term V an ***umbilicus*** of W and the projection ***umbilical*** in analogy with such a process adopted in 4-space [5] and elsewhere ([13], [14]) with some advantage. The orthogonality of 3-spheres is implied by the conjugacy, for W, of the corresponding primes whose sections of W project into them and whose poles for W project to their centres. Cross-ratios and harmonic relations, in particular, are unaltered. Harmonic inversion w.r.t. a point and a prime gives us inversion [5] w.r.t. the corresponding 3-sphere into which the section of W by the prime projects. The vertices of the hypercones [4] through a pair of prime sections of W project into the centres of similitude of the corresponding 3-spheres [5].

2. n-Dimensional space

Now we are in a position to state without proof the analogous results for $n+1$ mutually intersecting hyperspheres (A_i) in an n-space giving rise to $n+1$ pairs of points of intersection like (λ) (§ 1a) and thus to 2^n pairs of complementary simplexes and 2^n pairs of complementary hyperspheres of intersection. We may analogously define their central and radical simplexes S, S' (§ 1b) too such that S, S' are polar reciprocal for their common orthogonal hypersphere (H). We may also similarly define their associated $n+1$ hyperspheres (A'_i) orthogonal to (H) and n (A_j) $j \neq i$), the 2 sets (A_i), (A'_i) being mutually related such that the radical simplex of one set is the central simplex of the second set (§ 5a). Thus we have a general

Proposition. *The joins of the corresponding vertices of the central and radical simplexes S, S' of $n+1$ intersecting hyperspheres (A_i) in an n-space form an associated set* [10] *of $n+1$ lines such that any $(n-2)$-space meeting n of them also meets the $(n+1)$th (Theorem* 1).

Any (A_i) gives rise to 2^n pairs of complementary simplexes of intersection, each pair perspective from their orthogonal centre H. The 2^n primes of perspectivity form their $(S-C)$ of intersection [6] *and its $\binom{n+1}{2}$ pairs of opposite vertices are conjugate for their common orthogonal hypersphere (H) and the circumhypersphere (S') of S' (Theorem* 2).

(H) is a common hypersphere of antisimilitude of the 2^n pairs of complementary hyperspheres of intersection of (A_i) and their 2^n radical primes coin-

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cide with the 2^n *primes of perspectivity of the corresponding pairs of complementary simplexes of intersection inscribed in them; the* 2^n *joins of their centres therefore* concur *at their common centre of similitude* H *and are normal to their respective radical primes (Theorem* 3). *Their* 2^n *second centres of similitude form the dual configuration* $(R.S-C)$, *the polar reciprocal of the* $(S-C)$ *for* (H) *with* S *as its diagonal simplex* [6]; *thus they form a closed set of* 2^n *points such that* [8] *any quadric, for which* S *is selfpolar, passing through one of them passes through all of them (Theorem* 4).

The $(S-C)$ *is inscribed in the* $(R.S-C)$ *(Theorem* 5).

The pairs of opposite vertices of the $(S-C)$ *are the* $\binom{n+1}{2}$ *pairs of the centres of similitude of the associated* $n+1$ *hyperspheres* (A'_i) *of* (A_i) *and its primes are their* 2^n *primes of similitude (Theorem* 6).

The join of H *to the centre of* (S') *is normal to the Newtonian prime* [6] *of* (A'_i) *(Theorem* 7).

Each pair of complementary hyperspheres of intersection of (A_i) *is orthogonal to half of the* $n(n+1)$ *hyperspheres of antisimilitude of* (A'_i) *with centres in the radical prime of the complementary hyperspheres considered, and every* (A_i) *is orthogonal to* $n(n-1)$ *of them with centres in the radical prime of* (H) *and the hypersphere considered (Theorem* 8).

If S, S' *be in perspective with the prime of perspectivity* u *the same as that of a pair of complementary simplexes of intersection* I, I' *of* (A_i), *then:* u *is one of the* 2^n *primes of similitude of* (A_i); I, I' *are both isodynamic with their Lemoine primes* [12] *coinciding with* u; *the* $\binom{n+1}{2}$ *hyperspheres of antisimilitude of* (A_i) *with centres in* u *coincide with those of* (A'_i) *and form the common Neuberg hyperspheres of* I, I'; *the Lemoine points of* I, I' *lie on their common Brocard diameter normal to* u *and meeting their* $2(n+1)$ *altitudes (Theorem* 9).

If a simplex of intersection I *of* (A_i) *be isodynamic, the same is also true of its complementary one* I'; *their Lemone primes are then coaxal with their prime of perspectivity* u *and may coincide with* u; *in this case* S, S' *are in perspective with the same prime of perspectivity as* u *(Theorem* 10).

If 2 *non-complementary simplexes of intersection of* (A_i) *be isodynamic in such a way that their Lemoine primes coincide respectively with those of their complementary ones, the same is true of every such pair of complementary simplexes. Consequently* (A_i) *are mutually orthogonal such that: the* 2^n *Brocard diameters of the* 2^n *pairs of complementary simplexes of intersection of* (A_i) concur *at* H; *the* 2^n *primes of the* $(S-C)$ *of intersection of* (A_i) *coincide with their* 2^n *primes of similitude; the radical prime of* (H) *and the circumhypersphere* (S) *of* S *(*S, S' *coincide here) coincide with the Newtonian prime of* (A_i); *each pair of hyperspheres of intersection of* (A_i) *is orthogonal to* $\binom{n+1}{2}$ *of their* $n(n+1)$ *hyperspheres of antisimilitude with centres in the radical prime* *of the complementary hyperspheres considered, and every* (A_i) *is orthogonal to* $n(n-1)$ *hyperspheres of antisimilitude with centres in the radical prime of* (H) *and* (A_i) *considered.* (*Theorem* 11).

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References

[1] Baker, H. F, Principles of Geometry, vol. 4, Cambridge, 1940.
[2] Court, N. A., Modern pure solid geometry, New York, 1935.
[3] Coxeter, H. S. M., Regular Polytopes, London, 1948.
[4] Mandan, S. R., Hypercones through 2 quadrics with a common conic, *Jour. Lahore Phil. Soc.* 8 (1946), 59.
[5] Mandan, S. R., Umbilical projection, *Proc. Indian Acad. Sci.* 28A (1948), 166-172.
[6] Mandan, S. R., An *S*-Configuration in Euclidean & elliptic *n*-space, *Canadian J. Math.* 10 (1958), 489—501.
[7] Mandan, S. R., Altitudes of a simplex in 4-space, *Bull. Calcutta Math. Soc.* Supp. 50 (1958), 8—20.
[8] Mandan S. R., Harmonic inversion, *Math. Mag.* 32 (1959) 71—78.
[9] Mandan, S. R., On four intersecting spheres, *Jour. Indian Math. Soc.* 23 (1959), 151—167.
[10] Mandan, S. R., Polarity for a quadric in *n*-space, *Rev. Fac. Scs.* Univ. Istanbul, Series A 24 (1959), 21—40.
[11] Mandan, S. R., Cevian simplexes, *Proc. Amer. Math. Soc.* 11 (1960), 837—845.
[12] Mandan, S. R., Isodynamic & Isogonic simplexes, *Annali Mat. pura appl.*, Series (4) 53 (1961), 45—56.
[13] Mandan, S. R., Semi-inverse simplexes, *Jour. Indian Math. Soc.* 25 (1961), 163—171.
[14] Mandan, S. R., Orthogonal hyperspheres, *Acta Math.* (Hungary) 13 (1962), 25—34.
[15] Mandan, S. R., Polarity for a simplex (in press).
[16] Salmon, G., Analytical Geometry of three dimensions, New York, 1927.
[17] Salmon, G., Higher plane curves, New York, 1934.
[18] Semple, J. G. & Roth, L., Algebraic Geometry, Oxford, 1949.
[19] Somerville, D. M. Y., Analytical conics, London, 1949.

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