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ON AUTOMORPHISM GROUPS OF DIVISIBLE DESIGNS

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0. Introduction. A (group) divisible design is a tactical configuration for which the v points are split into m classes of n each, such that points have joining number λ_1 (resp. λ_2) if and only if they are in the same (resp. in different) classes. We are interested in such designs with a nice automorphism group. We first investigate divisible designs with equally many points and blocks admitting an automorphism group acting regularly on all points and on all blocks, i.e., with a Singer group (Singer [**50**] obtained the first result in this direction for the finite projective spaces).

As in the case of block designs, one may expect a divisible design with a Singer group to be equivalent to some sort of difference set; as it turns out, one here obtains a generalisation of the relative difference sets of Butson and Elliott [**11**] and [**20**]. The Singer group will have a normal subgroup acting regularly on each point class if and only if each of its elements either fixes no or all point classes. In this case, the dual of the divisible design is also divisible with the same parameters and the same Singer group. These results are obtained in Section 2, after reviewing some properties of divisible designs in Section 1. Sections 3, 4 and 5 give constructions of divisible designs with Singer groups, using e.g. affine geometry, difference sets and uniform Hjelmslev matrices. In particular it is shown that the examples obtained from affine spaces by discarding either all hyperplanes parallel to a given line or one point and all hyperplanes through it have a Singer group. In Sections 6 and 7 we study divisible designs with $\lambda_1 = 0$ (but not necessarily $b = v$) admitting an automorphism group acting regularly on each point class; these are equivalent to "partial difference matrices" which generalise difference matrices, (generalised) Hadamard matrices and (generalised) balanced weighing matrices. This approach is in analogy to the one taken in the case of transversal designs by the author in [**30**]. Finally, we introduce semisymmetic designs in Section 8, generalising the semibiplanes of Hughes [**25**]. We prove some basic results and apply the constructions of the previous sections to obtain classes of examples.

1. Divisible designs. Divisible designs have first been studied in [**6**]. In this section, we repeat the necessary definitions and some known results on such structures.

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1.1 *Definition.* An incidence structure $\Delta = (\mathscr{P}, \mathscr{B}, I)$ is called a *divisible design* with parameters n, m, k, λ_1 and λ_2 if the following conditions are satisfied:

(1.1) The point set $\mathscr{P}$ is split into m classes of n points each. One writes $p \sim q$ if p and q are in the same class.

(1.2) For distinct points p, q, we have

$$[p, q] = \begin{cases} \lambda_1 & \text{if } p \sim q \\ \lambda_2 & \text{if } p \nsim q \end{cases}.$$

(Here $[p, q]$ denotes the number of blocks through p and q as in [**16**].)

(1.3) Each block contains exactly k points.

The $\sim$-classes are called the *point classes.* If $b := |B| = mn =: v$, then Δ is called *square.* If furthermore the dual structure Δ^* of Δ is also a divisible design with parameters n, m, k, λ_1 and λ_2, then Δ is called *symmetric.*

We warn the reader that what we have called "square" is often called "symmetric" in the literature; "symmetric" in our sense then is called "with the dual property". It is immediate from Definition 1.1 that a divisible design is a block design if and only if either $n = 1$ or $\lambda_1 = \lambda_2$. We will assume always $n \neq 1$ and $\lambda_1 \neq \lambda_2$ unless the contrary is stated. Such divisible designs may be called *proper.* In the literature, one often encounters the term "group" instead of "point class" and the term "group divisible design"; we avoid this terminology, as there is no connection to the algebraic group notion and as we will deal with groups acting on the point classes.

1.2 PROPOSITION (Bose and Connor). *Let Δ be a divisible design with parameters n, m, k, λ_1 and λ_2. Then:*

(1.4) *Each point is on exactly r blocks, where*

$$r(k - 1) = n(m - 1)\lambda_2 + (n - 1)\lambda_1;$$

(1.5) $nmr = bk;$

(1.6) $rk \geqq nm\lambda_2.$

Proof. (1.4) is essentially proved in [**6**], (3.1) and follows in fact by counting all flags (q, B) with p I B for a fixed point p in two ways. So Δ is a tactical configuration, which implies (1.5). (1.6) is [**6**], Theorem 3.

1.3 *Definition.* A divisible design with parameters n, m, k, λ_1 and λ_2 is called *singular*, if $r = \lambda_1$; *semi-regular*, if $r > \lambda_1$ and $rk = nm\lambda_2$; and *regular* if $r > \lambda_1$ and $rk > nm\lambda_2$.

This is the classification of Bose and Connor [**6**], where it is shown that singular divisible designs are equivalent to block designs (where each

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point is taken n times) (see [**6**], Theorem 2). As this case is rather uninteresting, we assume for the rest of the paper that all divisible designs considered are non-singular. One then has:

1.4 PROPOSITION (Bose, Connor, Agrawal). *Let Δ be a semiregular divisible design with parameters n, m, k, λ_1 and λ_2. Then:*

(1.7) *m divides k and each block meets each point class precisely $c := k/m$ times.*

(1.8) $\lambda_2 > \lambda_1$;

(1.9) $b \geqq v - m + 1$;

(1.10) *Two distinct blocks have at most*

$$\frac{2r(k-1) + \lambda_1}{b} - k + r - \lambda_1$$

points in common.

Proof. (1.7) and (1.9) are [**6**], Theorems 4 and 5; (1.8) is in [**13**], Lemma 7.1; and (1.10) is due to Agrawal, see [**43**], Theorem 8.5.9. The case $c = 1$ is of particular interest: it is equivalent to $\lambda_1 = 0$.

1.5 *Definition.* A semiregular divisible design with parameters $n, m, k, 0$ and λ_2 (so $c = 1, m = k$) is called a *transversal design* with parameters n, k and $\lambda = \lambda_2$. (The dual structure then is an $(n, k; \lambda)$-net in the sense of [**19**].) A symmetric transversal design with parameters $n, k = n\lambda$ and λ (the value for k follows from (1.4)) is denoted an ST (n, λ).

1.6 PROPOSITION. *Let Δ be a transversal design with parameters n, k and λ. Then:*

(1.11) $k \leqq (n^2\lambda - 1)/(n - 1)$

with equality if and only if Δ is the dual of an affine design.

Assume furthermore that Δ is resolvable, i.e., there is a partition of the block set into parallel classes such that each point is on precisely one line of each parallel class. Then

(1.12) $k \leqq n\lambda$

with equality if and only if Δ is symmetric.

These results have been proved independently by many authors (cf. the survey paper on affine designs and symmetric transversal designs by Mavron [**38**], who uses the term "hypernet" instead of "symmetric transversal design"). The inequalities are in fact consequences of [**6**], Theorem 5 and Corollary. A proof of (1.11) may be found in [**19**], Section 5 and one of (1.12) in [**30**], Section 3.

1.7 PROPOSITION (Bose, Connor). *Let Δ be a regular divisible design with parameters n, m, k, λ_1 and λ_2. Then:*

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(1.13) *$r \geqq k$ (equivalently, $b \geqq v$);*

(1.14) *If Δ is square, then $[B, C] \leqq \max\{\lambda_1, \lambda_2\}$ for all $B \neq C$.*

(1.15) *If Δ is square and $(k^2 - nm\lambda_2, \lambda_1 - \lambda_2) = 1$, then Δ is symmetric.*

Proof. (1.13) is [**6**], Theorem 6 and (1.14) and (1.15) are [**13**], Theorems 5.1 and 6.2.

1.8 PROPOSITION. (Bose and Connor). *Let Δ be a square regular divisible design with parameters n, m, k, λ_1 and λ_2.*

(1.16) *If m is even, then k^2-$nm\lambda_2$ is a square. If furthermore $m \equiv 2 \bmod 4$, then $k - \lambda_1$ is the sum of two squares.*

(1.17) *If m is odd and n is even, then $k - \lambda_1$ is a square and the equation*

$$(k^2 - mn\lambda_2)x^2 + (-1)^{m(m-1)/2}n\lambda_2 y^2 = z^2$$

has a non-trivial solution in integers x, y, z.

(1.18) *If both m and n are odd, then the equation*

$$(k - \lambda_1)x^2 + (-1)^{n(n-1)/2}ny^2 = z^2$$

has a nontrivial solution in integers x, y, z.

Proof. This is [**6**], Theorem 9 (cf. also [**16**], 7.1.13).

In the square case with $\lambda_1 = 0$, there is a further non-existence result:

1.9 PROPOSITION (Bose, Mielants). *Let Δ be a square divisible design with parameters n, m, k, 0 and λ_2. Then there exists a symmetric block design with parameters m, $m - k$, $m - 2k + n\lambda_2$ and*

(1.19) $n\lambda_2 \neq k, k - 1 \Rightarrow n\lambda_2 \leqq k - \sqrt{k}$.

Proof. The first assertion is in [**5**] Theorem 3.2 as well as in [**41**] where (1.19) has been obtained as a consequence of the first assertion.

Further results on the intersection structure of square divisible designs with $\lambda_1 = 0$ may be found in [**35**] including another proof of Bose's result 1.9. We finally mention a few more basic results which do not seem to be in the literature.

1.10 LEMMA. *A square divisible design has no repeated blocks.*

Proof. If Δ is regular, then the assertion is clear from (1.14), as $\lambda_1 < r = k$ and $\lambda_2 < k$ from $k^2 > nm\lambda_2$. Now assume that Δ is semi-regular, i.e., $k^2 = nm\lambda_2$. Using (1.7) and counting all flags (q, B) where BIp for a fixed point p and where q is from a point class not containing p, we get $\lambda_2 = kc/n$; then from (1.4)

$$\lambda_1 = (n\lambda_2 - k)/(n - 1) = k(c - 1)/(n - 1).$$

Using $k^2 = nm\lambda_2$, it is easy to see from (1.10) that two distinct blocks

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intersect in less than $2\lambda_2 - \lambda_1$ points. But

$$2\lambda_2 - \lambda_1 = k(cn - 2c + n)/n(n - 1)$$

and trivially $cn - 2c + n < n(n - 1)$, as $c < n$.

1.11 LEMMA. *Let α be an automorphism of a square regular divisible design Δ. Then α has as many fixed points as fixed blocks.*

Proof. Let N be an incidence matrix for Δ. As Δ is regular, N is non-singular by the results of Bose and Connor (see [**6**], (3.7)). Now α induces permutation matrices P, Q with $PNQ = N$ and thus $P = NQ^{-1}N^{-1}$. Hence trace P = trace Q; but trace P is the number of fixed points of α and trace Q is the number of fixed blocks of α.

This is in fact a standard argument due to E. T. Parker who first used it for symmetric designs. Using 1.11 and Burnside's lemma on the number of orbits of a permutation group (see e.g. [**27**], Result 1.14) we obtain at once

1.12 PROPOSITION. *Let G be an automorphism group of a square regular divisible design. Then the number of orbits of G on the point set equals that of orbits on the block set.*

1.13 COROLLARY. *Let G be an automorphism group of a square regular divisible design. Then G acts regularly on the point set if and only if it acts regularly on the block set.*

2. Singer groups and relative difference sets. In this Section we consider square divisible designs admitting a Singer group and investigate the corresponding difference method. It may be remarked that Bose, Shrikhande and Bhattacharya [**7**] were the first to use difference methods for the construction of divisible designs.

2.1 *Definition.* Let Δ be a square divisible design and let G be an automorphism group of Δ acting regularly on both the point and block sets of Δ. Then G is called a *Singer group* for Δ. If G is abelian or cyclic, then Δ is also called *abelian* or *cyclic.*

Remark. We would like to call a divisible design with a Singer group "regular", but refrain from doing so as this would collide with the well-established use of "regular" as in Definition 1.3.

In view of Corollary 1.13, it is sufficient to postulate one of the two conditions in Definition 2.1 provided that Δ is regular. Note that Corollary 1.13 fails if Δ is not regular; a counterexample is obtained by considering the symmetric transversal design obtained from an affine translation plane by discarding a parallel class of lines and taking G to be the translation group (and similarly for affine spaces).

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2.2 PROPOSITION. *Let Δ be a square divisible design with a Singer group G and choose a point p of Δ. Then $N := \{g \in G: p^g \sim p\}$ is a subgroup of G. Furthermore, the point class of the point $q = p^h$ is p^{N+h}. Also, N is normal in G if and only if each $g \in G$ satisfies the following condition:*

(2.1) $\quad x \sim x^g$ *for some point x implies* $y \sim y^g$ *for all points y.*

Proof. As $\lambda_1 \neq \lambda_2$ and as the point classes are defined by the joining number λ_1, it is clear that every automorphism of Δ respects the relation $\sim$. Thus N is a subgroup of G. Now let $q = p^h$ and $n \in N$; then $p \sim p^n$ implies $q = p^h \sim p^{n+h}$; from the regularity of G we see that the class of q is precisely p^{N+h}. Then the group N_q defined in analogy to $N_p = N$ is obviously

(2.2) $\quad N_q = -h + N + h.$

Now if (2.1) holds, we must have $N_q = N$ because of the regularity of G; thus N is normal in G in this case by (2.2). Conversely, if N is normal in G, then $N_q = N$ for all q which implies (2.1).

We remark that (2.1) is automatically satisfied if G is abelian, as then any subgroup is normal. In this case N is uniquely determined, and we will use the symbol N then always in this sense. If a non-abelian Singer group nevertheless satisfies (2.1), it will be called *normal*. The divisible design Δ then will also be called *normal*. We will exhibit some examples not satisfying (2.1) in Section 4. Our next goal is to show the equivalence of divisible designs with a Singer group and of a generalisation of the relative difference sets of Butson and Elliott.

2.3 *Definition*. Let G be a group of order nm and let N be a subgroup of G of order n. Then a k-subset D of G is called a *relative difference set* with parameters n, m, k, λ_1 and λ_2 (relative to N) or briefly an $(n, m, k, \lambda_1, \lambda_2)$-RDS, provided that

(2.3) The differences $d - d'$ $(d, d' \in D, d \neq d')$ contain each element of N (excepting 0) precisely λ_1 times and each element of $G \backslash N$ exactly λ_2 times.

Elliott and Butson [**20**] have considered the special case of a normal subgroup N and of $\lambda_1 = 0$. This had earlier been introduced for cyclic groups G by Butson [**11**]. Before proving the announced equivalence result, we will give some examples:

2.4 *Examples*. (i) $\{0, 1\}$ is a $(2, 2, 2, 0, 1)$-RDS in $\mathbf{Z}_4$.

(ii) $\{1, i, j, k\}$ is a $(2, 4, 4, 0, 2)$-RDS in the quaternion group.

(iii) $\{0, 1, 4, 6\}$ is a $(2, 7, 4, 0, 1)$-RDS in $\mathbf{Z}_{14}$.

(iv) $\{0, 1, 2, 4, 9\}$ is a $(2, 6, 5, 0, 2)$-RDS in $\mathbf{Z}_{12}$.

(v) $\{(0, 0), (1, 0), (0, 1), (1, 1), (1, 2), (2, 2)\}$ is a $(3, 3, 6, 3, 4)$-RDS in EA (9).

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(vi) $\{(0, 0), (0, 1), (1, 0), (1, a), (3, 0), (3, 1 + a)\}$ is a (4, 7, 6, 2, 1)-RDS in $\mathbf{Z}_7 \oplus$ EA (4).

(vii) $\{(0, 0), (a, 0), (0, 1), (1, 1), (0, 2), (1 + a, 2)\}$ is a (4, 3, 6, 2, 3)-RDS in EA (4) $\oplus \mathbf{Z}_3$.

Here (and in the remainder of the paper) $\mathbf{Z}_w$ denotes the cyclic group of residues modulo w, and EA (q) denotes the elementary abelian group of order q (for prime powers q).

We note the following equation which follows trivially from counting the differences arising from D:

2.5 LEMMA. *Let D be an $(n, m, k, \lambda_1, \lambda_2)$-RDS in G. Then*

$$(2.4) \qquad k(k - 1) = n(m - 1)\lambda_2 + (n - 1)\lambda_1.$$

2.6 *Definition.* Let G be a group and D a subset of G. Then the incidence structure

$$\text{dev } D := (G, \{D + g: g \in G\}, \in)$$

is called the *development* of D.

2.7 THEOREM. *Let G be a group of order nm and D a k-subset of G. Then* dev D *is a divisible design with parameters n, m, k, λ_1 and λ_2 admitting G as a Singer group if and only if D is an $(n, m, k, \lambda_1, \lambda_2)$*-RDS *in G relative to some subgroup N of G of order n. Furthermore, all divisible designs with a Singer group may be represented in this way.*

Proof. Clearly dev D has constant block size k. Consider any two distinct elements g, g' of G and a block $D + x$ of dev D. Then $g, g' \in D + x$ if and only if $g = d + x$ and $g' = d' + x$ for some $d, d' \in D$, i.e., if and only if $g - g' = d - d'$ and $g = d + x$. Thus we have shown

$$(2.5) \qquad [g, g'] = |\{(d, d'): d, d' \in D, g - g' = d - d'\}|.$$

Now assume that D is an $(n, m, k, \lambda_1, \lambda_2)$-difference set in G. Take as point classes of dev D the right cosets $N + x$. Then by (2.5) and the definition of a relative difference set, we have $[g, g'] = \lambda_2$ if and only if $g - g' \in G\backslash N$ and

$$[g, g'] = \lambda_1 \Leftrightarrow g - g' \in N \Leftrightarrow g \sim g'.$$

So dev D is a divisible design with the desired parameters. Also, G acts regularly on the point set of dev D by defining the action of $g \in G$ by

$$(2.6) \qquad x \mapsto x + g, D + x \mapsto D + x + g.$$

(So N is the group determined from the point 0 according to Proposition 2.2.) If we consider blocks $D + x$, $D + y$ as distinct if and only if $x \neq y$, then G is obviously also regular on the blocks of dev D and thus a Singer group. Lemma 1.10 then shows that in fact distinct blocks are also distinct as point sets. Conversely, let Δ be any divisible design with parameters

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n, m, k, λ_1 and λ_2 admitting G as a Singer group. Choose a "base point" p; then any point q of Δ may be uniquely coordinatised as $q = p^g$, $g \in G$, as G acts regularly on the point set of Δ. Similarly, any block C may be uniquely coordinatised as $C = B^h$, $h \in G$, where B is a fixed "base block". Let D be the set of coordinates of all points q of B and let N be the subgroup of order n of G, determined by the point p as in Proposition 2.2 so that the point classes of Δ are the right cosets $N + x$ of N. Also G acts on $\Delta \cong \text{dev } D$ as described in (2.6). Then from (2.5) and the properties of a divisible design, we at once obtain that D is an $(n, m, k, \lambda_1, \lambda_2)$-RDS in G relative to N.

Our next aim is to show that divisible designs with a normal Singer group are symmetric. We first prove

2.8 PROPOSITION. *Let D be an $(n, m, k, \lambda_1, \lambda_2)$-difference set in G relative to the normal subgroup N. Then D also satisfies*

(2.7) *The differences $-d + d'$ $(d, d' \in D, d \neq d')$ contain each element of N (excepting 0) exactly λ_1 times and each element of $G \backslash N$ exactly λ_2 times.*

Proof. Consider the group algebra $\mathbf{Q}G$ of G over the field of rational numbers, where we write G multiplicatively for notational convenience. Also, put

$$S := \sum_{d \in D} d,\ S^* := \sum_{d \in D} d^{-1},\ A := \sum_{g \in G} g \quad \text{and} \quad B := \sum_{n \in N} n.$$

With these notations, the condition (2.3) for an RDS reads

(2.8) $$SS^* = k + \lambda_2 A + (\lambda_1 - \lambda_2)B =: C;$$

similarly, the validity of (2.7) is equivalent to

(2.9) $$S^*S = C.$$

As N is a normal subgroup of G, it is easy to see that C is in the center of $\mathbf{Q}G$. We try to find an inverse of the form $x + yA + zB$ of C in $\mathbf{Q}G$. This yields the equation

(2.10) $$(x + yA + zB)C = xk + (x\lambda_2 + yk + ynm\lambda_2 + yn(\lambda_1 - \lambda_2) + zn\lambda_2)A + (x(\lambda_1 - \lambda_2) + zk + (\lambda_1 - \lambda_2)zn)B = 1.$$

Now (2.10) has a (unique) solution provided that

$$k + n(\lambda_1 - \lambda_2) \neq 0 \quad \text{and} \quad k + nm\lambda_2 + n(\lambda_1 - \lambda_2) \neq 0.$$

The last term is always $\neq 0$, as it equals $k^2 + \lambda_1$ by (2.4); again by (2.4) $k + n\lambda_1 - n\lambda_2 = 0$ is equivalent to $k^2 = nm\lambda_2 - \lambda_1$. But by Theorem 2.7 and (1.6), $k^2 \geqq nm\lambda_2$. Thus if $k + n\lambda_1 - n\lambda_2 = 0$, then

(2.11) $$\lambda_1 = 0,\ k = n\lambda_2,\ k^2 = nm\lambda_2.$$

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We first consider the case $k + n\lambda_1 - n\lambda_2 \neq 0$. Then C is invertible and thus from (2.8) S is invertible too and in fact $S^* = S^{-1}C$. (Note that left inverses and right inverses coincide in $\mathbf{Q}G$.) But then $S^*S = S^{-1}CS = C$ as C is in the center of $\mathbf{Q}G$, which proves the assertion (2.9). It remains to consider the case (2.11). Then dev D is a transversal design admitting G as Singer group by Theorem 2.7. We want to show that dev D is symmetric, in this case. As dev D is square, it is sufficient to show that it is resolvable because of (1.12). In fact the block orbits of dev D under N define a resolution of dev D: Clearly no two blocks in the same orbit can intersect as they both meet the same point classes exactly once each (because $\lambda_1 = 0$) and as G acts regularly on the point set of dev D. But then each point is on a block in each orbit for reasons of cardinality. Thus two blocks $D + x$ and $D + y$ intersect exactly λ_2 times if and only if $x - y \in G\backslash N$ and not at all otherwise. The assertion (2.7) now follows from

$$(2.12) \quad [D + x, D + y] = |\{(d, d'): d, d' \in D, -d + d' = x - y\}|$$

which is proved similarly as (2.5).

2.9 COROLLARY. *Let G be a group of order nm, N a normal subgroup of order n and D a subset of G. Then conditions* (2.3) *and* (2.7) *are equivalent.*

These results are trivial if G is abelian. For the case $n = 1$, they have been proved (in the non-abelian case) by Bruck [**9**] whose proof in fact inspired our arguments for Proposition 2.8. Using Proposition 2.8 and Theorem 2.7, we now immediately obtain

2.10 THEOREM. *Any divisible design Δ with a normal Singer group G is symmetric. Furthermore, the normal subgroup N of G acting regularly on each point class of Δ also acts regularly on each block class of Δ.*

We remark that Theorem 2.10 does not in general follow from Connor's result (1.15) which demands that Δ is regular and also satisfies the condition

$$(k^2 - nm\lambda_2, \lambda_1 - \lambda_2) = 1.$$

It would be interesting to decide whether or not the hypothesis of normality in Theorem 2.10 is essential; the author knows no examples of non-symmetric divisible designs with a Singer group. We now list some consequences of Theorems 2.7 and 2.10 and the results of Section 1.

2.11 COROLLARY. *The existence of a (symmetric) divisible design with parameters n, m, k, λ_1, λ_2 admitting a normal Singer group is equivalent to that of an* $(n, m, k, \lambda_1, \lambda_2)$-RDS *relative to a normal subgroup. In particular, the existence of an* ST (n, λ) *admitting a normal Singer group is equivalent to that of an* $(n, n\lambda, n\lambda, 0, \lambda)$-RDS *relative to a normal subgroup.*

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2.12 COROLLARY. *Let D be an $(n, m, k, 0, \lambda)$-RDS relative to a normal subgroup. Then*:

(2.13) $n\lambda \neq k, k - 1 \Rightarrow n\lambda \leqq k - \sqrt{k}$

and if $k^2 > nm\lambda$, then (1.16) *to* (1.18) *hold.*

We will also require the following result in the case $\lambda_1 = 0$:

2.13 PROPOSITION (Elliott and Butson). *Let D be an $(n, m, k, 0, \lambda)$-RDS in G relative to N. Then*

(2.14) $k \leqq m$;

also, if M is any normal subgroup of G contained in N, then $\{d + M: d \in D\}$ is an $(n/s, m, k, 0, \lambda s)$-RDS in G/M relative to N/M, where s is the order of M. In particular, if N is normal in G, then there exists an ordinary difference set with parameters $m, k, n\lambda$ in G/N.

Proof. These results are (in the case of normal subgroups N) in [**20**], Section 2. (2.14) follows, as the elements of D have to belong to mutually distinct cosets of N (otherwise we would obtain elements of N as differences from D). The same argument is used in proving the remaining assertion.

Proposition 2.13 gives further restrictions on the existence of relative difference sets with $\lambda_1 = 0$. It might be mentioned here that Shrikhande [**48**] has obtained existence conditions for cyclic relative difference sets (with arbitrary λ_1 but only in the regular case) and that Elliott and Butson [**20**] have obtained multiplier theorems which also give necessary existence conditions (only for $\lambda_1 = 0$). A recent paper on multiplier theorems is [**36**]. We will not make use of any multiplier results in this paper.

2.14 COROLLARY. *Let Δ be a (symmetric) divisible design with parameters $n, m, k, 0, \lambda$ admitting a normal Singer group G (with normal subgroup N acting on the point classes). Furthermore let M be any normal subgroup of G contained in N. Then Δ admits an epimorphism α onto a symmetric divisible design with parameters $n/s, m, k, 0, \lambda s$ (where s is the order of M) having G/M as a Singer group. Also, α satisfies*

(2.15) $p \sim q \Rightarrow p^\alpha \sim q^\alpha$ *for all points p, q (and dually).*

In particular, Δ is always the pre-image of a symmetric block design admitting a Singer group.

We finally mention special cases of two well-known results (see e.g. [**16**], 1.2.13 and 1.2.14).

2.15 PROPOSITION. *Any abelian divisible design admits a polarity.*

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2.16 PROPOSITION. *Let Δ be a divisible design admitting a Singer group, represented in the form* dev D *as in Theorem* 2.7. *Then a permutation* $x \mapsto x^\varphi$ *of the points of* Δ *is contained in the normaliser of* G *in the full automorphism group of* Δ *if and only if there exist* $a, b \in G$ *and an automorphism* α *of* G *satisfying*

$$(2.16) \quad D^\alpha = -a + D + b$$

and

$$(2.17) \quad p^\varphi = -a + p^\alpha, \ (D + x)^\varphi = D + b + x^\alpha \text{ for all } p, x \in G.$$

In fact, the proof of Bruck [9], Theorem 3.1 may be immediately adapted to the situation considered here. It is natural to call α a "multiplier" for D, in analogy to the case for symmetric block designs. As already mentioned, Elliott and Butson [20] have in fact studied "numerical multipliers" of relative difference sets. We will not make use of this concept in this paper.

In the following three Sections, we will show that various classes of divisible designs admit Singer groups and thus construct examples of relative difference sets (or vice versa). We will first consider the case $\lambda_1 = 0$ in Section 3 (for transversal designs) and in Section 4 (for $k \neq n\lambda_2$). Constructions for $\lambda_1 \neq 0$ will be given in Section 5. Only some of these examples of relative difference sets will be new, but in the remaining cases the proofs are (in our opinion) considerably simpler.

A relative difference set with parameters $n, m, k, \lambda_1 = 0$ and $\lambda_2 = \lambda$ will henceforth simply be called an (n, m, k, λ)-RDS, omitting the parameter which is 0.

3. Constructions for $\lambda_1 = 0$, $k = n\lambda$. By the results of Section 2, relative difference sets with $\lambda_1 = 0$, $k = n\lambda$ correspond to symmetric transversal designs ST (n, λ) and have $m = k$. Examples 2.4 (i) and (ii) are of this type. We give one more non-abelian example, in fact a pre-image of Example 2.4 (ii): $\{(0, 1), (0, i), (0, j), (1, k)\}$ is a (4, 4, 4, 1)-RDS in the direct sum of $\mathbf{Z}_2$ with the quaternion group relative to $N = \mathbf{Z}_2 \oplus \{1, -1\}$. We will obtain more non-abelian examples below. Of the examples given up to now, only the (2, 2, 2, 1)-RDS was in a cyclic group. This is no coincidence:

3.1 PROPOSITION (Elliott and Butson). *The only* $(n, n\lambda, n\lambda, \lambda)$-RDS *in a cyclic group G is the* (2, 2, 2, 1)-RDS.

Proof. See [20], Theorem 6.2. The case $\lambda = 1$ is already in [23].

We now start with the case $\lambda = 1$, i.e., with relative difference sets corresponding to affine planes with one parallel class discarded. (It is

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easy to see that any ST $(n, 1)$ can be extended to a unique affine plane of order n by considering the point classes as new lines.)

3.2 THEOREM. *Let* Π *be any affine plane coordinatized by a division ring* R *of order* n. *Then the* ST $(n, 1)$ Σ *obtained from* Π *by discarding one parallel class of lines of* Π *admits a normal Singer group* G. G *is abelian if and only if the division ring* R *is commutative.*

Proof. We may assume Π to be defined on the point set (x, y), $x, y \in R$, with lines

$$[m, k] = \{(x, y): mx + y = k\} \ (m, k \in R) \quad \text{and} \quad [k] = \{(k, y): y \in R\}$$

(see e.g. [**27**], Chapters V and VI). We may also assume without loss of generality that the parallel class discarded consists of the lines $[k] (k \in R)$. For $(a, b) \in R$, consider the mapping α_{ab} with

$$\alpha_{ab}: (x, y) \mapsto (x + a, y + ax + b).$$

α_{ab} is an automorphism of Σ as

$$\alpha_{ab}: [m, k] \mapsto [m - a, k + ma + b - a^2].$$

It is easily checked that

$$\alpha_{ab}\alpha_{a'b'} = \alpha_{a+a', b+b'+a'a};$$

thus all $\alpha_{ab}(a, b \in R)$ form a group G which obviously is abelian if and only if R is commutative. It is trivial to check that G operates regularly on both the point and block sets of Σ. Also, the α_{ab} satisfy (2.1) as α_{ab}: $[k] \mapsto [k + a]$. Thus G is a normal Singer group for Δ.

The division ring property is used in proving that α_{ab} is an automorphism; here both distributive laws are needed. Theorem 3.2 is essentially due to Hughes who gave this construction in [**24**] from the point of view of quasiregular collineation groups of projective planes. The result is for two reasons particularly interesting. First, there is no affine plane with a cyclic collineation group acting regularly on all points (by [**23**]) excepting the plane of order 2; but any plane over a commutative division ring admits a group regular on the points and regular on all lines except for those of one parallel class. (Omitting the last condition, this holds of course for every translation plane.)

Secondly, it is widely conjectured that the only projective resp. affine planes admitting a Singer group (i.e., those corresponding to a difference set resp. an affine difference set) are the desarguesian ones. Theorem 3.2 shows that the corresponding conjecture for symmetric transversal designs with $\lambda = 1$ is not true. We now use Theorem 3.2 to obtain relative difference sets.

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3.3 THEOREM. *Let q be a prime power. Then there exists a* $(q, q, q, 1)$-RDS *in G, where $G =$* EA (q^2) *is the elementary abelian group of order q^2 if q is odd and $G = \mathbf{Z}_4 \oplus \ldots \oplus \mathbf{Z}_4$ if q is even.*

Proof. We consider the desarguesian affine plane of order q in Theorem 3.2, i.e., we take $R = GF(q)$. By Corollary 2.11 and the proof of Theorem 3.2, we obtain the existence of a $(q, q, q, 1)$-RDS in G, where G is defined on $GF(q) \times GF(q)$ by

$$(3.1) \qquad (a, b) * (a', b') = (a + a', b + b' + a'a).$$

It remains to show that G has the structure asserted in the statement of the theorem. Now it is easily shown by induction on m that

$$(a, b)^m = \left(ma, mb + \frac{m(m-1)}{2} a^2\right) \quad \text{in } (G, *).$$

Thus if $q = p^r$ (where p is a prime) then (a, b) has order p for all $(a, b) \neq (0, 0)$ if p is odd; and for $p = 2$, $(0, b)$ has order 2 for all $b \neq 0$ and order 4 for all (a, b) with $a \neq 0$. This implies the assertions on G.

For $q = p$ or p^2, $p \neq 2$, relative difference sets with these parameters in EA (q^2) have already been obtained in [**20**], Theorems 3.1 and 3.2. The arguments given in the proof of Theorem 3.3 remain valid for any commutative division ring. If $q = p^{2r}$, $r \neq 1$, $p \neq 2$, then there exists a commutative division ring which is not a field (see [**27**], Theorem 9.12). The symmetric transversal designs constructed from such a division ring and from the Galois field are clearly not isomorphic (as the extension to an affine plane is unique and the affine planes are not isomorphic). This gives

3.4 THEOREM. *If $q = p^{2r}$ is a prime power with $r \neq 1$, $p \neq 2$, then there are at least two non-isomorphic symmetric transversal designs* ST $(n, 1)$ *admitting* EA (q^2) *as a Singer group.*

Proof. It only remains to check that the subgroup N is the same in both cases. But from the proof of Theorem 3.2 we see that N is the subgroup of G of all α_{0b} $(b \in R)$ and thus $N =$ EA (q), as the additive groups of $GF(q)$ and of any division ring of order q are isomorphic.

If $q = p^r$, $r \geqq 3$, $p \neq 2$ then there are also non-commutative division rings of order q (see [**16**], 5.3.8). We will not consider the more complex situation for $p = 2$. We thus have

3.5 THEOREM. *If $q = p^r$ is a prime power with $r \geqq 3$, $p \neq 2$, then there exists a normal* $(q, q, q, 1)$-RDS *in a non-abelian group.*

We finally remark that the unique ST $(4, 1)$ (unique as the affine plane of order 4 is unique) admits both an abelian Singer group (i.e., $\mathbf{Z}_4 \oplus \mathbf{Z}_4$

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by Theorems 3.2 and 3.3; explicitly, we may take

$$\{(0, 0), (1, 0), (0, 1), (3, 3)\}$$

as relative difference set) and a non-abelian normal Singer group as shown in the beginning of this section. Thus isomorphic ST's may admit non-isomorphic Singer groups (as in the case of projective planes).

Next, we apply Proposition 2.13 to Theorem 3.3 for $q = p^{i+j}$, p a prime. Then we obtain (taking a normal subgroup of order p^j):

3.6 THEOREM. *There exists a* $(p^i, p^{i+j}, p^{i+j}, p^j)$-RDS *in an abelian group* G *whenever* p *is a prime and* i, j *are non-negative integers. Here, if* $p \neq 2$, $G = \text{EA}\ (p^{2i+j})$; *if* $p = 2$, $G = \text{EA}\ (2^j) \oplus \mathbf{Z}_4 \oplus \ldots \oplus \mathbf{Z}_4$.

Again, for odd p, the cases $i = 1$ and $i = 2$, j even, have already been obtained in [**20**], Theorems 3.1 and 3.2. As for $p = 2$, Elliott and Butson have obtained examples with $i = 1$ and j odd in EA (2^{j+2}) in [**20**], Theorem 4.1. They have also shown (see [**20**], Lemma 6.1.3) that for $p = 2$ no relative difference set with the parameters given above can exist in an elementary abelian group when $i + j$ is odd. Note that this agrees nicely with Theorem 3.3. We will now generalise the construction of [**20**], Theorem 4.1 to obtain further relative difference sets with $k = n\lambda$ and $n = 2$.

3.7 THEOREM. *Assume the existence of an ordinary difference set* D_1 *with parameters*

$$(3.2) \qquad v = 4u^2,\ k = 2u^2 - u,\ \lambda = u^2 - u$$

in a group H. *Then* $D := (\{0\} \times D_1) \cup (\{1\} \times \overline{D_1})$ *is a normal relative difference set with parameters*

$$(3.3) \qquad m = k = n\lambda = 4u^2,\ n = 2,\ \lambda = 2u^2$$

in $G = \mathbf{Z}_2 \oplus H$. *Here* $\overline{D_1}$ *denotes the complement of* D_1.

Proof. First of all it is obvious that (0, 0) and (1, 0) do not occur as differences from D. Also, it is well-known that $\overline{D_1}$ is a difference set with parameters $4u^2$, $2u^2 + u$, $u^2 + u$. Using this fact the assertion is easily checked.

In [**40**], Menon has constructed difference sets with parameters $4u^2$, $2u^2 - u$, $u^2 - u$ for all values $u = 2^s3^r$ with $s \geqq r - 1$ by means of a direct product construction and by exhibiting examples for $u = 1$ and $u = 3$. Applying these results to Theorem 3.7, we in fact obtain

3.8 COROLLARY. *There exist relative difference sets with parameters* (3.3) *in* $G = \mathbf{Z}_2 \oplus H$ *for all values* $u = 2^s3^r$ *with* $s \geqq r - 1$. *Here* H *is the direct product of* r *groups of order* 36 *and of* $s - r + 1$ *groups of order* 4; *each factor*

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of order 36 *may be chosen to be either* $\mathbf{Z}_6 \oplus \mathbf{Z}_6$ *or* $S_3 \oplus S_3$ (*where* S_3 *denotes the symmetric group on* 3 *elements*) *and each factor of order* 4 *may be chosen to be either* $\mathbf{Z}_4$ *or* EA (4).

Corollary 3.8 includes both the case $p = 2$, $i = 1$ of Theorem 3.6 and Theorem 4.1 of [**20**]. We will see later that the case $n = 2$, $k = 2\lambda$ is intimately connected with Hadamard matrices. We conclude this section with some investigations concerning the possibility of relative difference sets with $n = 2$ not having parameters (3.3). We will see in Section 6 that λ has to be even. So we have parameters

(3.4) $\quad n = 2, \lambda = 2a, k = m = n\lambda = 4a$

and G is of order $8a$. We first assume that $G = \mathbf{Z}_2 \oplus H$ where $N = \mathbf{Z}_2 \times \{0\}$. We assert that in this case a is a square and thus the parameters reduce to (3.3). For let t denote the number of elements of D of the form $(1, x)$; then we obtain $2t(4a - t)$ differences with first coordinate 1, which should cover the $4a - 1$ elements $(1, x)$ with $x \neq 0$ each $2a$ times. But this yields $t = 2a \pm \sqrt{a}$ and thus the assertion. In fact it may be shown that the elements of D with first coordinate 0 and those with first coordinate 1 induce a pair of complementary difference sets in H, one of which has parameters $4u^2$, $2u^2 - u$, $u^2 - u$ (if $a = u^2$). Thus we have:

3.9 Proposition. *Assume the existence of a relative difference set D with parameters* (3.4) *in* $G = \mathbf{Z}_2 \oplus H$ *relative to* $N = \mathbf{Z}_2 \times \{0\}$. *Then* $a = u^2$, *so that D has in fact parameters* (3.3), *and either D or $\bar{D}$ can be constructed as in Theorem* 3.7.

We next assume that $G = \mathbf{Z}_4 \oplus H$ and $N = \{(0, 0), (2, 0)\}$. Let x, y, z denote the number of elements of D with first coordinate 0, 1, 2 respectively. Counting all differences with first coordinate 0, 1, 2 we obtain the three equations

$$(3.5) \quad x(x - 1) + y(y - 1) + z(z - 1) + (4a - x - y - z) \times (4a - x - y - z - 1) = 2a(2a - 1);$$

$$(3.6) \quad (4a - x - y - z)(x + z) + xy + yz = 4a^2;$$

$$(3.7) \quad 2xz + 2y(4a - x - y - z) = 2a(2a - 1).$$

Note that the differences with first coordinate 3 again yield (3.6). From (3.6), we get at once $x + z = 2a$. Then (3.5) and (3.7) are equivalent and reduce to

$$(3.8) \quad (x - a)^2 + (y - a)^2 = a.$$

Hence in this case a is the sum of two squares. Thus we have:

3.10 Proposition. *Assume the existence of a relative difference set D with*

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parameters (3.4) *in* $G = \mathbf{Z}_4 \oplus H$ *where* $N = \{(0, 0), (2, 0)\}$. *Then* a *is the sum of two squares and the numbers* x, y, z *of elements of* D *with first coordinates* 0, 1, 2 *respectively satisfy* $x + z = 2a$ *and* (3.8).

The results now obtained suffice to exclude the possibility of abelian relative difference sets with parameters (3.4) for many odd values of a. In this case, G has order $8a$, a odd. As G is abelian, it splits into a group K of order 8 and a group L of order a. Assume that a is squarefree; then L is cyclic and thus K cannot be cyclic too by Proposition 3.1. By Proposition 3.9, we cannot have $K = \mathbf{Z}_2 \oplus \mathbf{Z}_4$ and $N = \mathbf{Z}_2 \times \{0\}$ or $K = \text{EA }(8)$. Thus the only remaining possibility is $K = \mathbf{Z}_2 \oplus \mathbf{Z}_4$ and $N = \{(0,0),(0,2)\} \subset K$. But then a is a sum of two squares by Proposition 3.10. Thus we have shown

3.11 PROPOSITION. *There exists no relative difference set with parameters* (3.4) *in an abelian group whenever* a *is squarefree, odd and not the sum of two squares.*

This excludes for example the values $a = 3, 7, 11, 15, 19, 21$, etc. On the other hand, we know the existence of a relative difference set with parameters (3.4) for $a = 1, 4, 9, 16, 36, \ldots$ by Corollary 3.8. So the smallest open cases are $a = 2$ and $a = 5$. Using Proposition 3.10, one finds a solution for $a = 2$ in $\mathbf{Z}_4 \oplus \mathbf{Z}_4$ relatively easily:

3.12 *Examples.* The following is a (2, 8, 8, 4)-RDS in $\mathbf{Z}_4 \oplus \mathbf{Z}_4$:

$$D = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 3), (3, 3)\}$$

where $N = \{(0, 0), (2, 0)\}$. Similarly,

$$D' = \{(0, 0), (0, x), (0, y), (1, 0), (1, x), (1, y), (2, x + y), (3, x + y)\}$$

is a (2, 8, 8, 4)-RDS in $\mathbf{Z}_4 \oplus \text{EA }(4)$ (with $\text{EA }(4) = \{0, x, y, x + y\}$).

Using Proposition 3.10, it also becomes possible to decide the case $a = 5$ in some hours without the use of a computer. D has to have the form

$$D = (\{0\} \times A) \cup (\{1\} \times B) \cup (\{2\} \times \bar{A}) \cup (\{3\} \times \bar{B})$$

in $\mathbf{Z}_4 \oplus \mathbf{Z}_{10}$ and without loss of generality one may assume $|A| = 7$ and $|B| = 6$. As each difference with first coordinate 0 (except for (0, 0)) has to appear 10 times each, the differences from A plus those from $\bar{A}$ plus those from B plus those from $\bar{B}$ have to contain each $x \neq 0$ ($x \in \mathbf{Z}_{10}$) exactly 10 times. The 120 choices for A are in 12 orbits under $\mathbf{Z}_{10}$ and the 210 choices for B in 20 orbits of 10 and 2 orbits of 5 sets. Clearly the difference distribution only depends on the orbits. Working this out only leaves 10 feasible orbit pairs (A, B) all of which are then seen to be impossible by checking the differences with first coordinate 1. Thus we have

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3.13 PROPOSITION. *There is no abelian relative difference set with parameters* (3.4) *for* $a = 5$.

We now turn our attention to the case $k \neq n\lambda$.

4. Constructions for $\lambda_1 = 0$, $k \neq n\lambda$. In the case $k = n\lambda$, cyclic relative difference sets are ruled out (with the exception of the $(2, 2, 2, 1)$-RDS) by Proposition 3.1. In contrast to this, we will now obtain a large class of cyclic examples for $k \neq n\lambda$. It may be remarked that there are non-existence results for cyclic relative difference sets with $k \neq n\lambda$, too, which go beyond the general conditions of 2.12. These have been obtained by Shrikhande [**48**].

4.1 THEOREM. *Let* Σ *be the affine space* $AG(d, q)$ *and* p *any point of* Σ. *Then the incidence structure* Δ *obtained by removing* p *and all hyperplanes containing* p *from* Σ *is a symmetric divisible design with parameters*

$$n = q - 1, m = (q^d - 1)/(q - 1), k = q^{d-1}, \lambda = q^{d-2} \tag{4.1}$$

admitting a cyclic Singer group G.

Proof. We may assume without loss of generality that $p = 0$. It is well-known that Δ is a divisible design with parameters (4.1); this is easily checked realizing that the point classes are just the lines of Σ through 0. Now the points of Σ may be identified with the elements of the field $K = GF(q^d)$. Let ω be a generating element of K^*; then the bijection $x \mapsto \omega x$ of K onto itself is linear (where K is considered as the d-dimensional vectorspace over its subfield $GF(q)$) and thus induces an automorphism α of Σ of order $q^d - 1$. As α fixes 0 and all hyperplanes through 0, $G = \langle \alpha \rangle$ clearly is an automorphism group of Δ acting regularly on the point set. Now assume that α^t fixes a block B of Δ and let l be the order of α^t. Then $l \mid q^d - 1$; but as G is regular on the set of points, α^t has to permute the points of B in cycles of length l, and therefore also $l \mid q^{d-1}$. This implies $l = 1$ and thus G acts regularly on the blocks of Δ, too. Hence G is a Singer group for Δ.

This proof imitates the proof of Singer's classical theorem on finite projective spaces (see [**50**]). The special case $d = 2$ is due to Bose [**4**]. Using Corollary 2.11, we therefore obtain a series of cyclic difference sets which for $d = 2$ are the affine difference sets of [**4**] which have also been studied by Hoffman [**23**]. The general case has first been obtained by Butson [**11**], Theorem 5.3 (for primes q) and then in general by Elliott and Butson [**20**], Theorem 5.1 with the aid of linear recurring sequences. Recently, Berman has given another proof in [**3**], Theorem 5.2 using ω-circulant generalised weighing matrices.

4.2 THEOREM. *Let* q *be a prime power and* $d \geqq 2$. *Then there exists a cyclic relative difference set with parameters* (4.1).

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Using Proposition 2.13 we have the following result:

4.3 THEOREM. *Let $q = nt + 1$ be a prime power and $n, d \geqq 2$. Then there exists a cyclic relative difference set with parameters*

(4.2) $\quad n, m = (q^d - 1)/(q - 1), k = q^{d-1}, \lambda = tq^{d-2}.$

It is worthwhile to state some individual cases separately.

4.4 COROLLARY. *Let q be an odd prime power. Then there exists a cyclic relative difference set with parameters*

(4.3) $\quad n = 2, m = q + 1, k = q, \lambda = (q - 1)/2.$

Delsarte, Goethals and Seidel have shown in [**15**], Theorem 5.1 that the existence of a cyclic relative difference set with parameters (4.3) is equivalent to that of a "negacyclic" conference matrix. They also conjectured that q always has to be a prime power.

4.5 COROLLARY. *Let q be an odd prime power. Then there exists a cyclic relative difference set with parameters*

(4.4) $\quad n = 2, m = q^2 + q + 1, k = q^2, \lambda = (q^2 - q)/2.$

Divisible designs with these parameters have been obtained by Seberry [**45**] by a considerably more involved method which yields no information on their automorphism group. It has the advantage, though, of working also for powers of 2. A cyclic RDS with parameters (4.4) for $q = 2$ is exhibited in Example 2.4 (iii).

4.6 COROLLARY. *Let $q = nt + 1$ be a prime power. Then there exists a cyclic relative difference set with parameters*

(4.5) $\quad n, m = q + 1, k = q, \lambda = t.$

These difference sets yield generalised conference matrices which will be discussed in Sections 6 and 7.

4.7 COROLLARY. *Let q be a prime power and $d \geqq 2$. Then there exists a cyclic relative difference set with parameters*

(4.6) $\quad n = q + 1, m = (q^{2d} - 1)/(q^2 - 1), k = q^{2d-2}, \lambda = q^{2d-4}(q - 1).$

These are obtained by replacing q by q^2 in Theorem 4.3 and taking $n = q + 1$, $t = q - 1$. In fact, these relative difference sets admit a rather nice interpretation within projective geometry:

4.8 PROPOSITION. *Let q be a prime power, $d \geqq 2$ and consider the projective space $\Pi = PG(2d - 1, q)$. Then Π contains a symmetric divisible design with parameters* (4.6) *as a sub-structure admitting a cyclic Singer group.*

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Proof. First note that mn is the number of points resp. hyperplanes of Π. We will construct the desired design Δ by omitting a suitable $(2d - 3)$-dimensional subspace from each hyperplane. By the theorem of Singer [**50**], $\mathbf{Z}_{mn}$ acts as a collineation group G of Π regularly on both the point and hyperplane sets. As $2d$ is even, Π contains $GF(q^2)$ as a line; from the proof of Singer's theorem it is clear that this line L consists of the images of the point $p = GF(q)$ of Π under the subgroup N of order $q + 1$ of G. Here we consider Π as the collection of subspaces of the vector space $GF(q^{2n})$ over $GF(q)$. Choose any fixed hyperplane H of Π. Then the points and hyperplanes of Π can be uniquely represented in the form p^γ resp. H^γ with $\gamma \in G$. It is well-known that the mapping $p^\gamma \leftrightarrow H^{-\gamma}$ defines a polarity π of Π (see [**16**], 1.2.13); as $L = \{p^\alpha : \alpha \in N\}$ is a line, its image $L^\pi = \{H^\alpha : \alpha \in N\}$ is a pencil of hyperplanes, i.e., the intersection of these hyperplanes is a $(2d - 3)$-dimensional flat U of Π and L^π contains all hyperplanes containing U. For $\gamma \in G$, $L^{\gamma\pi}$ will be the pencil of hyperplanes determined by U^γ. Thus the hyperplanes of Π are partitioned into the m pencils of n each determined by the $(2d - 3)$-dimensional flats U^β with $\beta \in B$, where B is a system of coset representatives of N in G. Now omit U^β from each hyperplane in the corresponding pencil; clearly then hyperplanes of the same pencil do not intersect in the incidence structure Δ thus determined. But hyperplanes from distinct pencils intersect in a $(2d - 3)$-flat W in Π, which intersects the corresponding U^β's in $(2d - 4)$-flats W_1 and W_2, which in turn meet in a $(2d - 5)$-flat. So these hyperplanes will have

$$\frac{q^{2d-2} - 1}{q - 1} - \frac{2(q^{2d-3} - 1)}{q - 1} + \frac{q^{2d-4} - 1}{q - 1} = (q - 1)q^{2d-4} = \lambda$$

points in common (considered in Δ). Hence the dual Δ^* of Δ clearly is a cyclic divisible design with parameters (4.6) and therefore the same holds for Δ itself by Theorem 2.10. (In fact, it is also quite easy to see that the point classes of Δ are the lines $L^\beta (\beta \in B)$ of Π.)

This construction has been used by Berman [**3**], Section 5 in somewhat disguised form to obtain certain generalised weighing matrices; it is not clear from [**3**] whether or not these correspond to relative difference sets. Proposition 4.8 provides a nice interpretation but does not give anything really new compared with the affine space construction. It is in fact not too difficult to see how to obtain the design Δ of Proposition 4.8 from the design Ω constructed from $AG(d, q^2)$ as in Theorem 4.1: Consider both $AG(d, q^2)$ and $PG(2d - 1, q)$ in the vector space $GF(q^{2d})$. Then a block class of Ω consists of the cosets of a $(2d - 2)$-dimensional (linear) subspace U; factoring out the subgroup of order $q - 1$ of the cyclic group of order $q^{2d} - 1$ acting on Ω splits the block classes into $q + 1$ classes each (determined by the images of $GF(q)$ in $GF(q^2)$). Then a "small" class

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together with U determines a hyperplane and so a block class of Ω yields a pencil of hyperplanes of $PG(2d - 1, q)$. The assertion should now be obvious, as the blocks of Ω after all did not contain the subspaces U. Thus Ω admits an epimorphism onto Δ, as was to be shown.

We now construct the examples mentioned in Section 2 where condition (2.1) is not satisfied.

4.9 PROPOSITION. *Let* Π *be the desarguesian projective plane of order* q^2 *and* Π_0 *a Baer subplane of* Π. *Then removing the points and lines of* Π_0 *induces a divisible design* Δ *with parameters*

(4.7) $$n = q^2 - q, m = q^2 + q + 1, k = q^2, \lambda = 1.$$

For $q = 2, 3, 4$, Δ *admits a Singer group.*

Proof. The first part of the assertion is well-known (see [**16**], 7.1.14). By Singer's theorem, Π may be represented as dev D, where D is a difference set in $Z: = \mathbf{Z}_{q^4+q^2+1}$; then D will contain a subset D_0 which forms a difference set for a Baer subplane Π_0 in the subgroup H of order $q^2 + q + 1$ of Z. Also, we may assume that both D and D_0 are fixed by the multiplier p, where $q = p^r$, p a prime (for the notion and results on multipliers, the reader may consult [**1**] and [**22**]). Now removal of Π_0 yields Δ, where the points are now the elements of $Z\backslash H$, the blocks are the sets $(D + x)\backslash H$ $(x \in Z\backslash H)$, the point classes are the sets $(D\backslash D_0) + h$ $(h \in H)$, and similarly for the block classes. Now H clearly acts semiregularly on the points resp. blocks of Δ and any orbit contains in fact precisely one element from each point resp. block class. We now try to find an automorphism group N of Δ acting regularly on the elements of some class; we also want H to be normal in $G: = \langle H, N\rangle$ as this will ensure that G will not become too large. This last condition will be met by any subgroup of the group P generated by the multiplier p; here P clearly is an automorphism group of Δ, as P fixes both D and D_0. We now use the tables of [**1**] to settle the cases $q = 2, 3, 4$. For $q = 2$, $D = \{3, 6, 7, 12, 14\}$, $D_0 = \{7, 14\}$ and $N = \langle p^3\rangle = \langle 8\rangle$ is regular on $\{7, 14\}$. As multiplication by 8 and addition of 3 in fact commute in $Z = \mathbf{Z}_{21}$, Δ is cyclic in this case. For $q = 3$, we may take

$$D = \{0, 1, 3, 7, 9, 27, 49, 56, 61, 77, 81\} \text{ and } D_0 = \{0, 49, 56, 77\};$$

the multiplier 3 has order 6 and $\langle 3\rangle$ is regular on the point class $D\backslash D_0$. But as G is clearly not abelian in this case, $N = \langle 3\rangle$ cannot be normal in G (as H is normal in G). A similar argument works for $q = 4$, where we have

$$D = \{1, 2, \ldots, 2^{11}, 91, 182, 117, 234, 195\}$$

with the last 5 entries forming D_0; 2 has order 12 in $\mathbf{Z}_{273}$.

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It is easy to see that this construction cannot work for $q > 4$, as then the multiplier group of D is too small: q^2 has order 3 modulo $q^4 + q^2 + 1$ and so p has order $6r$ modulo $q^4 + q^2 + 1$ (recall $q = p^r$). So we need $6r \geqq p^r(p^r - 1)$ which implies $p^r \leqq 4$. But from the orders of $PGL(3, q^2)$ resp. $PGL(3, q)$ it seems conceivable that Δ nevertheless always has a Singer group. We show that this is not true and that in fact our construction is the only possible one whenever $q^2 + q + 1$ is a prime (e.g. for $q = 2, 3, 5, 8, 13, \ldots$). By a result of [**16**], p. 317 Δ allows the reconstruction of Π in a unique way (see also [**51**] for a proof of an in fact considerably stronger statement). Then it is easy to see that each automorphism of Δ has to extend to an automorphism of Π fixing Π_0 as a whole (using that the point classes are respected). Note for this, that the blocks of Π_0 may be viewed as the point classes of Δ and the points of Π_0 as block classes of Δ or alternatively as pencils of $q + 1$ point classes of Δ; and that a block class and a pencil of point classes determine the same point of Π_0 if and only if each of the blocks misses all these point classes in Δ. Thus any Singer group G of Δ (of order $(q^2 - q)(q^2 + q + 1)$) acts on Π_0, too. But as $q^2 + q + 1$ is a prime, G contains a unique normal subgroup of this order (by the Sylow theorems), say H. Then H is a (cyclic) Singer group for Π_0; also the group N of Proposition 2.2 has order $q^2 - q$ and is a complement for H in G. As N normalises H, N is a subgroup of the multiplier group of H by [**9**], Theorem 3.1 and thus a subgroup of Aut H $= \mathbf{Z}_{q^2+q}$. Hence $q^2 - q = |N|$ divides $|\text{Aut } H| = q^2 + q$ which implies $q = 2$ or 3. This argument in fact works also if Π is not desarguesian.

Applying Theorem 2.7 to Proposition 4.9, we note

4.10 COROLLARY. *There exist relative difference sets with parameters* (4.7) *for $q = 3$ and $q = 4$ in the semidirect product of $\mathbf{Z}_{q^2+q+1}$ with $\mathbf{Z}_{q^2-q}$, relative to a non-normal subgroup.*

We return once more to the proof of Proposition 4.9 and observe that for $q = 3$ resp. $q = 4$ multiplication by 3^3 and addition of 7 resp. multiplication by 2^6 and addition of 13 commute; hence $\langle 3^3 \rangle$ resp. $\langle 2^6 \rangle$ are normal subgroups of G contained in N in this situation and application of Proposition 2.13 to Corollary 4.10 also yields

4.11 COROLLARY. *There exist relative difference sets with parameters*

$$(4.8) \qquad n = (q^2 - q)/2, m = q^2 + q + 1, k = q^2, \lambda = 2$$

for $q = 3$ or 4 (relative to a non-normal subgroup).

5. Constructions for $\lambda_1 \neq 0$. We now give constructions for $\lambda_1 \neq 0$ and begin with a trivial one:

5.1 LEMMA. *Let D be an ordinary (v, k, λ)-difference set in a group N and let G be any group containing N as a subgroup of index m. Then D is a*

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$(v, m, k, \lambda, 0)$-RDS in G relative to N; all relative difference sets with $\lambda_2 = 0$ are obtained in this way.

Of course, the situation of Lemma 5.1 is completely uninteresting, as it just means taking m disjoint copies of a symmetric design with a Singer group. The next construction is a generalisation of Theorem 3.7.

5.2 THEOREM. *Let D_1 be an ordinary (n, a, λ)-difference set in a group N and let D_2 be a difference set with parameters (3.2) in a group H. Then $D: = (D_1 \times \overline{D_2}) \cup (\overline{D_1} \times D_2)$ is a relative difference set with parameters*

$$(5.1) \qquad n, m = 4u^2, k = 2u^2n + 2au - un, \lambda_1 = (2u^2 - u)(n - 2a) + 4u^2\lambda \quad \text{and} \quad \lambda_2 = u^2n - un + 2au$$

in $G = N \oplus H$ relative to N.

Proof. First note that $\overline{D_1}$ is an $(n, n - a, n - 2a + \lambda)$-difference set in N and that $\overline{D_2}$ is a $(4u^2, 2u^2 + u, u^2 + u)$-difference set in H. From this it is easily checked that elements of the form $(x, 0)$ with $x \neq 0$ of G occur precisely

$$\lambda(2u^2 + u) + (n - 2a + \lambda)(2u^2 - u) = \lambda_1$$

times from D; that elements of the form $(0, y)$ with $y \neq 0$ of G occur precisely

$$a(u^2 + u) + (n - a)(u^2 - u) = \lambda_2$$

times as differences from D; and that elements (x, y) with $x, y \neq 0$ occur precisely

$$\lambda(u^2 + u) + (n - 2a + \lambda)(u^2 - u) + 2(a - \lambda)u^2 = \lambda_2$$

times. It is trivial to check the value for k.

It can in fact be shown along similar lines that $(D_1 \times \overline{D_2}) \cup (\overline{D_1} \times D_2)$ (where D_1, D_2 are ordinary difference sets) will be a relative difference set if and only if one of them has parameters (3.2) and an ordinary difference set if and only if both of them have parameters of the type (3.2); the last assertion is due to Menon [**40**]. We note that Theorem 3.7 is the special case $N = \mathbf{Z}_2, D_1 = \{0\}$ (so $a = 1, n = 2, \lambda = 0$). We note some further consequences:

5.3 COROLLARY ([**40**]). *The existence of difference sets with parameters $u = u_1$ and $u = u_2$ in (3.2) implies the existence of a difference set with parameters (3.2) for $u = 2u_1u_2$.*

5.4 COROLLARY. *The existence of an (n, a, λ)-difference set implies that*

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of normal relative difference sets with parameters

(5.2) $n, m = 4, k = n + 2a, \lambda_1 = n - 2a + 4\lambda, \lambda_2 = 2a;$

(5.3) $n, m = 16, k = 6n + 4a, \lambda_1 = 6n - 12a + 16\lambda, \lambda_2 = 3n + 4a;$

(5.4) $n, m = 36, k = 15n + 6a, \lambda_1 = 15n - 30a + 36\lambda, \lambda_2 = 6n + 6a.$

In particular, these series always exist for $n, n - 1, n - 2$; $(q^d - 1)/(q - 1)$, $(q^{d-1} - 1)/(q - 1)$, $(q^{d-2} - 1)/(q - 1)$ *for prime powers* q *and* $d \geqq 3$; $(4t - 1, 2t - 1, t - 1)$ *for prime powers* $4t - 1$ *and for values* $4t - 1 = q(q + 2)$ *where both* q *and* $q + 2$ *are prime powers.*

This follows from 5.2 by taking $u = 1, 2, 3$ and choosing known series of difference sets (cf. [**22**], [**40**]). We now use uniform Hjelmslev matrices in a further construction. We first recall

5.5 *Definition.* A *uniform c-(t, r)-Hjelmslev matrix* over a group N of order t^2/c is a collection of $r + 1$ subgroups $A_0, \ldots, A_r$ of order t of N satisfying:

(5.5) For $i \neq j$ ($i, j \in \{0, \ldots, r\}$) the set of all differences $a_i - a_j$ ($a_i \in A_i, a_j \in A_j$) contains each element of N precisely c times;

(5.6) The differences $a_i - b_i$ ($a_i, b_i \in A_i, a_i \neq b_i, i = 0, \ldots, r$) contain each non-zero element of N precisely $t\mu$ times for some constant μ.

Of course, μ is determined from the equation

$$t(t - 1)(r + 1) = t\mu((t^2/c) - 1).$$

5.5 is an adaptation of the definitions in [**19**], Sections 7 and 8. "Hjelmslev matrices" have first been introduced by the author in [**31**] to study and construct projective Hjelmslev planes with particularly pleasant Singer groups.

5.6 THEOREM. *Assume the existence of an ordinary difference set* D_1 *with parameters* $(m, r + 1, \lambda)$ *in a group* H *and of a uniform c-(t, r)-Hjelmslev matrix in* N. *Then there exists a normal relative difference set* D *with parameters*

(5.7) $n = t^2/c, m, k = t(r + 1), \lambda_1 = t\mu$ *and* $\lambda_2 = c\lambda$

in $G = H \oplus N$ *relative to* N.

Proof. Let $D_1 = \{d_0, \ldots, d_r\}$ and let $A_0, \ldots, A_r$ be the components of the matrix. Put

$$D: = \bigcup_{i=0}^{r} \{d_i\} \times A_i;$$

the assertion is then easily checked.

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Theorem 5.6 is implicit in the (much more general) work of [**19**]. An analogous construction for symmetric divisible designs without the use of Singer groups has been given in [**18**]. To apply Theorem 5.6 we need examples of uniform c-Hjelmslev-matrices. The following Lemma is due to [**19**], Proposition 8.3.

5.7 LEMMA. *Let q be a prime power and d a non-negative integer. Then there exists a uniform $q^d - (q^{d+1}, q^{d+1} + q^d + \ldots + q)$-Hjelmslev matrix in $N = \mathrm{EA}\ (q^{d+2})$. Here $\mu = q^d + \ldots + q + 1$.*

Sketch of proof. Consider N as the $(d + 2)$-dimensional vector space over $GF(q)$ and choose as the components of the matrix all $(d + 1)$-dimensional subspaces.

We remark that it has been shown conversely in [**33**], Theorem 5.4 that all uniform c-(t, r)-Hjelmslev matrices in an abelian group N have the parameters stated in 5.7 and are defined in an elementary abelian group. We note some corollaries of 5.6 and 5.7:

5.8 COROLLARY ([**39**]). *Let q be a prime power, d a non-negative integer and H any group of order $q^{d+1} + q^d + \ldots + q + 2$. Then there exists an ordinary difference set with parameters*

$$(5.8) \qquad v = q^{d+2}(q^{d+1} + q^d + \ldots + q + 2), k = q^{d+1}(q^{d+1} + \ldots + q + 1)$$

and

$$\lambda = q^{d+1}(q^d + \ldots + q + 1)$$

in $G = H \oplus \mathrm{EA}\ (q^{d+2})$.

Proof. Use 5.7 in Theorem 5.6 for the trivial $(m, m - 1, m - 2)$-difference set in H.

5.9 COROLLARY. *Let q be a prime power. Then there exists a relative difference set with parameters*

$$(5.9) \qquad n = q^2, m = q^2 + q + 1, k = q(q + 1), \lambda_1 = q \text{ and } \lambda_2 = 1$$

in $G = \mathbf{Z}_{q^2+q+1} \oplus \mathrm{EA}\ (q^2)$ relative to $\mathrm{EA}\ (q^2)$.

Proof. Use 5.7 with $d = 0$ in Theorem 5.6 for the Singer difference sets of desarguesian projective planes (see [**50**] or [**22**]).

This series in fact already appears in [**31**] and corresponds to uniform Hjelmslev planes. In fact, any normal relative difference set with parameters (5.9) and $q \neq 2$ corresponds to a uniform Hjelmslev plane by Theorem 2.7 and [**32**]. This implies

5.10 PROPOSITION. *If D is a relative difference set with parameters (5.9) in an abelian group $G = H \oplus N$ relative to N and if $q > 2$, then q is a prime*

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power and $N = \text{EA}(q^2)$. *Also, the set of* $d \in H$ *for which there is an* $n \in N$ *with* $(d, n) \in D$ *forms an ordinary difference set for a projective plane of order* q.

Proof. dev D then is a Hjelmslev plane which is "regular" in the sense of [**31**]. The assertion follows from [**31**], Section 2 and [**34**], Theorem 3.1.

There are counter-examples to Proposition 5.10 for $q = 2$, due to [**32**]. We give them here:

5.11 *Examples.* The sets

$$D_1 = \{(0, 1), (0, a), (0, 1 + a), (1, 0), (2, 0), (4, 0)\}$$

and

$$D_2 = \{(0, 1), (0, 2), (0, 3), (1, 0), (2, 0), (4, 0)\}$$

are relative difference sets with parameters (5.9) for $q = 2$ in $\mathbf{Z}_7 \oplus \text{EA}(4)$ resp. in $\mathbf{Z}_7 \oplus \mathbf{Z}_4$ which do not correspond to a uniform Hjelmslev plane.

Using [**9**], page 475 in the same construction as in 5.9, we also obtain:

5.12 PROPOSITION. *Let* q *be a prime power* $\equiv 1 \bmod 3$. *Then there exists a relative difference set with parameters* (5.9) *in a group* $G = H \oplus \text{EA}(q^2)$, *where* H *is non-abelian.*

We now generalise the construction in 5.9 by using a Singer difference set for the projective space $PG(d, q)$ together with a uniform q^{d-2}-$(q^{d-1}, q^{d-1} + \ldots + q)$-$H$-matrix and obtain

5.13 COROLLARY. *Let* q *be a prime power and* d *a positive integer with* $d \geqq 2$. *Then there exists a relative difference set with parameters*

$$(5.10)\quad n = q^d,\ m = q^d + \ldots + q + 1,\ k = q^{d-1}(q^{d-1} + \ldots + q + 1),$$
$$\lambda_1 = q^{d-1}(q^{d-2} + \ldots + q + 1) \quad and \quad \lambda_2 = q^{d-2}(q^{d-2} + \ldots + q + 1)$$

in $\mathbf{Z}_{q^d + \ldots + q + 1} \oplus \text{EA}(q^d)$.

We give 5 more examples using the known difference sets with $k \leqq 10$ and $\lambda > 1$ (see Appendix I in [**22**]):

5.14 *Examples.* There are abelian (16, 11, 20, 4, 2)-, (25, 16, 30, 5, 2)-, (8, 15, 28, 12, 6)-, (64, 37, 72, 8, 2)- and (64, 19, 72, 8, 4)-relative difference sets.

The examples constructed up to now in this section all yield regular divisible designs. We have exhibited two relative difference sets yielding semi-regular divisible designs in Examples 2.4 (v) and (vii). Using Theorem 5.6 for a trivial (m, m, m)-difference set and Lemma 5.7, we get a family of examples:

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5.15 COROLLARY. *Let q be a prime power and d a positive integer. Then there exists a normal relative difference set with parameters*

$$(5.11)\quad n = q^{d+1},\ m = q^d + \ldots + q + 1,\ k = q^d(q^d + \ldots + q + 1),$$
$$\lambda_1 = q^d(q^{d-1} + \ldots + q + 1) \text{ and } \lambda_2 = q^{d-1}(q^d + \ldots + q + 1)$$

in $H \oplus \mathrm{EA}(q^{d+1})$, where H is any group of order m.

To the knowledge of this author, this is the first series of symmetric semi-regular divisible designs with $\lambda_1 \neq 0$ in the literature. An example of a semi-regular design with parameters (5.11) for $q = 2$ and $d = 1$ is No. 1 on page 187 of [7]. In view of the preceding comment, it seems worthwhile to exhibit some more series of symmetric semi-regular divisible designs (which may be constructed with the aid of projective spaces), though this will not yield any more relative difference sets.

5.16 THEOREM. *Let Π be the projective space $PG(d, q)$, where $d \geqq 3$, and let a be a positive integer $\geqq (d-1)/2$. Then the incidence structure Δ obtained from Π by discarding an a-flat U with all its points and also all hyperplanes through a fixed $(d-a-1)$-flat $S \subset U$ is a symmetric semi-regular divisible design with parameters*

$$(5.12)\quad n = q^{d-a},\ m = q^a + q^{a-1} + \ldots + q^{2a-d+1},\ k = q^{d-1} + \ldots + q^a,$$
$$\lambda_1 = q^{d-2} + \ldots + q^a \quad \text{and} \quad \lambda_2 = q^{d-2} + \ldots + q^{a-1}.$$

So $v = q^d + \ldots + q^{a+1}$ and $c = q^{d-a-1}$, where c has the meaning of (1.7).

Proof. Note first that $S \subset U$ is possible in view of the restriction on a. Let H be any block of Δ; then H meets U in Π in exactly $q^{a-1} + \ldots + q + 1$ points, as H does not contain U (otherwise it would have been discarded). This observation yields the value of k. We now define a point class to consist of all those points which together with S span the same $(d-a)$-flat T, where $S \subset T \not\subset U$. Clearly each point class has $(q^{d-a} + \ldots + q + 1) - (q^{d-a-1} + \ldots + q + 1) = n$ elements. Also, the number of $(d-a)$-flats of Π through S is $q^{a+1} + \ldots + q + 1$; exactly $q^{2a-d} + \ldots + q + 1$ of these are contained in U, which yields the value of m. Now the number of hyperplanes of Π containing a given $(d-a)$-flat T (T as above) is $q^{a-1} + \ldots + q + 1$ which yields λ_1, as any two points of Π are on $q^{d-2} + \ldots + q + 1$ common hyperplanes. Finally, points in distinct classes span (together with S) a $(d-a+1)$-flat W of Π; the number of hyperplanes of Π containing W is $q^{a-2} + \ldots + q + 1$ which yields λ_2. Hence Δ is a semi-regular divisible design with parameters (5.12). If we define a block class of Δ to consist of all those blocks meeting U in a given $(a-1)$-flat V with $S \not\subset V$, then the corresponding assertions for blocks may be verified dually. Thus Δ is symmetric.

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Note that for $a = d - 1$ we discard an incident point-hyperplane pair which is a well-known construction for a symmetric transversal design ST (q, q^{d-2}). Note also that replacement of d in (5.12) by $2d + 1$ and of a by d yields the parameters (5.11). We conclude this section with the following consequence of Theorem 5.16:

5.17 COROLLARY. *Let q be a prime power and d and a positive integers satisfying $d \geqq 3$ and $a \geqq (d - 1)/2$. Then there exists a semi-regular divisible design with parameters*

$$(5.13) \quad n = q^{d-a},\ m \leqq q^a + q^{a-1} + \ldots + q^{2a-d+1},$$
$$k = mq^{d-a-1},\ \lambda_1 = q^{d-2} + \ldots + q^a \text{ and } \lambda_2 = q^{d-2} + \ldots + q^{a-1}.$$

For $m = q^a$, we obtain an affine resolvable semi-regular divisible design (i.e., the design is resolvable and any two non-parallel blocks intersect in the same number of points, here in fact in q^{d-2} points).

Proof. It is clear from (1.7) that removal of some point classes of a semi-regular divisible design again yields a semi-regular divisible design; apply this to Theorem 5.16 to obtain (5.13). Now it is known (see e.g. [**43**] 8.5.10.1) that a semi-regular divisible design is affine resolvable if and only if $b = v - m + r$ and k^2/v is an integer; both conditions are satisfied for $m = q^a$. (In fact, if one discards all point classes of a fixed hyperplane of Δ the resulting semi-regular divisible design is contained in the affine space $AG(d, q)$ and the remaining blocks of Δ are hyperplanes of this affine space, from which the assertion is geometrically obvious.)

One may try to use the method of Proposition 4.9 to obtain Singer groups for the divisible designs of Theorem 5.16; this then forces d to be odd and a to be $(d - 1)/2$ (which yields the parameters (5.11), as already mentioned). In fact, the multiplier group will again be too small to yield examples in general. Also, the parameters would not be new anyway, so that a detailed discussion of this situation seems not too interesting.

6. Class regular divisible designs and partial difference matrices. Consider a symmetric transversal design with a normal Singer group G; we have seen in Proposition 2.2 that G has a normal subgroup N acting regularly on each point class and (as seen in the proof of Proposition 2.8) also on each block class. This situation has already been considered by the author in [**30**] where such a symmetric transversal design has been called "regular" (with respect to N). We will change this term to "class regular" here to avoid a collision with the terminology of Definition 1.3 and will actually generalise the concept of [**30**] to arbitrary divisible designs with $\lambda_1 = 0$. We have shown in [**30**] that class regular symmetric transversal designs are equivalent to generalised Hadamard matrices; in the more general case of class regular symmetric divisible designs we

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will obtain the equivalence with generalised balanced weighing matrices. But we will also briefly consider the non-symmetric case. In this section, a divisible design with parameters n, m, k, λ will denote (in the old terminology) one with parameters n, m, k, $\lambda_1 = 0$ *and* $\lambda_2 = \lambda$.

6.1 *Definition.* Let Δ be a divisible design with parameters n, m, k, λ and let N be an automorphism group of Δ acting regularly on each point class. Then Δ is called *class regular* (with respect to N).

As already remarked, this definition has been given in the special case of transversal designs in [**30**] where we asked in addition that N should act semiregularly on the block set of Δ. This is in fact redundant, as S. S. Sane has pointed out:

6.2 LEMMA. *Let Δ be a divisible design that is class regular with respect to N. Then N acts semiregularly on the block set of Δ.*

Proof. Let B and $C = B^n$ be two blocks in the same orbit of N and assume the existence of a point p I B, C. Now p^n I C is a point in the class of p and thus $p = p^n$, as C meets each point class at most once (recall $\lambda_1 = 0$). Thus n is the identity automorphism and $B = C$, as N acts regularly on each point class.

The combinatorial equivalent of class regular divisible designs is as follows:

6.3 *Definition.* Let N be a group of order n. A *partial difference matrix* with parameters n, m, k, λ over N is a matrix $D = (d_{ij})$ with m rows, β columns and entries from $N \cup \{\infty\}$ satisfying the following conditions:

(6.1) Each column of D has precisely k entries $\neq \infty$.

(6.2) For any two distinct rows i and j of D, the differences $d_{ih} - d_{jh}$ ($h = 1, \ldots, \beta$) contain each element of N exactly λ times (here $\infty - x = x - \infty = \infty$).

If $k = m$ (i.e., if no entry of D is ∞), D is called a *difference matrix* or an (n, k, λ, N)-difference matrix.

6.4 LEMMA. *Let D be a partial difference matrix with parameters n, m, k, λ over N. Then replacing each entry ∞ of D by* 0 *and each entry $\neq \infty$ of D by* 1, *yields the incidence matrix of an $(m, k, n\lambda)$-block design. Thus*

(6.3) *each row of D contains exactly $r = (m - 1)n\lambda/(k - 1)$ entries $\neq \infty$.*

(6.4) $\beta = m(m - 1)n\lambda/k(k - 1)$.

This follows immediately from (6.1) and (6.2) and well-known properties of block designs. Note that the symmetric case ($m = \beta$) of our partial difference matrices is equivalent to the "orthogonal configurations over a group" of Delsarte (see [**12**], Section 3) and to the "generalised balanced

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weighing matrices" of Seberry [**46**]. A generalisation of partial difference matrices (where the entries ∞ are replaced by symbols $x_1, \ldots, x_s$) with $\lambda = 1$ has been used in the construction of mutually orthogonal Latin squares (see [**8**] and [**54**]); difference matrices with $\lambda = 1$ are equivalent to sets of mutually orthogonal Latin squares with the group N acting in a certain way (see [**29**]) and have also been used for the construction of such squares (see [**28**] and [**42**]). Hadamard matrices are partial difference matrices with parameters 2, 2λ, 2λ, λ; conference matrices have parameters 2, $2\lambda + 2$, $2\lambda + 1$, λ; and balanced weighing matrices have $n = 2$. Accordingly, we use the following terminology:

6.5 *Definition*. Let D be a partial difference matrix with parameters n, m, k, λ over N and assume that $r = k$. Then D is called a *generalised balanced weighing matrix* GBW (n, m, k). (Note that λ is determined from (6.3)); if furthermore $m = k$, D is called a *generalised Hadamard matrix* GH (n, λ) and if $m = k + 1$, a *generalised conference matrix* GC (n, λ).

6.6 LEMMA. *The existence of a partial difference matrix with parameters n, m, k, λ over N implies that of a partial difference matrix with parameters n/s, m, k, λs over N/M, whenever M is a normal subgroup of N of order s.*

This follows trivially by taking the image of D under the natural epimorphism from N onto N/M.

6.7 THEOREM. *Let D be a partial difference matrix with parameters n, m, k, λ over N. Then:*

(6.5) $m \leqq \beta$;

(6.6) $m = \beta$ *implies that* $D^* = -D^T$ *is also a partial difference matrix with parameters* n, m, k, λ.

Proof. By Lemma 6.4, D determines an $(m, k, n\lambda)$-block design. Now if $r = \lambda n(m - 1)/(k - 1) > n\lambda$, then the Fisher inequality "$b \geqq v$" applied to this design yields (6.5). But if $r = n\lambda$, then every point of the block design is incident with every block and thus in particular $k = m$. So D is a difference matrix then and the assertion follows from [**30**], Theorem 2.2. Finally, (6.6) is [**12**], Theorem 3.3; an alternative proof using 6.8 below is given in [**35**].

6.8 THEOREM. *The existence of a partial difference matrix with parameters n, m, k, λ over N is equivalent to that of a class regular divisible design with parameters n, m, k, λ (with respect to N).*

Proof. Let D be a partial difference matrix with parameters n, m, k, λ over N and define an incidence structure Δ as follows: The point set of Δ is the union of the point classes $\mathscr{P}_1, \ldots, \mathscr{P}_m$ where

$$\mathscr{P}_i := \{(i, x) : x \in N\} \quad \text{for } i = 1, \ldots, n.$$

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The block set of Δ is the union of the classes $\mathscr{B}_1, \ldots, \mathscr{B}_\beta$ with

$$\mathscr{B}_j = \{B_{jx}: x \in N\} \ (j = 1, \ldots, \beta),$$

where

$$B_{jx} = \{(j, d_{ij} + x): j = 1, \ldots, m, d_{ij} \neq \infty\}.$$

Incidence is given by the membership relation. Then clearly points in the same class are not joined. Consider points (i, x) and (j, y) with $i \neq j$; there are precisely λ indices k with $d_{ik} - d_{jk} = x - y$. But then (i, x) and (j, y) are on the blocks $B_{k,-d_{ik}+x}$ for all these k. Conversely, it is easily seen that $(i, x), (j, y) \in B_{ku}$ implies $d_{ik} - d_{jk} = x - y$. Thus Δ is a divisible design with parameters n, m, k, λ which is clearly class regular with respect to N by letting $g \in N$ act on Δ by

(6.7) $\quad (i, x) \mapsto (i, x + g) \quad$ and $\quad B_{jy} \mapsto B_{j,y+g}.$

Now assume that Δ is any divisible design with parameters n, m, k, λ that is class regular with respect to N. In each point class $\mathscr{P}_i$ $(i = 1, \ldots, m)$ choose a "base point" p_i and coordinatise the point $q \in \mathscr{P}_i$ as (i, x) if x is the unique element of N mapping p onto q (which is well-defined by the regularity of N on point classes). By Lemma 6.2, N acts semi-regularly on the block set of Δ; hence there will be $b/n = \beta$ orbits under N. In each of these orbits $\mathscr{B}_j$ $(j = 1, \ldots, \beta)$ choose a "base block" B_{j0} and coordinatise the block C of $\mathscr{B}_j$ as B_{jy} if y is the unique element of N mapping B_{j0} onto C. Define D as follows: If B_{j0} contains no point of $\mathscr{P}_i$, put $d_{ij} = \infty$; if it contains the point (i, x), put $d_{ij} = x$. It is then easily checked that D is a partial difference matrix with parameters n, m, k, λ over N.

6.9 Corollary. *Let Δ be a class regular divisible design with parameters n, m, k, λ. Then $b \geqq v$; and if Δ is actually square, then it is in fact symmetric.*

Proof. Use Theorems 6.7 and 6.8.

Theorem 6.8 is in complete analogy to [**30**], Theorem 1.5 where the case of transversal designs has been considered. We note:

6.10 Proposition. *If $|N| \geqq 2$, then the existence of a generalised Hadamard matrix $GH(n, 1)$ over N is equivalent to that of a projective plane of order n of Lenz type at least* II *having N as the group of all (p, L)-elations for some flag (p, L). Also, the existence of a generalised conference matrix* GC $(n, 1)$ *over N is equivalent to that of a projective plane of order $n + 1$ having N as the group of (p, L)-homologies for some antiflag (p, L).*

Proof. The first part is [**30**], Proposition 1.6. The second part is similar and we only sketch one direction: Given the projective plane, discard p

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(together with all lines through p) and L (with all points on L). This gives a symmetric divisible design with parameters $n, n + 2, n + 1, 1$ which is class regular with respect to N. Then apply Theorem 6.8. These steps can be reversed to yield the converse.

We conclude this section by listing the known generalised balanced weighing matrices.

6.11 *Examples.* The following generalised balanced weighing matrices are known to exist:

(i) Generalised Hadamard matrices $GH(n, \lambda)$ for:

a) $n = 2$: Ordinary Hadamard matrices, cf. [**52**].

b) $n = p^i, \lambda = p^j, N = \text{EA}\ (p^i)$ where p is a prime, $i > 0, j \geqq 0$ ([**17**], Corollary 1.9);

c) n a prime power, $\lambda = 2$, $N = \text{EA}\ (n)$ ([**30**] Theorem 2.4; for n a prime, this is already in [**10**]);

d) n and $\lambda = n - 1$ both prime powers, $N = \text{EA}\ (n)$ ([**44**], see also [**47**]);

e) The existence of GH (n, λ) and GH (n, λ') over N implies that of $GH(n, n\lambda\lambda')$ over N ([**49**]);

(ii) Generalised conference matrices GC (n, λ) for:

a) $n = 2$: ordinary conference matrices, see [**15**], [**21**] and [**37**];

b) $n\lambda + 1$ a prime power, $N = \mathbf{Z}_n$ ([**14**], [**46**]);

(iii) Generalised balanced weighing matrices GBW (n, k, λ) for:

a) $n = q - 1, m = (q^d - 1)/(q - 1), k = q^{d-1}, N = \mathbf{Z}_n$ for q a prime power and $d \geqq 2$ ([**3**]) and images as in Lemma 6.6;

b) $n = 6, m = 13, k = 9, N = S_3$ the symmetric group on 3 elements (see [**46**]).

To the author's knowledge, this list is complete. It should be remarked that Berman's "Generalised weighing matrices" are a much wider class than the GBW-matrices considered here (though Berman only considers cyclic groups); it can be seen from his proofs that the only matrices of Berman which are balanced in our sense are those given in (iii)a) above and, of course, images according to Lemma 6.6.

In view of Definition 6.5 and Theorems 6.7 and 6.8 we will call any symmetric divisible design with parameters n, m, k, λ where $m = k + 1$ (equivalently, $k = n\lambda + 1$) a *conference design*; note that in fact by (1.15) any square design with $k = n\lambda + 1$ is symmetric. Using this terminology and the observation that the transversal designs are characterised by $k = n\lambda$ (from (1.4)), we may restate (1.19) as

6.12 LEMMA. *Let Δ be a symmetric divisible design with parameters n, m, k, λ. Then Δ is a transversal design or a conference design or*

$$(6.8) \qquad n\lambda \leqq k - \sqrt{k}.$$

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Similarly, a generalised balanced weighing matrix is a generalised Hadamard matrix or a generalised conference matrix or satisfies (6.8).

We already observed at the beginning of this section that divisible designs with a normal Singer group are class regular. In the next section we will study the relations between normal relative difference sets and generalised balanced weighing matrices.

7. Relative difference sets and generalised balanced weighing matrices. We repeat our observation about normal divisible designs.

7.1 LEMMA. *Any divisible design with* $\lambda_1 = 0$ *and a normal Singer group* G *is class regular with respect to the subgroup* N *of* G *acting regularly on each point class.*

7.2 COROLLARY. *The existence of a relative difference set with parameters* n, m, k, λ *in* G *relative to the normal subgroup* N *implies that of a generalised balanced weighing matrix* GBW (n, m, k) *over* N.

This follows immediately from Theorems 2.7 and 6.8 together with Lemma 7.1. Note that application of 7.2 to our results in Sections 3 and 4 yields Examples 6.11 (i)b, (ii)b and (iii)a) as well as some of (i)(a) and (ii)a. Here we again omit the parameters $\lambda_1 = 0$. Also note that Corollary 4.3 yields three examples not covered by Examples 6.11. We now study those generalised balanced weighing matrices that come from a relative difference set in more detail. We will give a necessary and sufficient condition for this to happen in two important special cases. A general criterion can be obtained, but it is so awkward that it does not seem worthwhile stating it. The first criterion is for the case where G splits over N. We first need a definition.

7.3 *Definition.* Let M be a generalised balanced weighing matrix GBW (n, m, k) over N and let H be any group of order m. M is called *H-invariant* if the columns and rows of M may be labelled by the elements of H in such a way that $m_{fg} = m_{f+h,g+h}$ for all $f, g, h \in H$.

If H is cyclic this means that M may be put into circulant form. We then have:

7.4 THEOREM. *Let* H *and* N *be groups of orders* m *resp.* n. *Then there exists an* H*-invariant generalised balanced weighing matrix* GBW (n, m, k) *over* N *if and only if there exists a relative difference set with parameters* n, m, k, λ *in* $G = H \oplus N$ *relative to* N.

Proof. Assume first that $D = \{(h_1, n_1), \ldots, (h_k, n_k)\}$ is a relative difference set as described above. Consider the divisible design dev D of Theorem 2.7 which is symmetric by Theorem 2.10. Its block classes then may be represented by the blocks $D + h$ $(h \in H)$. Now construct M from

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dev D as in Theorem 6.8, choosing the $D + h$ as the "base blocks". Then the element n_i of N will appear precisely in the cells $(h_i + h, h)$ of M $(i = 1, \ldots, k)$ and all remaining entries will be ∞. Thus M is clearly H-invariant; it is a GBW (n, m, k) by Theorem 6.8. Conversely, let M be an H-invariant GBW (n, m, k) and consider the divisible design Δ constructed from M as in Theorem 6.8. Let

$$D: = \{(g, m_{g,0}): g \in H, m_{g,0} \neq \infty\}.$$

Then the block class belonging to column h of M may be represented by

$$D + h = \{(g + h, m_{g,0}): g \in H, m_{g,0} \neq \infty\}$$

as M is H-invariant. But then it is easily seen that the mapping

$$(g, x) \mapsto (g + h, x + n) \text{ and } D + (f, y) \mapsto D + (f + h, y + n)$$

is an automorphism of Δ for all $(h, n) \in H \oplus N$. Thus $G = H \oplus N$ is a normal Singer group for Δ and the assertion follows from Theorem 2.7.

We remark that H-invariant generalised balanced weighing matrices are "regular" in the sense of [**12**], i.e., the number of entries n (for a given $n \in N$) is the same in each column (and also in each row). We give some corollaries of Theorem 7.4 and results of Sections 3 and 4.

7.5 COROLLARY. *The existence of a circulant generalised balanced weighing matrix* GBW (n, m, k) *over N is equivalent to that of an* (n, m, k, λ)-RDS *in* $\mathbf{Z}_m \oplus N$.

7.6 COROLLARY. *There exist H-invariant generalised Hadamard matrices* GH (n, λ) *with* $n = p^i$, $\lambda = p^j$, $N =$ EA (p^i), $H =$ EA (p^{i+j}) *for all primes $p \neq 2$ and positive integers i and non-negative integers j. There exist H-invariant Hadamard matrices of order $4u^2$ for all values $u = 2^s3^r$ with $s \geqq r - 1$ for those H described in Corollary* 3.8.

7.7 COROLLARY. *There exist circulant generalised balanced weighing matrices* GBW (n, m, k) *over* $\mathbf{Z}_n$ *for all values* $m = (q^d - 1)/(q - 1)$, $k = q^{d-1}$ *for which q is a prime power* $\equiv 1$ *modulo n and for which* $(n, m) = 1$. *In particular, there exist circulant generalised conference matrices* GC (n, λ) *whenever* $n\lambda + 1$ *is a prime power and* $n > 2$.

7.8 COROLLARY. *There exists a* (6, 13, 9, 1)-RDS *in* $\mathbf{Z}_{13} \oplus S_3$.

Proof. Use Corollary 7.5 and Example 6.11 (iii)b.

We next consider the case of cyclic relative difference sets. The following definition is due to Berman [**3**] and generalises that of "negacyclic" matrices of [**15**].

7.9 *Definition.* Let M be a generalised balanced weighing matrix GBW (n, m, k) over $\mathbf{Z}_n$ and let $t \in \mathbf{Z}_n$. M is called *t-circulant* if it satisfies

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$m_{i+1,j+1} = m_{i,j}$ whenever $i \neq m - 1$ and $m_{0,j+1} = m_{m-1,j} + t$. Here we assume that the rows and columns of M are labelled by the elements of $\mathbf{Z}_m$.

The following result generalises Theorem 5.1 of [**15**]; one of its parts has been proved in a special case by [**3**], Theorem 4.2.

7.10 THEOREM. *The existence of a cyclic* (n, m, k, λ)-RDS *is equivalent to that of a* 1-*circulant generalised balanced weighing matrix* GBW (n, m, k) *over* $\mathbf{Z}_n$.

Proof. Assume first that the cyclic relative difference set $D = \{d_1, \ldots, d_k\}$ is given. Then D is relative to $N = \{mi: i = 0, \ldots, n - 1\} \cong \mathbf{Z}_n$. Consider the symmetric divisible design dev D; its point classes are $N, N + 1, \ldots, N + m - 1$ and its block classes may be represented by the sets $D + j$ $(j = 0, \ldots, m - 1)$. We construct the GBW (n, m, k) over N as in the proof of Theorem 6.8, using the $D + j$ as the "base blocks". Now

$$D + j = \{d_1 + j, \ldots, d_k + j\};$$

for $h = 1, \ldots, k$, put $d_h + j = ma_h + r_h$, where $r_h \in \{0, \ldots, m - 1\}$. Then set $m_{ij} = ma_h$ if and only if $r_h = i$, and $m_{ij} = \infty$ otherwise. But column $j + 1$ of M is constructed from $D + j + 1$ and clearly $d_h + j + 1 = ma_h + r_h + 1$. Thus

$$m_{i+1,j+1} = ma_h = m_{ij} \Leftrightarrow r_h = i < m - 1;$$
$$m_{i+1,j+1} = \infty = m_{ij} \Leftrightarrow i \neq r_h \quad \text{for all } h;$$

and

$$m_{0,j+1} = m(a_h + 1) \Leftrightarrow r_h = m - 1 \text{ and } = \infty \text{ otherwise.}$$

Thus M may be considered as a 1-circulant matrix over $\mathbf{Z}_n$. Conversely, assume that M is a 1-circulant GBW (n, m, k) over N, where $\mathbf{Z}_n \cong N \subset \mathbf{Z}_{mn}$ as above. Consider the divisible design Δ determined by M as in Theorem 6.8 and identify the point (i, x) of Δ with $mx + i \in \mathbf{Z}_{mn}$. By Theorem 2.7 it will be sufficient to show that the mapping $mx + i \mapsto mx + i + 1$ is an automorphism of Δ. Consider one of the "base blocks" B_j; then we have

$$B_j = \{mm_{ij} + i: m_{ij} \neq \infty\}$$

and as M is 1-circulant, $B_{j+1} = B_j + 1$. As the block classes are the sets $B_j + a$ $(a \in N)$, the assertion follows immediately.

Our proofs of Theorems 7.4 and 7.10 utilize the geometric interpretations of Theorems 2.7 and 6.8. In our opinion these proofs are much more transparent than direct proofs (avoiding the corresponding divisible design) would be. We now note a corollary of Theorem 7.10:

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7.11 COROLLARY. *There exist* 1-*circulant generalised balanced weighing matrices* GBW (n, m, k) *over* $\mathbf{Z}_n$ *for all values* $m = (q^d - 1)/(q - 1)$, $k = q^{d-1}$ *for which* q *is a prime power* $\equiv 1$ *modulo* n. *In particular, there exist* 1-*circulant generalised conference matrices* GC (n, λ) *whenever* $n\lambda + 1$ *is a prime power.*

We conclude this section with a remark. If Theorem 4 of [**46**] is true, then the existence of the GBW $(q - 1, (q^d - 1)/(q - 1), q^{d-1})$ would imply the existence of the series of symmetric block designs with parameters $v = q(q^d - 1) + 1$, $k = q^d$ and $\lambda = q^{d-1}$, strongly generalising results of [**2**] and [**44**]. Unfortunately, her proof uses the existence of a GH $(d, 1)$ in $\mathbf{Z}_d$ which is only known if d is a prime (meaning in this situation, that $q - 1$ is a prime). The author has been unable to find an alternative proof of the result in question.

8. Some applications to divisible semisymmetric designs. In this section, we follow a suggestion of D. R. Hughes and consider some basic properties of semisymmetric designs, which generalise his semibiplanes (see [**25**]). We then apply the results of this paper to obtain some existence results for divisible semisymmetric designs.

8.1 *Definition.* An incidence structure Δ is called a *semisymmetric design* of *index* λ if it satisfies:

(8.1) $[p, q] = 0$ or λ for any two points p, q;

(8.2) $[G, H] = 0$ or λ for any two blocks G, H;

(8.3) Δ is connected.

In case of $\lambda = 1$, we additionally require

(8.4) There is a constant k with $[p] = [G] = k$ for all points p and all blocks G.

8.2 LEMMA. *Any semisymmetric design is a tactical configuration with* $r = k$ (*so* $b = v$).

Proof. This is obvious for $\lambda = 1$ in view of (8.4). Now let $\lambda > 1$ and consider any flag (p, G). Count all flags (q, H) with $p\text{I}H$ and $q\text{I}G$ in two ways to obtain the equation

$$([p] - 1)(\lambda - 1) = ([G] - 1)(\lambda - 1),$$

so $[p] = [G]$ for any flag (p, G). As Δ is connected, this gives the assertion.

In view of Lemma 8.2, let k denote the constant block size of the semisymmetric design and v the number of points. Then we call v, k, λ the *parameters* of the design.

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8.3 *Definition*. A semisymmetric design is called *divisible* if it is simultaneously a divisible design and if there are points not joined. In this case, we use parameters n, k, λ instead of v, k, λ (cf. (8.5) below).

8.4 PROPOSITION. *Let Δ be a semisymmetric design with parameters v, k, λ. If Δ is divisible, then necessarily $\lambda_1 = 0$ and $\lambda_2 = \lambda$. Also then*

(8.5) $\quad v = n + k(k - 1)/\lambda, m = 1 + k(k - 1)/n\lambda,$

so that a necessary condition for divisibility is $n\lambda | k(k - 1)$. Furthermore, a divisible semisymmetric design is symmetric in the sense of Definition 1.1, i.e., the dual design is also divisible.

Proof. Assume that Δ is divisible; then by definition, either $\lambda_1 = 0$, $\lambda_2 = \lambda$ or $\lambda_1 = \lambda$, $\lambda_2 = 0$. But as there are points not joined, the last case would mean that Δ is not connected, contradicting (8.3). Now it is easily seen that for any given point p there are precisely $k(k - 1)/\lambda$ points joined to it; this gives (8.5). By the same counting, it is seen that for any given block B there are precisely $k(k - 1)/\lambda$ blocks C meeting it and precisely $v - 1 - k(k - 1)/\lambda$ blocks D not meeting it. Now consider two blocks B_1, B_2 that do not meet. B_1 contains precisely 1 point of each of k point classes and B_2 contains at most 1 point of each of these k point classes determined by B_1. Thus the number of blocks C intersecting both B_1 and B_2 is at least $k(k - 1)\lambda/\lambda^2 = k(k - 1)/\lambda$ which is the number of blocks meeting B_1 and also the number of blocks meeting B_2; thus each of the remaining $n - 2$ blocks has to miss both B_1 and B_2, which proves the assertion.

8.5 PROPOSITION. *Let Δ be a divisible semisymmetric design with parameters n, k, λ. Then Δ is a (symmetric) transversal design if and only if $k = n\lambda$ and Δ is a conference design if and only if $k = n\lambda + 1$. In all remaining cases, Δ satisfies*

(8.6) $\quad 2 \leqq n \leqq (k - \sqrt{k})/\lambda.$

Furthermore, any semisymmetric design with $v = 2 + k(k - 1)/\lambda$ is divisible (with $n = 2$).

Proof. The assertions are just a restatement of Lemma 6.12. Finally, if $v = 2 + k(k - 1)/\lambda$, then for each point there is a unique other point not joined to it. It is clear that this then defines a divisible design with $n = 2$.

Using some of the existence results of the previous sections, we get:

8.6 COROLLARY. *There exists a divisible semisymmetric design with parameters n, k, λ in at least the following cases:*

(i) *$\lambda = p^i$ and $k = p^j$ for some prime p with $j > i$, $n = p^{j-i}$;*

(ii) *k a prime power $\equiv 1$ modulo λ, $n = (k - 1)/\lambda$;*

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(iii) $\lambda = tq^{d-1}$ *and* $k = q^d$ (*where* q *is a prime power and* t *a divisor of* $q - 1$), $n = (q-1)/t$.
In all these cases, we may also assume the existence of a Singer group.

We state the case of semibiplanes ($\lambda = 2$) individually (cf. [**25**]):

8.7 COROLLARY. *There exists a divisible semibiplane with block size* k *on* v *points in at least the following cases:*
(i) $k = 2^i$, $n = 2^{i-1}$;
(ii) $(n, k) = (3, 9)$ *or* $(6, 16)$;
(iii) k *a prime power* $\equiv 1$ *modulo* 2, $n = (k-1)/2$;
(iv) $k = 2q$ *where* q *is an odd prime power*, $n = q$.
In cases (i) *to* (iii), *we may assume the existence of a Singer group; in case* (iv), *we may assume class regularity.*

We remark that our results give quite a lot of information on the existence problem for divisible semibiplanes, if one makes use of Propositions 8.4, 8.5 and 1.8. We give two examples.

8.8 *Example.* The only pairs (n, k) with $k < 20$ for which the existence of a divisible semibiplane with parameters (n, k) is in doubt are (2, 9), (3, 10), (3, 12), (6, 12), (2, 13), (3, 13), (3, 15), (7, 15), (2, 16), (3, 16), (4, 16), (5, 16), (6, 16).

8.9 *Example.* Let Δ be a divisible semibiplane with parameters (n, k) where $k \equiv 2 \bmod 4$ is not the sum of two squares and where $k < 330$. Then either Δ is a transversal design or a conference design, or (k, n) is one of the following pairs:

(42, 3)	(110, 5)	(190, 45)	(238, 21)
(46, 5)	(118, 9)	(206, 5)	(246, 105)
(54, 9)	(126, 45)	(210, 7)	(266, 35)
(66, 15)	(150, 25)	(210, 55)	(294, 49)
(78, 21)	(166, 33)	(222, 13)	(326, 65).

Proof. As $k \equiv 2 \bmod 4$ one sees from (8.5) that n must be odd while m must be even; thus Proposition 1.8 implies

(8.7) $\quad m \equiv 0 \bmod 4$

as k is not the sum of two squares. Now (8.5) implies that $2v = 2nm = 2n + k^2 - k$, so

(8.8) $\quad 2n = k - (k^2 - 2nm).$

By Proposition 1.8 $k^2 - 2nm$ is a square, so $k^2 - 2nm \equiv 4 \bmod 8$ (using (8.7) and $k \equiv 2 \bmod 4$). Then

(8.9) $\quad k^2 - 2nm = 4(2t + 1)^2$

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for some integer t. Conditions (8.8) and (8.9) yield

(8.10) $2n = k - 4(2t + 1)^2$,

so (8.5) implies that

(8.11) $k - 4(2t + 1)^2 \mid k(k - 1)$.

Assume $t = 0$. Then $k - 4 \mid k(k - 1)$. Then every prime power factor of $k - 4$ divides k or $k - 1$ (as well as $k - 4$), so $k - 4 \mid 12$. By hypothesis, we now get

$$2 \equiv k = 4 + 2^i 3^j \bmod 4,$$

so i is odd, hence $i = 1$. Then $k - 4$ is 2 or 6, and k is 6 or 10. If $k = 6$, condition (8.10) would imply $n = 1$, contradicting (8.6). If $k = 10$ our hypothesis is contradicted. Then $t \neq 0$.

If $t = 1$, condition (8.11) implies that $k - 36$ divides $k(k - 1)$. Arguing as before one sees that $k - 36 = 2.3^i 5^j 7^r$ where $i \leqq 2$ and $j, r \leqq 1$. Using the fact that k is not the sum of two squares, one sees that there are only 7 possible pairs (k, n) for $t = 1$; namely, $(k, n) = (42, 3), (46, 5), (54, 9), (66, 15), (78, 21), (126, 45), (246, 105)$. If $t \geqq 4$, conditions (8.8) and (8.9) imply

$$k - 2n = k^2 - 2nm \geqq 324.$$

Since $n \geqq 2$, $k \geqq 328$, hence $k \geqq 330$, a contradiction. Treating the cases $t = 2, 3$ as we treated the case $t = 1$, we obtain the result stated above.

We finally remark that a detailed study of semibiplanes is in [**53**] and that semisymmetric designs are also studied in [**26**].

Note. After finishing this research, the author has obtained a copy of Delsarte's unpublished paper [**14**]. There is some overlap between his results and Chapters 6 and 7 of this paper. In particular, his paper contains a result similar to our Theorem 6.8 and also Corollary 7.7.

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Notes added in proof. 1) Topics related to those in the present paper are also discussed by H. P. Ko and D. K. Ray-Chaudhuri in their paper "Intersection theorems for group divisible difference sets" (Discr. Math. *39* (1982), 37–58).

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2) Dina Ghinelli Smit has considered non-existence results for automorphism groups of divisible designs which in particular include a strengthening of the Bose-Connor theorem for symmetric divisible designs and also apply to relative difference sets (see her paper "Automorphisms and generalized incidence matrices of point-divisible designs", to appear in the Proc. of the Int. Conf. on Combinatorial Geometries and their applications (Annals of Discr. Math.), and her Ph.D. thesis (University of London, 1982)).

3) After submitting this paper the author has become aware of the fact that certain families of relative difference sets (with respect to a normal subgroup) have been studied from the point of view of quasiregular collineation groups of projective planes, see e.g. F. C. Piper, "On relative difference sets and projective planes", Glasgow Math. J. *15* (1974), 150–154, and M. J. Ganley and E. Spence, "Relative difference sets and quasi-regular collineation groups", J. Comb. Th. A *19* (1975), 134–153. In particular, it has been shown by C. W. H. Lam (using rather involved arguments and a computer search) that no normal relative difference set with parameters (4.7) can exist for $q = 4$ or 5 (see his paper "On relative difference sets", in Proc. Seventh Manitoba Conference on Numerical Math. and Computing 1977, pp. 445–474); note that the remarks given after the proof of Proposition 4.9 immediately rule out the case $q = 5$ even for relative difference sets with respect to a not necessarily normal subgroup.

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