LOCAL PROPERTIES OF THE EMBEDDING OF A GRAPH IN A THREE-MANIFOLD

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1. Introduction and definitions. Let G be a finite graph topologically embedded in the interior of a 3-manifold M. Doyle **(4)** and Debrunner and Fox **(3)** have noted that the following local homotopy condition at each point $p \in G$ is necessary in order for the embedding of G to be tame:

For each sufficiently small open set U containing p, there is an open set V such that $p \in V \subset U$ and if W is any connected open set such that $p \in W \subset V$, then the image under the inclusion homomorphism i^*: $\pi_1(W - G) \to \pi_1(U - G)$ is a free group on $n - 1$ generators.

Here, n denotes the *order* of p in G, i.e., the smallest integer n such that p has arbitrarily small neighbourhoods (in G) each with a boundary consisting of exactly n points. The base point is any point of $W - G$. We abbreviate the previous paragraph by saying that "$M - G$ has 1-FLG at p" (*free local fundamental groups at p*). If p is a point of order two (one), this condition is equivalent to "$M - G$ has 1-ALG at p" ("$M - G$ is 1-LC at p"), as defined in **(11)**. Another statement equivalent to "$M - G$ has 1-ALG at p" which we shall use is the following: *For each sufficiently small open set U containing p there is an open set V such that $p \in V \subset U$ and each loop in $V - G$ which "bounds"* (see **11**) *in $U - G$ is contractible in $U - G$.*

We are concerned here with the question of whether G is tame if $M - G$ has 1-FLG at each point of G. Lemma 5 reduces the problem to the case where G is an arc, and it follows from Theorem 1 that such an arc is tame if each of its subarcs pierces a 2-cell. The outstanding question left unanswered is whether each 1-ALG arc pierces a 2-cell; see §3.

Most of our terminology is well known. For example, for a discussion of what it means for an arc to pierce a 2-cell, see **(11)**. We remark here for later reference that if an arc A in S^3 pierces a 2-cell at a point $p \in \text{Int}\, A$, then (by Bing's polyhedral approximation theorem) it pierces a 2-cell D that is locally polyhedral except at p and is such that $D \cap A = p$. For the terms "locally polyhedral," "locally tame," "tame," etc., see **(1)**. We use the following notations: Δ^n is the closed n-simplex; Z is the infinite cyclic group; $B(x; \epsilon)$ denotes the ϵ-neighbourhood of x; and "piecewise linear" is abbreviated as

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"pwl." We assume that the term "manifold" refers to a connected space unless explicitly stated otherwise.

Finally, if M is a 3-manifold-with-boundary, we say that M is *irreducible* if for each component C of Bd M the kernel of the inclusion-induced homomorphism $i^*: \pi_1(C) \to \pi_1(M)$ is trivial. It follows from **(14)** and **(15)** that M is reducible (i.e., not irreducible) if and only if there is a polyhedral 2-cell $D \subset M$ such that $D \cap \text{Bd } M = \text{Bd } D$ and Bd D is not contractible in Bd M.

2. A special case: Isolated singular points. The aim of this section is to reduce the problem of taming a 1-FLG graph to the problem of taming a 1-ALG arc. We first need a result about untangling a collection of arcs. This turns out to be an application of a theorem in **(16)**. This result is stated here as Lemma 1 for the reader's convenience.

LEMMA 1 (Stallings). *Let M be a compact 3-manifold with connected boundary. Let T_0 and T_1 be disjoint homeomorphic compact 2-manifolds-with-boundary in* Bd M *such that T_0 and T_1 are polyhedral and each component of the closure of*

$$(\text{Bd } M) - (T_0 \cup T_1)$$

is an annulus joining one component of Bd T_0 *to one component of* Bd T_1. *Suppose further that each polyhedral 2-sphere in M bounds a 3-cell in M and that the inclusion of T_0 into M induces an isomorphism of $\pi_1(T_0)$ onto $\pi_1(M)$. Then there is a piecewise-linear homeomorphism h of M onto $T_0 \times [0, 1]$ such that $h(x) = (x, 0)$ for $x \in T_0$ and $h(T_1) = T_0 \times 1$.*

The proof is contained in the proof of **(16**, Theorem 2**)**. The relevant sections are 7–10, on pp. 97 and 98. Our notation is the same as that of Stallings, and our hypotheses are exactly those required in this portion of the proof. We remark that he also obtains (§§11 and 12, pp. 98 and 99) the analogue of the above result when Bd M consists of two homeomorphic closed surfaces that are neither 2-spheres nor projective planes.

Now let $C \subset E^3$ be the cube

$$\{(x_1, x_2, x_3) \mid 0 \leqslant x_i \leqslant 1\},$$

with opposite faces

$$A = \{(x_1, x_2, x_3) \in C \mid x_3 = 1\}$$

and
$$B = \{(x_1, x_2, x_3) \in C \mid x_3 = 0\}.$$

Let $\pi: C \to B$ be the projection $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$ and let $a_1, \ldots, a_n$ be distinct points of Int A and $b_1, \ldots, b_n$ be points of B such that $\pi(a_i) = b_i$. Let L_i be the straight-line segment from a_i to b_i. Choose a base point in

$$(\text{Int } A) - \bigcup_{i=1}^{n} a_i$$

for the fundamental groups appearing in the following lemma and its proof.

LEMMA 2. *Let $J_1, \ldots, J_n$ be a disjoint collection of polygonal arcs in C such that $J_i \cap \operatorname{Bd} C = a_i \cup b_i$. If the inclusion-induced homomorphism*

$$\alpha^*: \pi_1\left(A - \bigcup_{i=1}^{n} a_i\right) \to \pi_1\left(C - \bigcup_{i=1}^{n} J_i\right)$$

is onto, then there is a piecewise-linear homeomorphism H of C onto itself which is the identity on A, which maps B onto itself, and which throws the collection $J_1, \ldots, J_n$ onto the collection $L_1, \ldots, L_n$. In fact, if h is any piecewise-linear homeomorphism of A onto itself such that $h(a_i) = a_i$, for each i, then H can be chosen so that $H|A = h$.

Proof. The last assertion follows immediately from the first part of the lemma, since C and the collection of arcs $L_1, \ldots, L_n$ can be considered as the "product" with $[0, 1]$ of A and the collection of points $a_1, \ldots, a_n$. Hence, if I is the identity map on $[0, 1]$, $h \times I$ maps C onto itself in a "level-preserving" manner, is the identity on each L_i, and extends h.

The first part of the lemma will be obtained from Lemma 1. Let $C_1, \ldots, C_n$ be a disjoint collection of polyhedral 3-cells in C such that $C_i \cap \operatorname{Bd} C$ consists of two polyhedral 2-cells, $F_i \subset \operatorname{Int} A$, and $\pi(F_i) = G_i \subset \operatorname{Int} B$, and such that $J_i \subset \operatorname{Bd} C_i$. Let M be the 3-manifold which is the closure of

$$C - \bigcup_{i=1}^{n} C_i$$

and let

$$T_0 = A - \bigcup_{i=1}^{n} \operatorname{Int} F_i, \qquad T_1 = B - \bigcup_{i=1}^{n} \operatorname{Int} G_i.$$

Then M, T_0, and T_1 will satisfy the hypotheses of Lemma 1 when we have shown that the inclusion $T_0 \to M$ induces a monomorphism on fundamental groups, since we have an epimorphism by hypothesis.

If the kernel of this homomorphism were non-trivial, we would obtain from the Loop Theorem **(14)** and Dehn's Lemma **(15)** a polyhedral 2-cell

$$D \subset C - \cap_{i=1}^{n} J_i$$

such that $D \cap \operatorname{Bd} C = \operatorname{Bd} D \subset A$ and $\operatorname{Bd} D$ is not contractible on $A - \cup_{i=1}^{n} a_i$. Let E be the 2-cell in A bounded by $\operatorname{Bd} D$. Clearly, $E \cap \cup_{i=1}^{n} a_i \neq \emptyset$, say $a_1 \in E$. Let J be an arc joining a_1 to b_1 such that $\operatorname{Int} J \subset S^3 - C$. Then $J \cup L_1$ and $\operatorname{Bd} D$ are linking simple closed curves in S^3. This is impossible, since $\operatorname{Bd} D$ is contractible in $S^3 - (J \cup L_1)$.

Let the symbols C_i' and M' have the same meaning with respect to the arcs $L_1, \ldots, L_n$ that C_i and M have with respect to the arcs $J_1, \ldots, J_n$. We assume that $C_i' \cap A = F_i$ and $C_i' \cap B = G_i$. By Lemma 1, there is a pwl homeomorphism H of M onto M' such that H is the identity on T_0 and $H(T_1) = T_1$. Since $H(\operatorname{Bd} G_i) = \operatorname{Bd} G_i$, we may also assume that $H(b_i) = b_i$, for each i. Extend H by the identity to take each F_i onto itself, and by some pwl homeomorphism to take each G_i onto itself. Finally, extend H to take C_i

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piecewise-linearly and homeomorphically onto C_i', for each i. This gives a pwl homeomorphism H of C onto C which is the identity on $A \cup \bigcup_{i=1}^{n} b_i$, which maps B onto itself, and is such that $H(J_i) \subset \text{Bd } C_i'$, for each i. It is now an easy matter to "straighten" the $H(J_i)$'s, one by one, with a pwl homeomorphism of C onto C, moving no point of Bd C. This completes the proof.

Let S^2 be the boundary of a 3-simplex and let $a_1, \ldots, a_n$ be distinct points of S^2. The following lemma is a corollary to the proof of Lemma 2.

LEMMA 2′. *Let $J_1, \ldots, J_n$ be a disjoint collection of polygonal arcs in $S^2 \times I$ such that*

$$J_i \cap \text{Bd}(S^2 \times I) = (a_i \times \{0\}) \cup (a_i \times \{1\}) \qquad \textit{for each } i.$$

If the inclusion-induced homomorphism

$$\alpha^*\colon \pi_1(S^2 \times \{0\} - \bigcup_{i=1}^{n} a_i \times \{0\}) \to \pi_1\left(S^2 \times I - \bigcup_{i=1}^{n} J_i\right)$$

is onto, then there is a piecewise-linear homeomorphism H of $S^2 \times I$ onto itself which is the identity on $S^2 \times \{0\}$ and which is such that $H(J_i) = a_i \times I$ for each i. In fact, if h is any piecewise-linear homeomorphism of $S^2 \times \{0\}$ onto itself such that $h(a_i \times \{0\}) = a_i \times \{0\}$, for each i, then H can be chosen so that $H|S^2 \times \{0\} = h$.

Following **(3)**, if $n \geqslant 2$, an *n-frame* will mean any space homeomorphic to the 1-complex formed by taking the join of a point with n distinct points. Note that if $n \geqslant 3$ there is a unique point of order n in an n-frame, called the *branch point*. A branch point of an arc (which we shall consider a 2-frame) will mean any interior point.

LEMMA 3. *Let G be an n-frame in S^3 such that G is locally polyhedral except at the branch point p. If $S^3 - G$ has* 1-FLG *at p, then G is tame.*

Proof. It will be shown first that p can be enclosed in a polyhedral 2-sphere S of arbitrarily small diameter such that $S \cap G$ consists of exactly n points, one on each "branch," at each of which G pierces S. To this end, let U be any open spherical neighbourhood about p, and sufficiently small that the open set V (which we also assume to be an open spherical neighbourhood) in the definition of the 1-FLG property exists. We also assume U to be so small that the points of order one of G are in $S^3 - U$. Finally, choose another open spherical neighbourhood W of p such that $W \subset V$ and W has, with respect to V, the properties guaranteed in the definition of 1-FLG.

Let

$$i\colon V - G \to U - G,$$

$$j\colon W - G \to V - G$$

be the inclusions and i^*, j^* the corresponding homomorphisms on fundamental groups (with any base point in $W - G$). Since the images of j^* and i^*j^* are free groups on $n - 1$ generators and since a free group of finite rank cannot be mapped onto a free group of the same rank by a homomorphism with non-trivial kernel, we have: the kernel of $i^*|(\text{Image } j^*)$ is trivial. This fact will be needed later. Let us now suppose that there is no 2-sphere S in V of the type desired. Most of the rest of the proof of Lemma 3 will be devoted to showing that this assumption leads to a contradiction.

Consider the collection Σ^* of polyhedral subsets Σ of S^3 with the following three properties:

(1) Σ is a compact 3-manifold-with-boundary and $p \in \text{Int } \Sigma \subset \Sigma \subset V$.

(2) Exactly one component of Bd Σ meets G, and its intersection with G consists of n points, at each of which G pierces this component in general position.

(3) If k is the inclusion $\Sigma - G \to V - G$ and k^* the induced homomorphism on fundamental groups, then $(\text{Image } k^*) \subset (\text{Image } j^*)$, for any base point not in G but sufficiently close to p.

To see that Σ^* is non-empty, we enclose p in a polyhedral 3-cell A such that $A \subset W$ and Bd A is in general position relative to G, so that $G \cap \text{Bd } A$ consists of a finite number of points. Further, A is chosen so small that each component of $G - A$ that does not contain a point of order one of G is contained in W. That this last can be done follows from the fact that G is uniformly locally arcwise-connected. We now add each of these "small" components of $G - A$ to A and take a small closed-star neighbourhood of the resulting complex to obtain an element of Σ^*. This particular manifold is actually a cube-with-handles and lies in W, but we shall not require these properties for all elements of Σ^*.

If $\Sigma \in \Sigma^*$, we define a non-negative integer $c(\Sigma)$ measuring the "complexity" of Σ. Let

$$c(\Sigma) = \sum_{n \geqslant 0} n^2 g(n)$$

where $g(n)$ is the number of components of genus n in Bd Σ. Let Σ_0 be an element of Σ^* with $c(\Sigma_0)$ minimal. By the hypothesis made earlier about the non-existence of S, $c(\Sigma_0) > 0$.

The rank of $H_1(\Sigma_0 - G; Z)$ is at least n. To see this, choose some (new) pwl embedding of Σ_0 in S^3 such that the complement in S^3 of the interior of the image under the new embedding is a regular neighbourhood of a finite (possibly not connected) polyhedral graph H. Such an embedding is guaranteed by (**6**, Main Theorem) and, since $c(\Sigma_0) > 0$, some component of H contains a simple closed curve. Hence, $\Sigma_0 - G$ has the homotopy type of $S^3 - H'$, where H' is a (non-polyhedral) finite graph with first Betti number at least n (recall that G meets only one component of Bd Σ_0). By Alexander duality, the rank of $H_1(\Sigma_0 - G; Z)$ is at least n.

Now by property (3) of Σ_0 and an earlier remark, the kernel of $i^*|(\text{Image } k_0^*)$ is trivial, where k_0 is the inclusion $\Sigma_0 - G \to V - G$. Hence if k_0^* were a monomorphism, $i^*k_0^*$ would be also. But the image of $i^*k_0^*$ is free of rank

$n - 1$ by our choice of U and V, so $\pi_1(\Sigma_0 - G)$ would then be free of rank $n - 1$ and $H_1(\Sigma_0 - G; Z)$ would then be free abelian of rank $n - 1$, a contradiction to the previous paragraph. We conclude that k_0^* is not a monomorphism.

Let f: Bd $\Delta^2 \to \Sigma_0 - G$ be a pwl homeomorphism representing a nontrivial element of the kernel of k_0^*. Then f can be extended to Δ^2 so as to be pwl, to map Δ^2 into $V - G$, and (calling the extension f) to be "transverse" to each component of Bd Σ_0, so that each component of f^{-1}(Bd Σ_0) is a polyhedral simple closed curve.

If $\Sigma_0 - G$ and all of the components of $(V - \text{Int } \Sigma_0) - G$ were irreducible 3-manifolds, then we could eliminate the components of f^{-1}(Bd Σ_0) by a familiar process. Namely, let J be an "inside" component of f^{-1}(Bd Σ_0) and (using the irreducibility) redefine f on the interior of the 2-cell D in Δ^2 bounded by J so as to map D into (Bd Σ_0) $- G$. Then push the new map slightly to one side of the appropriate component of Bd Σ_0, thus producing a mapping h of Δ^2 into $V - G$ which extends the original f and is such that h^{-1}(Bd Σ_0) has fewer components than does f^{-1}(Bd Σ_0). Continuing thus, after a finite number of steps we would find that f is null-homotopic in $\Sigma_0 - G$, a contradiction. We thus conclude from the remark about irreducible manifolds in §1 that there is a polyhedral 2-cell $E \subset V - G$ such that

$$\text{Bd } E \subset \text{Bd } \Sigma_0, \qquad (\text{Bd } \Sigma_0) \cap (\text{Int } E) = \emptyset,$$

and Bd E is not contractible on (Bd Σ_0) $- G$.

Note now that Bd E is not even contractible in Bd Σ_0. If it were, it would bound a 2-cell $F \subset$ Bd Σ_0. Clearly, $F \cap G \neq \emptyset$. In fact, it follows from an easy linking argument (similar to the one in the proof of Lemma 2) that $G \cap \text{Bd } \Sigma_0 \subset F$. We could then take $S = E \cup F$ to be the 2-sphere mentioned earlier, a contradiction. Hence, Bd E is not contractible in Bd Σ_0.

There are now two possibilities to consider: either $E \subset \Sigma_0$, or

$$\Sigma_0 \cap \text{Int } E = \emptyset.$$

In the first case, we cut Σ_0 along E to obtain either one or two new manifolds. The component of the severed Σ_0 containing G is then an element of Σ^* that is less complex than Σ_0, a contradiction. In the second case, E is thickened nicely to form a 3-cell $K \subset V - G$ such that

$$K \cap \Sigma_0 = (\text{Bd } K) \cap (\text{Bd } \Sigma_0)$$

is an annulus. Then $\Sigma_0 \cup K$ is an element of Σ^* of smaller complexity than Σ_0 (if Bd E separates the component C of Bd Σ_0 containing Bd E, we can use a linking argument to show that all of $G \cap C$ lies in one component of $C -$ Bd E). This final contradiction forces us to admit the existence of the 2-sphere S mentioned at the beginning of the proof.

Using this result, and retaining the meaning of the symbols U and V, select a sequence $B_1, B_2, \ldots,$ of polyhedral 3-cells with diameters converging to zero such that

(1) $p \in B_{i+1} \subset \operatorname{Int} B_i \subset B_i \subset V$;

(2) Bd B_i is in general position with respect to G, and meets it in exactly n points; and

(3) $\pi_1(B_i - G)$ is freely generated by $n - 1$ loops in $B_i - B_{i+1} - G$.

Clearly there is a sequence $B_1, B_2, \ldots$, satisfying the first two properties. A subsequence satisfying all three properties can be chosen if we show that $\pi_1(B_i - G)$ is free on $n - 1$ generators. Since $B_i \subset V$, the image of $\pi_1(B_i - G)$ in $\pi_1(U - G)$ under the inclusion homomorphism is a free group on $n - 1$ generators. Hence, we need only the fact that this homomorphism has a trivial kernel. If the kernel were non-trivial, we would argue as before to show that either $(U - G - \operatorname{Int} B_i)$ or $B_i - G$ is reducible. Linking considerations show this to be impossible. Hence, the required sequence exists.

Let A_i be the annular region $B_i - \operatorname{Int} B_{i+1}$ so that

$$\operatorname{Bd} A_i = (\operatorname{Bd} B_i) \cup (\operatorname{Bd} B_{i+1}).$$

We claim that for $i \geqslant 2$, if a base point b_i is chosen in $(\operatorname{Bd} B_i) - G$, then $\pi_1(A_i - G)$ is generated by loops in $(\operatorname{Bd} B_i) - G$. To see this, note first that by (3) there are loops in $A_{i-1} - G$ based at b_i and generating $\pi_1(B_{i-1} - G)$. Now let an element of $\pi_1(A_i - G)$ be represented by a pwl map

$$f\colon (\operatorname{Bd} \Delta^2, v) \to (A_i - G, b_i).$$

Let σ be a 2-simplex in Δ^2 having v as one vertex and otherwise disjoint from Bd Δ^2. Since f also represents an element of $\pi_1(B_{i-1} - G)$, f extends to a pwl map (which we continue to call f)

$$f\colon (\Delta^2 - \operatorname{Int} \sigma, \operatorname{Bd} \sigma) \to (B_{i-1} - G, A_{i-1} - G),$$

such that $f^{-1}(\operatorname{Bd} B_{i+1})$ consists of a finite disjoint collection of polygonal simple closed curves in Int Δ^2 and $f^{-1}(\operatorname{Bd} B_i)$ consists of a finite disjoint collection of polygonal simple closed curves in Int Δ^2, plus another finite collection of polygonal simple closed curves (disjoint from the previous ones) all meeting at v and otherwise disjoint from each other and from Bd Δ^2. The following 3-manifolds are easily shown to be irreducible: $B_{i+1} - G$, $A_i - G$, and $A_{i-1} - G$. Hence, all of $f^{-1}(\operatorname{Bd} A_i)$ can be eliminated with the exception of those curves mentioned above that pass through v and separate $\sigma - v$ from $(\operatorname{Bd} \Delta^2) - v$ in Δ^2. There are such curves, since Bd B_i separates B_{i-1}. These remaining curves are linearly ordered by inclusion of the 2-cells they bound in Δ^2. The maximal curve in this ordering represents an element of

$$\pi_1((\operatorname{Bd} B_i) - G)$$

and clearly it is homotopic (rel v) in $A_i - G$ to f. Thus our assertion is proved.

We are now in a position to construct a homeomorphism h of S^3 onto itself throwing G onto a polyhedron. It will be clear from the construction that we can make h the identity outside an arbitrarily small neighbourhood of p. In fact, we shall need to move no point of S^3 outside a small neighbourhood of B_1. Also, h can be made locally pwl on $S^3 - p$.

First, since each A_i is homeomorphic to $S^2 \times [0, 1]$, we may suppose without loss of generality that each B_i is a closed 3-simplex with barycentre at p and that B_{i+1} is obtained from B_i by a radial contraction towards p. Further, we can adjust G near each Bd B_i so that if $\alpha_1, \ldots, \alpha_n$ are the arcs in G joining all the distinct points of order one of G to p, then

$$p \cup \left[\alpha_j \cap \left(\bigcup_{i \geqslant 1} \mathrm{Bd}\, B_i\right)\right]$$

is contained in a straight line L_j such that $L_j \cap L_k = p$ for $j \neq k$, and each $L_j \cap A_i$ is a straight-line interval.

Since $\pi_1(A_i - G)$ is generated by loops in $(\mathrm{Bd}\, B_i) - G$, $i \geqslant 2$, there is by Lemma 2′ a pwl homeomorphism h_2 of A_2 onto itself such that $h_2 \mid \mathrm{Bd}\, B_2$ is the identity and each component of $G \cap A_2$ is taken by h_2 onto the line segment of the form $L_j \cap A_2$ having the same end points. Now apply Lemma 2′ again to obtain a pwl homeomorphism h_3 of A_3 onto itself such that

$$h_3 \mid \mathrm{Bd}\, B_3 = h_2 \mid \mathrm{Bd}\, B_3$$

and each component of $G \cap A_3$ is taken by h_3 onto the line segment of the form $L_j \cap A_3$ having the same end points. Continue thus to define h_i for each $i \geqslant 2$. Finally, define h to be the identity on $[p \cup (S^3 - \mathrm{Int}\, B_2)]$, and let $h \mid A_i = h_i$ for $i \geqslant 2$. This defines a homeomorphism h and it is clear that $h(G)$ is a polyhedron. This completes the proof of Lemma 3.

The proof of the following is the same as the proof of Lemma 3, but it is convenient to state it separately.

LEMMA 4. *Let G be an arc in S^3 such that G is locally polyhedral except at the end point p. If $S^3 - G$ is $1 -$ LC at p, then G is tame.*

Finally, we state without proof the main result of this section. It is an easy consequence of Lemmas 3 and 4, (**12**, Theorem 2), and the fact that "locally tame sets are tame" (**1**, Theorem 8 and **13**, Theorem 8.1).

LEMMA 5. *Let G be a finite graph topologically embedded in the interior of a 3-manifold M. Suppose that $M - G$ has* 1-FLG *at each point of G and that G is locally tame except possibly at a finite set of points. Then G is tame in M.*

3. Some results on taming 1-ALG **arcs.** The chief problem considered in this section is that of showing that a 1-ALG arc is tame if each of its subarcs pierces a 2-cell. Since (Lemma 6) each such subarc is again a 1-ALG arc, the decisive point for determining whether or not each 1-ALG arc is tame is contained in the following:

QUESTION. *If A is an arc in S^3 such that $S^3 - A$ has* 1-ALG *at each point of A, does A pierce a 2-cell?*

First, we state some preliminary lemmas. The first two follow readily from the "chaining" techniques used in (**7**, Lemma 5.1 and **11**, Section 2), and their proofs will not be included.

LEMMA 6. *Let A be an arc topologically embedded in the interior of a* 3*-manifold M in such a way that $M - A$ has* 1-ALG *at each point of A. If $A^* \subset A$ is an arc, then $M - A^*$ has* 1-ALG *at each point of A^*.*

LEMMA 7. *Let A be an arc embedded in S^3 in such a way that $S^3 - A$ has* 1-ALG *at each point of A, and let $A^* \subset \operatorname{Int} A$ be a subarc. If U is an open subset of S^3 containing A^*, then there is an open set V such that $A^* \subset V \subset U$ and each loop in $V - A$ which bounds in $U - A$ is contractible in $U - A$.*

LEMMA 8. *Let D be a* 2*-cell in S^3 such that D is locally polyhedral except at p in $\operatorname{Int} D$. Suppose that A is an arc in S^3 such that $S^3 - A$ has* 1-ALG *at each point of A and that A pierces D at p. Then D is tame.*

Proof. By taking a subarc of A, if necessary, we may assume by Lemma 6 that $A \cap D = p$. Let ϵ, δ be positive numbers such that

$$B(p; \delta) - D = W_1 \cup W_2,$$

where W_1, W_2 are disjoint open sets such that any pair of points of $W_i (i = 1, 2)$ can be joined by an arc in $B(p; \epsilon) - D$. Since A pierces D at p, ϵ can also be taken so small that for some subarc A^* of A containing p in its interior, each of W_1, W_2 contains exactly one component of $A^* - p$. Finally, we require that $B(p; \epsilon)$ does not intersect $\operatorname{Bd} D$ or $\operatorname{Bd} A$ and (by the uniform local arcwise-connectivity of A) that each of the two components of $A - p$ meets exactly one of W_1, W_2.

Let B be the arc which is the closure of the component of $A - p$ containing the component of $A^* - p$ which is contained in W_2. Since, by Lemma 6, $S^3 - B$ has 1-ALG at each point of B, there are open sets U^*, U, and V such that

$$p \in V \subset U \subset U^* \subset B(p; \delta),$$

each loop in $V - B$ is null-homotopic in $U - B$, and $U^* \cap D$ is an open 2-cell. We assert that there is a polygonal loop in $(U^* \cap D) - p$ which is not contractible in $(U^* \cap D) - p$ but which is contractible in

$$[(U^* \cap D) - p] \cup [U^* \cap W_1].$$

If this is so, then by applying the Loop Theorem (**14**) and Dehn's Lemma (**15**) to this last set (which is a 3-manifold-with-boundary), there will be a polyhedral 2-cell E such that

$$\operatorname{Int} E \subset U^* \cap W_1, \qquad \operatorname{Bd} E \subset (U^* \cap D) - p,$$

and $\operatorname{Bd} E$ is not contractible in $(U^* \cap D) - p$. Repeating the argument for each element of a suitably chosen sequence of ϵ's converging to 0, we shall obtain a corresponding sequence of E's and we can argue as in (**10**, Theorem I)

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that, if F is the 2-cell in D with $\text{Bd}\, F = \text{Bd}\, E$, then the closure of the "small" component of $S^3 - (E \cup F)$ is a 3-cell. Hence, to show that D is tame from the "W_1 side" at p, we need only show the existence of a loop with the properties listed above.

We now make the assumption that each loop in $(U^* \cap D) - p$ that is contractible in $[(U^* \cap D) - p] \cup [U^* \cap W_1]$ is also contractible in $(U^* \cap D) - p$, and seek a contradiction. There is, by our choice of U^*, U, and V, a pwl mapping

$$f\colon (\Delta^2, \text{Bd}\, \Delta^2) \to (U^* - B, (U^* \cap D) - p),$$

such that $f \mid \text{Bd}\, \Delta^2$ is not contractible in $(U^* \cap D) - p$ and f is "transverse" to D, so that each component of $f^{-1}(D)$ is a polygonal simple closed curve. We assume that, among all such mappings, f is one with the number of components of $f^{-1}(D)$ minimal. Now if $f^{-1}(D) = \text{Bd}\, \Delta^2$, then either

$$f(\text{Int}\, \Delta^2) \subset U^* \cap W_1 \quad \text{or} \quad f(\text{Int}\, \Delta^2) \subset U^* \cap W_2.$$

The first alternative is outlawed by assumption, and the second by the fact that if we join the end points of A with an arc missing

$$D \cup B(p; \epsilon) \cup \text{Int}\, A,$$

we obtain a simple closed curve linking $f(\text{Bd}\, \Delta^2)$. Hence,

$$f^{-1}(D) \cap (\text{Int}\, \Delta^2) \neq \emptyset.$$

Let L be an "innermost" component of $f^{-1}(D)$, i.e., one bounding a 2-cell $G \subset \text{Int}\, \Delta^2$ such that $f(\text{Int}\, G) \cap D = \emptyset$. Then $f \mid L$ is a loop in $(U^* \cap D) - p$ which is contractible in $f(L)$ plus either $U^* \cap W_1$ or $U^* \cap W_2$. Arguing as in the previous paragraph, $f|L$ is contractible in $(U^* \cap D) - p$. Hence, f may be altered on a small neighbourhood of G so as to reduce the number of components of $f^{-1}(D)$ in $\text{Int}\, \Delta^2$, a contradiction; see the proof of Lemma 3.

Hence, D is tame from the "W_1 side" at p. By symmetry, D is tame from the "W_2 side" at p, hence locally tame at p, and hence tame. This completes the proof.

We state the following result here as a lemma for the reader's convenience. It was announced in a slightly weaker form than this in **(5)**. If each of H_1, H_2 is a cube-with-one-handle and H_2 is a tame subset of $\text{Int}\, H_1$, then we say that H_2 is *concentric* with H_1 if $H_1 - \text{Int}\, H_2$ is topologically the product of $S^1 \times S^1$ with the unit interval.

LEMMA 9 (Edwards). *Let J be a simple closed curve in S^3 each of whose subarcs pierces a 2-cell. Then J is tame if there exists a sequence $H_1, H_2, \ldots$, each element of which is a tame cube-with-one-handle such that:*

$$J = \bigcap_{i=1}^{\infty} H_i, \qquad H_{i+1} \subset \text{Int}\, H_i,$$

and H_{i+1} is concentric with H_i, for each i.

The following is our main result.

THEOREM 1. *Let G be a finite graph topologically embedded in the interior of a 3-manifold M in such a manner that $M - G$ has* 1-FLG *at each point of G. If each arc in G pierces a 2-cell, then G is tame in M.*

Proof. By Lemma 5, it suffices to show that G is locally tame at each of its points of order two. And to do this, it is enough to show that if $A \subset S^3$ is an arc such that $S^3 - A$ has 1-ALG at each point of Int A and each arc in A pierces a 2-cell, then A is locally tame at each of its interior points.

Accordingly, let A be such an arc and let $p \in \text{Int}\, A$. By hypothesis, we may find disjoint 2-cells D_1, D_2 such that for $i = 1, 2$, A pierces D_i at

$$A \cap D_i = p_i \in \text{Int}\, A,$$

D_i is locally polyhedral except at p_i, and the interior of the subarc of A joining p_1 and p_2 contains p. Let B be an arc in S^3 such that: $B \cap A = \{p_1\} \cup \{p_2\}$; B is locally polyhedral except at p_1, p_2; and for $i = 1, 2$, $B \cap D_i$ is an arc joining p_i to a point of Bd D_i. Then B and the subarc of A joining p_1, p_2 form a simple closed curve J that is locally polyhedral at each point of Int B and is such that $S^3 - J$ has 1-ALG at least at each point of $J - \{p_1\} - \{p_2\}$. Further, each arc in J pierces a 2-cell, and J contains a neighbourhood in A of p. Hence, the proof will be complete if we can show that J is tame.

First, we assert that $S^3 - J$ has 1-ALG at $p_i (i = 1, 2)$. To see this, let U be (relative to the 1-ALG condition for A at p_1) a "sufficiently small" open set in S^3 containing (say) p_1. Let us agree to call an open set W containing p_1 *normal* if there is a homeomorphism h of W onto E^3 such that $h(W \cap D_1)$ is the xy-plane, $h(W \cap B)$ is the non-negative part of the x-axis, and $h(W \cap J)$ is contained in the half-space H consisting of all points on or above the xy-plane. Such neighbourhoods exist by Lemma 8.

Let W and V be normal open sets such that $V \subset W \subset U$; the images of the inclusion-induced homomorphisms

$$H_1(W - J; Z) \to H_1(U - J; Z),$$
$$H_1(V - J; Z) \to H_1(W - J; Z),$$

are infinite cyclic; and V has, relative to W, the properties guaranteed by the fact that $S^3 - A$ has 1-ALG at p_1. Now let a loop f in $V - J$ be given which bounds in $U - J$. Then, by the above, f bounds in $W - J$. Using the "normal" homeomorphism h between V and E^3, f is homotopic in $V - J$ to a pwl loop f_1 such that hf_1 is a loop in (Int H) $- h(V \cap J)$ (note: the paths in the homotopy may meet $A - J$). We remark that f_1 bounds in $W - J$ and hence, by linking considerations, bounds also in $W - A$. Thus, f_1 is null-homotopic in $W - A$. The proof that $S^3 - J$ has 1-ALG at p_1 will be complete if we can show that f_1 is null-homotopic in $W - J$.

Let $F: \Delta^2 \to W - A$ be a pwl mapping such that $F \mid \text{Bd}\, \Delta^2 = f_1$ and each component of $F^{-1}(D_1)$ is a polygonal simple closed curve in Int Δ^2. Now, F restricted to such a curve represents a loop in $(W \cap D_1) - p_1$ which is contractible in $W - A$. We see easily that this loop is actually contractible in

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$(W \cap D_1) - p_1$ and hence F may be adjusted so as to retain the previous properties and yet one component of $F^{-1}(D_1)$ may be eliminated. Continuing by induction on the number of such components, we may suppose that $F(\Delta^2) \cap D_1 = \emptyset$, and our assertion follows. Hence, $S^3 - J$ has 1-ALG at each point of J.

Now by (**11**, Theorem 1) the arc $J \cap A$ is cellular in S^3. Hence, there is a mapping π of S^3 onto S^3 such that the only non-degenerate set of the form $\pi^{-1}(x)$ is $J \cap A$. Further $\pi(J)$ is a simple closed curve in S^3 that is locally tame except possibly at the point $\pi(J \cap A)$, and by Lemma 7, $S^3 - \pi(J)$ has 1-ALG at each point of $\pi(J)$. Thus by Lemma 5, $\pi(J)$ is tame.

Let $H_1, H_2, \ldots,$ be a sequence of tame subsets of S^3 such that H_i is a cube-with-one-handle ("solid torus" of genus one), $H_{i+1} \subset \operatorname{Int} H_i$, $\pi(J) = \bigcap_{i=1}^{\infty} H_i$, and H_{i+1} is concentric with H_i. Then the sequence $\pi^{-1}(H_1), \pi^{-1}(H_2), \ldots,$ enjoys exactly the same properties with respect to J. By Lemma 9, J is tame and the proof is complete.

COROLLARY. *Let G be a finite graph topologically embedded in the interior of a 3-manifold M in such a manner that $M - G$ has* 1-FLG *at each point of G. If G is locally tame except possibly at a zero-dimensional subset, then G is tame.*

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