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Abstract

Let p and ℓ be distinct primes, and let $\overline{\rho}$ be an orthogonal or symplectic representation of the absolute Galois group of an ℓ-adic field over a finite field of characteristic p. We define and study a liftable deformation condition of lifts of $\overline{\rho}$ 'ramified no worse than $\overline{\rho}$', generalizing the minimally ramified deformation condition for GL_n studied in Clozel *et al.* [*Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 1–181; MR 2470687 (2010j:11082)]. The key insight is to restrict to deformations where an associated unipotent element does not change type when deforming. This requires an understanding of nilpotent orbits and centralizers of nilpotent elements in the relative situation, not just over fields.

Contents

1. Introduction

Let ℓ and p be primes, L be a finite extension of $\mathbf{Q}_\ell$, and Γ_L be the absolute Galois group of L. Suppose $\mathcal{O}$ is the ring of integers in a p-adic field with residue field k. For a reductive group G, it is important to study deformations of a continuous representation $\overline{\rho} : \Gamma_L \to G(k)$. Information about the universal deformation ring, and quotients corresponding to restricted classes of deformations, have many applications, for example to producing congruences between modular forms, proving modularity lifting theorems, and understanding generalizations of Serre's conjecture and of the Breuil–Mézard conjecture. The case $G = \mathrm{GL}_n$ has received the most attention. In this paper, we assume $\ell \neq p$ and generalize the minimally ramified deformation condition for GL_n studied by Clozel, Harris and Taylor [CHT08, § 2.4.4] to symplectic and orthogonal groups.

This question was originally motivated by the problem of producing geometric deformations of representations of the absolute Galois group of a number field using a generalization of a method introduced by Ramakrishna [Ram99, Ram02]. For use in Ramakrishna's method, we

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would like to define a deformation condition of lifts which are 'ramified no worse than $\overline{\rho}$', such that the resulting deformation condition is liftable despite the fact that the unrestricted deformation condition for $\overline{\rho}$ may not be liftable. When $G = \mathrm{GL}_n$, the minimally ramified deformation condition defined in [CHT08, § 2.4.4] works. Attempting to generalize the argument of [CHT08, § 2.4.4] to groups besides GL_n leads to a deformation condition based on parabolics which is *not* liftable. Instead, inspired by the arguments of [Tay08, § 3] we define a deformation condition for symplectic and orthogonal groups based on deformations of a nilpotent element of $\mathrm{Lie}\, G_k$. This condition is liftable, which illustrates how genuinely new ideas are needed to study the deformation rings for representations valued in groups besides GL_n.

In § 5.2, we define a *minimally ramified deformation condition* for symplectic and orthogonal groups after extending the residue field k. This extension is harmless for the original application, and for that application it is also convenient to consider deformations with a fixed similitude character. Our main result is the following, which is precisely the local input needed in Ramakrishna's method in the $\ell \neq p$ case.

Theorem 1.1. *Let G be GSp_n or GO_n over $\mathbf{Z}_p$ with $p > n$, and let $\overline{\rho} : \Gamma_L \to G(k)$ be a continuous representation with $\ell \neq p$. After extending k, the minimally ramified deformation condition with fixed similitude character is a liftable deformation condition (in the sense of Definition 2.5), and its tangent space has dimension $\dim H^0(\Gamma_L, \mathrm{ad}^0(\overline{\rho}))$.*

This can equivalently be expressed as exhibiting a formally smooth quotient of the universal lifting ring $R^{\square}_{\overline{\rho}}$. In this paper, we study only the local theory: the applications to producing geometric lifts are discussed in [Boo19]. In the remainder of the introduction, we will sketch how to correctly generalize the minimally ramified deformation condition introduced for GL_n and analyze it. The strategy could work for general G, but several pieces of the argument are specific to orthogonal or symplectic groups (or GL_n), which was all that was needed for the original application.

The first step in [CHT08, § 2.4.4] is to reduce to studying certain tamely ramified representations. Clozel, Harris and Taylor reduce the problem to defining a nice class of deformations for representations of the group $T_q := \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$, where the first factor is generated by a Frobenius ϕ and the second by an element τ in the inertia group. They satisfy the relation $\phi\tau\phi^{-1} = q\tau$ for some q prime to p. This reduction generalizes without surprises to symplectic and orthogonal groups in § 6 (but the argument is genuinely restricted to orthogonal and symplectic groups as it relies heavily on the pairing).

The second step is to specify when a lift of $\overline{\rho} : T_q \to \mathrm{GL}_n(k)$ is 'ramified no worse than $\overline{\rho}$'. For a coefficient ring R, a deformation $\rho : T_q \to \mathrm{GL}_n(R)$ is *minimally ramified* according to [CHT08] when the natural k-linear map

$$\ker((\rho(\tau) - 1_n)^i) \otimes_R k \to \ker((\overline{\rho}(\tau) - 1_n)^i) \tag{1.1}$$

is an isomorphism for all i. The deformation condition is analyzed as follows.

- Defining $V_i = \ker((\overline{\rho}(\tau) - 1_n)^i)$ gives a flag

  $$0 \subset V_r \subset V_{r-1} \subset \cdots \subset V_1 \subset k^n.$$

  This flag determines a parabolic k-subgroup $\overline{P} \subset \mathrm{GL}_n$ (points which preserve the flag) such that $\overline{\rho}(\tau) \in (\mathcal{R}_u\overline{P})(k)$ and $\overline{\rho}(\phi) \in \overline{P}(k)$.

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- Lift $\overline{P}$ to a parabolic subgroup P of GL_n. The deformation functor of such lifts is formally smooth, and for any minimally ramified deformation ρ over R there is a choice of such P for which $\rho(\tau) \in (\mathcal{R}_u P)(R)$ and $\rho(\phi) \in P(R)$. Conversely, any ρ with this property is minimally ramified.
- Finally, for the standard block-upper-triangular choice of P, one shows the deformation functor
$$\{(T, \Phi) : T \in \mathcal{R}_u P, \Phi \in P, \Phi T \Phi^{-1} = T^q, \overline{T} = \overline{\rho}(\tau), \overline{\Phi} = \overline{\rho}(\phi)\}$$
is formally smooth by building the universal lift over a power series ring: this uses explicit calculations with block-upper-triangular matrices.

To generalize beyond GL_n, we need to replace (1.1) with a more group-theoretic criterion. The naive generalization is to associate a parabolic $\overline{P}$ to $\overline{\rho}$ and then use the following definition.

Definition 1.2. For a coefficient ring R, say a lift $\rho : T_q \to G(R)$ is *ramified with respect to* $\overline{P}$ provided that there exists a parabolic R-subgroup $P \subset G_R$ lifting $\overline{P}$ such that $\rho(\tau) \in (\mathcal{R}_u P)(R)$ and $\rho(\phi) \in P(R)$.

This idea does not work. Let us focus on the symplectic case to illustrate what goes wrong.

The first problem is to associate a parabolic subgroup to $\overline{\rho}$. Recall that parabolic subgroups of a symplectic group correspond to isotropic flags $0 \subset V_1 \subset \cdots \subset V_r \subset V_r^\perp \subset \cdots \subset V_1^\perp \subset k^{2n}$. There is no reason that the flag determined by (1.1) is isotropic, so we would need some other method of producing a parabolic $\overline{P}$ such that $\overline{\rho}(\tau) \in (\mathcal{R}_u \overline{P})(k)$. In [BT71], Borel and Tits give a natural way to associate to the unipotent $\overline{\rho}(\tau)$ a smooth connected unipotent k-subgroup of G. The normalizer of this subgroup is always parabolic and so gives a candidate for $\overline{P}$. However, working out examples in GL_n for small n shows that this produces a different parabolic than the one determined by (1.1). This raises the natural question of how sensitive the smoothness of the deformation condition is to the choice of parabolic.

This leads to the second, larger problem: there are examples such that for *every* parabolic $\overline{P}$ satisfying $\overline{\rho}(\tau) \in (\mathcal{R}_u \overline{P})(k)$, not all deformations ramified with respect to $\overline{P}$ are liftable.

Example 1.3. Take $L = \mathbf{Q}_{29}$ and $k = \mathbf{F}_7$. Consider the representation $\overline{\rho} : T_{29} \simeq \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_7 \to \mathrm{GSp}_4(\mathbf{F}_7)$ defined by
$$\overline{\rho}(\tau) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\rho}(\phi) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
The deformation condition of lifts ramified relative to a parabolic $\overline{P}$ of GSp_4 whose unipotent radical contains $\overline{\rho}(\tau)$ is not liftable for any choice of $\overline{P}$: there are lifts to the dual numbers that do not lift to $\mathbf{F}_7[\epsilon]/(\epsilon^3)$. This is easy to check with a computer algebra system such as [SAGE], since the existence of lifts can be reduced to a problem in linear algebra. This is a general phenomenon, which we will explain conceptually in § 5.3.

The correct approach is to define a lift $\rho : T_q \to G(R)$ to be minimally ramified if $\rho(\tau)$ has 'the same unipotent structure' as $\overline{\rho}(\tau)$. It is more convenient to work with nilpotent elements, using the exponential and logarithm maps (defined for nilpotent and unipotent elements since $p > n$). There are combinatorial parametrizations of nilpotent orbits of algebraic groups over an algebraically closed field, for example in terms of partitions or root data, which make precise the

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notion that the values of $N \in \mathfrak{g}_{\mathcal{O}}$ in the special and generic fiber lie in the same nilpotent orbit. In particular, for each nilpotent orbit σ, we use the results of § 3.1 to choose particular elements $N_\sigma \in \mathfrak{g}_{\mathcal{O}}$ with this property lifting $\overline{N} \in \mathfrak{g}_k$. In § 3.2, we define the *pure nilpotents lifting* $\overline{N}$ to be the $\widehat{G}(R)$-conjugates of N_σ for a coefficient ring R.

Example 1.4. For example, let $G = \mathrm{GL}_3$ and

$$\overline{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the lifts

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Both are nilpotent under the embedding of $\mathcal{O}$ into its fraction field K. The images of N_1 in $\mathfrak{g}_K$ and $\mathfrak{g}_k$ both lie in the nilpotent orbit corresponding to the partition $2+1$, so N_1 is an example of the type of nilpotent lift we want to consider. On the other hand, the image of N_2 in $\mathfrak{g}_K$ lies in the nilpotent orbit corresponding to the partition 3, while the image on $\mathfrak{g}_k$ lies in the orbit corresponding to $2+1$, so we do not want to use it. The pure nilpotents lifting $\overline{N}$ are $\widehat{G}(R)$-conjugates of N_1.

We finally define a lift $\rho : T_q \to G(R)$ to be *minimally ramified* provided $\rho(\tau)$ is the exponential of a pure nilpotent lifting $\log \overline{\rho}(\tau) = \overline{N}$. Proposition 5.6 shows that this deformation condition is liftable. The main technical fact needed to analyze this deformation condition is that the scheme-theoretic centralizer $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$ for N_σ as above. The smoothness of such centralizers over algebraically closed fields is well understood, and in § 4 we study $Z_G(N_\sigma)$ and show that $Z_G(N_\sigma)$ is *flat* over $\mathcal{O}$ and hence smooth. Lemma 4.4 gives a criterion for flatness that is easy to verify for classical groups, which suffices for our applications. We can reduce checking $\mathcal{O}$-flatness to the problem of finding elements $g \in Z_G(N_\sigma)(\mathcal{O})$ such that g_k lies in any specified component of $Z_{G_k}(\overline{N})/Z_{G_k}(\overline{N})^\circ$. There are difficulties beyond the classical cases due to the varied structure of $\pi_0(Z_G(\overline{N})_{\overline{k}})$ in general.

Remark 1.5. It is a fortuitous coincidence (for [CHT08]) that for GL_n the lifts minimally ramified in the preceding sense are exactly the lifts ramified with respect to a parabolic subgroup of G. This rests on the fact that all nilpotent orbits of GL_n are Richardson orbits (see § 5.3 for details).

1.1 Structure of the paper

Section 2 discusses deformation conditions, deformation rings, and lifting rings. Section 3 constructs integral representatives for nilpotent orbits, and defines the notion of a pure nilpotent lift. This notion requires a study of the $\mathcal{O}$-smoothness of $Z_G(N)$, which is carried out in § 4. Finally §§ 5 and 6 define and study the minimally ramified deformation condition, first in a special tamely ramified case and then in general.

1.2 Notation and assumptions

Throughout the paper, ℓ and p will be distinct primes, and $\mathcal{O}$ will be a discrete valuation ring with residue field k of characteristic p. We will ultimately work with orthogonal or symplectic (similitude) groups, or GL_n over $\mathcal{O}$, although the strategy of the argument (but not the details) would work in greater generality. Since reductive group schemes have connected fibers

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(a restriction going back to [SGA3, XIX, 2.7] to avoid the component group jumping across fibers), and since GO_m may be disconnected, the natural class of group schemes to work with are what we call *almost-reductive groups*. By this we mean a smooth separated group scheme over $\mathcal{O}$ such that the identity components of the fibers are reductive. Then G° is a reductive $\mathcal{O}$-subgroup scheme of G and G/G° is a separated étale $\mathcal{O}$-group scheme of finite presentation [Con14, Proposition 3.1.3 and Theorem 5.3.5]. Furthermore, by a result of Raynaud G is affine as it is a flat, separated, and of finite type with affine generic fiber over the discrete valuation ring $\mathcal{O}$ [PY06, Proposition 3.1].

We will often assume that p is very good for G: in cases of interest this means that $p \neq 2$ if G is orthogonal or symplectic, and $p \nmid n$ when $G = \mathrm{GL}_n$. To make uniform statements, we say that characteristic zero is very good for any G.

We will also work with a nilpotent $\overline{N} \in \mathfrak{g}_k$ associated to a continuous representation $\overline{\rho} : \Gamma_L \to G(k)$, where L is an ℓ-adic field. We will eventually impose additional hypotheses, including the following:

(A1) G is GSp_n, GO_n or GL_n and p is very good for G;

(A2) $p > n$;

(A3) k and $\mathcal{O}$ are large enough so that q, the size of the residue field of L, is a square in $\mathcal{O}^\times$, and $\mathcal{O}^\times$ includes square roots of -1 and 2;

(A4) k and $\mathcal{O}$ are large enough so there exists a pure nilpotent $N_\sigma \in \mathfrak{g}$ lifting $\overline{N}$ for which $Z_G(N_\sigma)$ is smooth.

2. Deformations of Galois representations

We recall some facts about the deformation theory for Galois representations: a basic reference is [Maz97], with the extension to algebraic groups beyond GL_n discussed in [Til96]. While we are mainly concerned with classical groups, there are no problems with doing so for any smooth group scheme G over a discrete valuation ring $\mathcal{O}$ with residue field k of characteristic p.

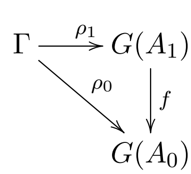
Let Γ be a pro-finite group satisfying the following finiteness property: for every open subgroup $\Gamma_0 \subset \Gamma$, there are only finitely many continuous homomorphisms from Γ_0 to $\mathbf{Z}/p\mathbf{Z}$. This is true for the absolute Galois group of a local field and for the Galois group of the maximal extension of a number field unramified outside a finite set of places.

Let $\widehat{\mathcal{C}}_{\mathcal{O}}$ be the category of coefficient $\mathcal{O}$-algebras: complete local Noetherian rings with residue field k, with morphisms local homomorphisms inducing the identity map on k and with the structure morphism a map of coefficient rings. Let $\mathcal{C}_{\mathcal{O}}$ denote the full subcategory of Artinian coefficient $\mathcal{O}$-algebras. Recall that a *small* surjection of coefficient $\mathcal{O}$-algebras $f : A_1 \to A_0$ is a surjection such that $\ker(f) \cdot \mathfrak{m}_{A_1} = 0$.

For $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$, define
$$\widehat{G}(A) := \ker(G(A) \to G(k)).$$

We are interested in deforming a fixed $\overline{\rho} : \Gamma \to G(k)$. Let $\mathfrak{g} = \mathrm{Lie}\, G$.

- Let $f : A_1 \to A_0$ be a morphism in $\widehat{\mathcal{C}}_{\mathcal{O}}$ and $\rho_0 : \Gamma \to G(A_0)$ a continuous homomorphism. A *lift* of ρ_0 to A_1 is a continuous homomorphism $\rho_1 : \Gamma \to G(A_1)$ such that the following diagram commutes.



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Define the functor $D^{\square}_{\overline{\rho},\mathcal{O}} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \text{Sets}$ by sending a coefficient $\mathcal{O}$-algebra A to the set of lifts of $\overline{\rho}$ to A.

- With the notation above, two lifts ρ and ρ' of $\overline{\rho}$ to $A_1 \in \mathcal{C}_{\mathcal{O}}$ are *strictly equivalent* if they are conjugate by an element of $\widehat{G}(A_1)$. A *deformation* of ρ_0 to A_1 is a strict equivalence class of lifts. Define the functor $D_{\overline{\rho},\mathcal{O}} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \text{Sets}$ by sending a coefficient $\mathcal{O}$-algebra A to the set of deformations of $\overline{\rho}$ to A.

We will drop the subscript $\mathcal{O}$ when it is clear from context.

Fact 2.1. The functor $D^{\square}_{\overline{\rho},\mathcal{O}}$ is representable. When $\mathfrak{g}_k^{\Gamma} = \text{Lie}(Z_G)_k$, the functor $D_{\overline{\rho},\mathcal{O}}$ is representable.

The first part is simple, the second is a reformulation of [Til96, Theorem 3.3].

The representing objects are denoted $R^{\square}_{\overline{\rho},\mathcal{O}}$ and (when it exists) $R_{\overline{\rho},\mathcal{O}}$. The former is called the universal lifting ring, while the latter is the universal deformation ring. While we usually care about deformations, it is technically easier to work with lifts.

This deformation theory is controlled by Galois cohomology. Let $\text{ad}(\overline{\rho})$ denote the representation of Γ on $\mathfrak{g}_k$ via the adjoint representation. Letting G' be the derived subgroup of G° with Lie algebra $\mathfrak{g}'$, we also consider the representation $\text{ad}^0(\overline{\rho})$ of Γ on $\mathfrak{g}'_k$. As p is very good, we have $\mathfrak{g}_k = \mathfrak{g}'_k \oplus \mathfrak{z}_{\mathfrak{g}}$ where $\mathfrak{z}_{\mathfrak{g}}$ is the Lie algebra of Z_G. The condition in Fact 2.1 is just that $H^0(\Gamma, \text{ad}(\overline{\rho})) = \mathfrak{z}_{\mathfrak{g}}$, or equivalently that $H^0(\Gamma, \text{ad}^0(\overline{\rho})) = 0$. In general, since p is very good the natural map $H^i(\Gamma, \text{ad}^0(\overline{\rho})) \to H^i(\Gamma, \text{ad}(\overline{\rho}))$ is injective for all i; we often use this without comment.

We can use the first order exponential map [Til96, § 3.5] to understand the tangent space. Recall that for a smooth $\mathcal{O}$-group scheme G, and a small surjection $f : A \to A/I$ of coefficient rings ($I \cdot \mathfrak{m}_A = 0$), smoothness gives an isomorphism

$$\exp : \mathfrak{g} \otimes_k I \simeq \ker(G(A) \to G(A/I)) = \ker(\widehat{G}(A) \to \widehat{G}(A/I)).$$

The tangent space $D_{\overline{\rho},\mathcal{O}}(k[\epsilon]/\epsilon^2)$ is identified with $H^1(\Gamma, \text{ad}(\overline{\rho}))$: Under this isomorphism, the cohomology class of a 1-cocycle τ corresponds to the lift $\rho(g) = \exp(\epsilon\tau(g))\overline{\rho}(g)$. For the universal lifting ring $R^{\square}_{\overline{\rho},\mathcal{O}}$, the tangent space is identified with the k-vector space $Z^1(\Gamma, \text{ad}(\overline{\rho}))$ of (continuous) 1-cocycles of Γ valued in $\text{ad}(\overline{\rho})$.

Remark 2.2. We also observe that

$$\dim_k Z^1(\Gamma, \text{ad}(\overline{\rho})) - \dim_k H^1(\Gamma, \text{ad}(\overline{\rho})) = \dim_k B^1(\Gamma, \text{ad}(\overline{\rho})) = \dim_k \mathfrak{g} - \dim_k H^0(\Gamma, \text{ad}(\overline{\rho}))$$

since the space of coboundaries admits a surjection from $\text{ad}(\overline{\rho})$ with kernel $\text{ad}(\overline{\rho})^{\Gamma}$. This will be useful when comparing dimensions of lifting rings and deformation rings that are smooth.

We will want to study special classes of deformations.

Definition 2.3. A *lifting condition* is a sub-functor $\mathcal{D}^{\square} \subset D^{\square}_{\overline{\rho},\mathcal{O}} : \mathcal{C}_{\mathcal{O}} \to \text{Sets}$ such that we have the following.

(i) For any coefficient ring A, $\mathcal{D}^{\square}(A)$ is closed under strict equivalence.

(ii) Given a Cartesian diagram in $\mathcal{C}_{\mathcal{O}}$

$$\begin{array}{ccc} A_1 \times_{A_0} A_2 & \xrightarrow{\pi_2} & A_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array}$$

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and $\rho \in D^{\square}_{\overline{\rho},\mathcal{O}}(A_1 \times_{A_0} A_2)$, we have $\rho \in \mathcal{D}^{\square}(A_1 \times_{A_0} A_2)$ if and only if $\mathcal{D}^{\square}(\pi_1) \circ \rho \in \mathcal{D}^{\square}(A_1)$ and $\mathcal{D}^{\square}(\pi_2) \circ \rho \in \mathcal{D}^{\square}(A_2)$.

As it is closed under strict equivalence, we naturally obtain a *deformation condition*, a subfunctor $\mathcal{D} \subset D_{\overline{\rho},\mathcal{O}}$.

By Schlessinger's criterion [Sch68, Theorem 2.11] being a lifting condition is equivalent to the functor $\mathcal{D}^{\square}$ being pro-representable. Likewise, the deformation condition $\mathcal{D}$ associated to a lifting condition $\mathcal{D}^{\square}$ is pro-representable provided that $D_{\overline{\rho},\mathcal{O}}$ is.

The tangent space of a deformation condition $\mathcal{D}$ is a k-subspace of $H^1(\Gamma, \mathrm{ad}(\overline{\rho}))$, and will be denoted by $H^1_{\mathcal{D}}(\Gamma, \mathrm{ad}(\overline{\rho}))$. For a small surjection $A_1 \to A_0$ and $\rho \in \mathcal{D}(A_0)$, the set of deformations of ρ to A_1 subject to $\mathcal{D}$ is a $H^1_{\mathcal{D}}(\Gamma, \mathrm{ad}(\overline{\rho}))$-torsor. This torsor-structure is compatible with the action of the unrestricted tangent space to $D_{\overline{\rho}}$ on the space of all deformations of ρ to A_1.

Example 2.4. Suppose G is almost-reductive, and let G' be the derived group of G°. The most basic examples of deformation conditions are the conditions imposed by fixing the lift of the homomorphism $\Gamma \to (G/G')(k)$. To be precise, for the quotient map $\mu : G \to G/G' =: S$, a fixed $\nu : \Gamma \to S(\mathcal{O})$ lifting $\mu \circ \overline{\rho}$, and $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$ with structure morphism $\imath : \mathcal{O} \to A$, we define a deformation condition $\mathcal{D}_{\nu} \subset \mathcal{D}_{\overline{\rho}}$ by

$$\mathcal{D}_{\nu}(A) = \{\rho \in \mathcal{D}_{\overline{\rho}}(A) : \mu_A \circ \rho = \imath \circ \nu_A\}.$$

One checks this is a deformation condition. Its tangent space is $H^1(\Gamma, \mathrm{ad}^0(\overline{\rho}))$ since p is very good. We define $\mathcal{D}^{\square}_{\nu}$ similarly.

Another important example is the *unramified* deformation condition for a non-archimedean place v where $\overline{\rho}$ is unramified: this consists of lifts that are unramified (possibly with a specified choice of ν). The tangent space is $H^1_{\mathrm{nr}}(\Gamma_v, \mathrm{ad}(\overline{\rho}))$ (respectively $H^1_{\mathrm{nr}}(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$).

Definition 2.5. A deformation condition $\mathcal{D}$ is *liftable* (over $\mathcal{O}$) if for all small surjections $f : A_1 \to A_0$ of coefficient $\mathcal{O}$-algebras the natural map

$$\mathcal{D}(f) : \mathcal{D}(A_1) \to \mathcal{D}(A_0)$$

is surjective.

A geometric way to check local liftability is to show that the corresponding deformation ring (when it exists) is smooth. Obviously it suffices to check liftability for lifts instead of deformations, so we can work with the lifting deformation ring and avoid representability issues for $\mathcal{D}_{\overline{\rho}}$.

Example 2.6. The unramified deformation condition is liftable: an unramified lift is completely determined by the image of Frobenius in $G(A_0)$, and G is smooth over $\mathcal{O}$.

When attempting to lift with a fixed lift ν of $\Gamma \to (G/G')(k)$, the obstruction to lifting is measured by a 2-cocycle $\mathrm{ob}(\rho_0)$ that lies in $H^2(\Gamma, \mathrm{ad}^0(\overline{\rho}))$. To see this, recall that the obstruction cocycle is defined by picking a continuous set theoretic lift ρ_1 of a given $\rho_0 : \Gamma_K \to G(A_0)$: the 2-cocycle records the failure of ρ_1 to be a homomorphism. By choosing the lift $\Gamma_K \to G(A_1)$ so that $\Gamma_K \to (G/G')(A_0)$ agrees with ν (as we may easily do since $\ker \rho_0$ is open in Γ_K), the obstruction cocycle takes values in $\mathrm{ad}^0(\overline{\rho})$.

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3. Representatives for nilpotent orbits and pure nilpotents

As a first step on the road to defining the minimally ramified deformation condition, we study integral representatives of nilpotent orbits and then define pure nilpotent lifts. Useful background about nilpotent orbits is collected in [Jan04]. We focus on classical groups, so consider $G = \mathrm{GL}_n$, $G = \mathrm{Sp}_m$, or $G = \mathrm{O}_m$ over a discrete valuation ring $\mathcal{O}$ with residue field k of characteristic $p > 0$. Assume p is very good for G_k. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and K be the field of fractions of $\mathcal{O}$.

3.1 Integral representatives

The nilpotent orbits for G over an algebraically closed field of good characteristic can be classified by combinatorial data $\mathcal{C}$ that is independent of the characteristic. For classical groups, nilpotent orbits can be classified by their Jordan canonical form in terms of partitions. For a partition $\sigma \in \mathcal{C}$, let $\mathcal{O}_{F,\sigma} \subset \mathfrak{g}_F$ denote the corresponding orbit over the algebraically closed field F. For $\sigma \in \mathcal{C}$, we seek elements

$$N_\sigma \in \mathfrak{g} \text{ such that } (N_\sigma)_k \in O_{\overline{k},\sigma} \text{ and } (N_\sigma)_K \in O_{\overline{K},\sigma}. \tag{3.1}$$

This makes precise the statement that N_K and N_k 'lie in the same nilpotent orbit'.

Remark 3.1. For a general reductive group scheme G, the Bala–Carter classification can be interpreted as giving a characteristic-free classification of nilpotent orbits, allowing a generalization of the condition in (3.1). One can obtain such N_σ in terms of root data following [SS70, III.4.29]. We need the additional information provided by the concrete description in the symplectic and orthogonal cases to analyze the centralizer $Z_G(N)$ as an $\mathcal{O}$-scheme, so do not use this.

Example 3.2. Nilpotent orbits for GL_n correspond to partitions $n = n_1 + n_2 + \cdots + n_r$. For a partition σ of n, Let $N_\sigma \in \mathfrak{g}$ be the nilpotent matrix in Jordan canonical form whose blocks (in order) are of sizes $n_1, n_2, \ldots, n_r$. Clearly N_σ has entries in $\mathcal{O}$ and satisfies (3.1).

For symplectic and orthogonal groups, we can produce the desired N_σ using a minor extension of the classical results known over algebraically closed fields [Jan04, § 1]. Let $G = \mathrm{Sp}_m$ with $m = 2n$, or $G = \mathrm{O}_m$ with $m = 2n$ or $m = 2n + 1$. We assume $n \geqslant 2$. Recall that Sp_m and O_m are defined using standard pairings on a free $\mathcal{O}$-module M of rank m. For $m = 2n$, the *standard alternating pairing* φ_{std} on $\mathcal{O}^m$ is the one given by the block matrix

$$\begin{pmatrix} 0 & I_n' \\ -I_n' & 0 \end{pmatrix},$$

where I_n' denotes the anti-diagonal matrix with 1s on the diagonal. The *standard symmetric pairing* φ_{std} on $\mathcal{O}^m$ is the one given by the matrix I_m'.

Remark 3.3. We chose to work with O_m instead of SO_m, as the classification is cleaner for O_m. The nilpotent orbits are almost the same for SO_m, except that certain nilpotent orbits of O_m (the ones where the partition contains only even parts) split into two SO_m-orbits [Jan04, Proposition 1.12] (conjugation by an element of O_m with determinant -1 carries one such orbit into the other).

Definition 3.4. Let σ denote a partition $m = m_1 + m_2 + \cdots + m_r$ of m. It is *admissible* if

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- every even m_i appears an even number of times when $G = \mathrm{O}_m$;
- every odd m_i appears an even number of times when $G = \mathrm{Sp}_m$.

The admissible partitions of m are in bijection with nilpotent orbits of Sp_m or O_m over any algebraically closed field of good characteristic [Jan04, Theorem 1.6]. The corresponding orbit is the intersection of $\mathfrak{g} \subset \mathfrak{gl}_m$ with the GL_m-orbit corresponding to that partition of m. Note that GL_m-orbit representatives in Jordan canonical form need not lie in $\mathfrak{g}$.

We will construct nilpotents together with a pairing, and then show how to relate the constructed pairing to the standard pairings used to define G. Let $\epsilon = 1$ in the case of O_m, and $\epsilon = -1$ in the case of Sp_m.

DEFINITION 3.5. Let $d \geqslant 2$ be an integer. Define $M(d) = \mathcal{O}^d$, with basis $v_1, \ldots, v_d$ and a perfect pairing φ_d such that

$$\varphi_d(v_i, v_j) = \begin{cases} (-1)^i & i + j = d + 1, \\ 0 & \text{otherwise} \end{cases}$$

(alternating for even d, symmetric for odd d). Define a nilpotent $N_d \in \mathrm{End}(M(d))$ by $N_d v_i = v_{i-1}$ for $1 < i \leqslant d$ and $N_d v_1 = 0$.

Similarly, define $M(d, d) = \mathcal{O}^{2d}$ with basis $v_1, \ldots, v_d, v'_1, \ldots, v'_d$ and a perfect ϵ-symmetric pairing $\varphi_{d,d}$ by extending

$$\varphi_{d,d}(v_i, v_j) = \varphi_{d,d}(v'_i, v'_j) = 0 \quad \text{and} \quad \varphi_{d,d}(v_i, v'_j) = \begin{cases} (-1)^i & i + j = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define a nilpotent $N_{d,d} \in \mathrm{End}(M(d, d))$ by $N_{d,d} v_i = v_{i-1}$ and $N_{d,d} v'_i = v'_{i-1}$ for $1 < i \leqslant d$, and $N_{d,d} v_1 = N_{d,d} v'_1 = 0$.

Note that the pairing $\varphi_{d,d}$ can be symmetric or alternating depending on the parity of d. It is straightforward to verify the pairings are perfect and that N_d (respectively $N_{d,d}$) is skew with respect to φ_d (respectively $\varphi d, d$) in the sense that for $v, w \in M(d)$ we have

$$\varphi_d(N_d v, w) = -\varphi_d(v, N_d w).$$

Given an admissible partition $\sigma : m = m_1 + m_2 + \cdots + m_r$, we will construct a free $\mathcal{O}$-module of rank m with an ϵ-symmetric perfect pairing and a nilpotent endomorphism that is skew with respect to the pairing such that the Jordan block structure of nilpotent endomorphism in geometric fibers is given by σ. Let $n_i(\sigma) = \#\{j : m_j = i\}$.

- If $G = \mathrm{O}_m$, then $n_i(\sigma)$ is even for even i, so we can define

$$M_\sigma = \bigoplus_{i\,\text{odd}} M(i)^{\oplus n_i(\sigma)} \oplus \bigoplus_{i\,\text{even}} M(i, i)^{\oplus n_i(\sigma)/2}.$$

- If $G = \mathrm{Sp}_m$, then $n_i(\sigma)$ is even for odd i, so we can define

$$M_\sigma = \bigoplus_{i\,\text{odd}} M(i, i)^{\oplus n_i(\sigma)/2} \oplus \bigoplus_{i\,\text{even}} M(i)^{\oplus n_i(\sigma)}.$$

Let φ_σ and N_σ denote the pairing and nilpotent endomorphism defined by the pairing and nilpotent endomorphism on each piece using Definition 3.5. In all cases, M_σ is a free $\mathcal{O}$-module of rank m. For each σ, let G_σ be the automorphism scheme $\underline{\mathrm{Aut}}(M_\sigma, \varphi_\sigma)$, so for an algebraically closed field F over $\mathcal{O}$ we have an isomorphism $(G_\sigma)_F \simeq G_F$ well defined up to $G(F)$-conjugation by using F-linear isomorphisms $(M_\sigma, \varphi_\sigma)_F \simeq (F^m, \varphi_{\mathrm{std}})$.

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Lemma 3.6. *For all admissible partitions of m, the specializations of the N_σ at geometric points ξ of* $\operatorname{Spec}\mathcal{O}$ *constitute a set of representatives for the nilpotent orbits of G_ξ, and the specializations lie in the orbit corresponding to σ.*

Proof. The set of admissible partitions of m is in bijection with the set of nilpotent orbits over any algebraically closed field. The N_σ we constructed are integral versions of the representatives constructed in [Jan04, § 1.7]. □

Let $e_1, e_2, \ldots, e_m$ be the standard basis for $\mathcal{O}^m$. The elements e_i and e_{m+1-i} pair non-trivially under the standard pairing. When $m = 2n+1$, e_{n+1} pairs non-trivially with itself under the standard pairing. We now relate the standard pairings to the pairings φ_σ.

Proposition 3.7. *Suppose that $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. Then φ_σ is equivalent to the standard pairing over $\mathcal{O}$. There exists an $\mathcal{O}$-basis $\{v_i\}$ of $\mathcal{O}^m$ with respect to which the pairing is given by φ_σ and N_σ satisfies the condition in (3.1) for $G = \mathrm{Sp}_m$ or $G = \mathrm{O}_m$.*

Proof. The standard pairings are very similar to φ_σ. In the case of Sp_m, each basis vector pairs trivially against all but one other basis vector, with which it pairs as ± 1. So after reordering the basis, φ_σ is the standard pairing. The case of O_m is slightly more complicated. Let $\sigma : m = m_1 + m_2 + \cdots + m_r$ be an admissible partition. The construction of M_σ and φ_σ gives a basis $\{v_{i,j}\}$ where $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m_i$. From the construction of φ_σ, we see that $v_{i,j}$ pairs trivially against all basis vectors except for v_{i,m_i+1-j}. So as long as $2j \neq m_i + 1$, we obtain a pair of basis vectors which are orthogonal to all others and which pair to ± 1. For each odd m_i, the vector $v_{i,(m_i+1)/2}$ pairs non-trivially with itself. The standard pairing with respect to the basis e_i has such a vector only when $m = 2n+1$ and then only for one e_i.

We must change the basis over $\mathcal{O}$ so that φ_σ becomes the standard symmetric pairing. Let $v = v_{i,(m_i+1)/2}$ and $v' = v_{j,(m_j+1)/2}$ be two distinct vectors which pair non-trivially with themselves. In particular, $\varphi_\sigma(v,v) = (-1)^{(m_i+1)/2} := \eta$ and $\varphi_\sigma(v',v') = (-1)^{(m_j+1)/2} := \eta'$. Define

$$w = \frac{\sqrt{\eta}v - \sqrt{-\eta'}v'}{\sqrt{2}} \quad \text{and} \quad w' = \frac{\sqrt{\eta}v + \sqrt{-\eta'}v'}{\sqrt{2}}.$$

Then we see that $\varphi_\sigma(w,w) = 0 = \varphi_\sigma(w',w')$ and $\varphi_\sigma(w,w') = 1$. Making this change of variable over $\mathcal{O}$ (which requires $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$), we have reduced the number of basis vectors which pair non-trivially with themselves by two, and produced a new pair of basis vectors orthogonal to the others and which pair to 1. By induction, we may therefore pick a basis $v'_1, \ldots, v'_m$ for which at most one basis vector pairs non-trivially with itself under φ_σ. After re-ordering, we may further assume that $\varphi_\sigma(v'_i, v'_j) = 0$ unless $i + j = m + 1$, in which case $\varphi_\sigma(v'_i, v'_j) = \pm 1$. Suppose $j = m + 1 - i$. If $i \neq j$, by scaling v'_i we may assume that $\varphi_\sigma(v'_i, v'_j) = 1$. If $i = j$, we already know that $\varphi_\sigma(v'_i, v'_j) = 1$. With respect to this basis, φ_σ is the standard pairing.

The last statement immediately follows from Lemma 3.6. □

3.2 Pure nilpotent lifts

For a nilpotent element $\overline{N} \in \mathfrak{g}_k$ of type $\sigma \in \mathcal{C}$, we will define the notion of a *pure nilpotent* lift of $\overline{N}$ in $\mathfrak{g}_k$ and study the space of such lifts, assuming there exists $N_\sigma \in \mathfrak{g}$ lifting $\overline{N}$ such that $(N_\sigma)_{\overline{K}} \in O_{\overline{K},\sigma}$ and such that $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$.

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Remark 3.8. Section 3.1 shows that for any nilpotent $\overline{N} \in \mathfrak{g}_k$, there exists $N'_\sigma \in \mathfrak{g}$ such that $(N'_\sigma)_{\overline{k}} \in O_{\overline{k},\sigma}$ and such that $(N'_\sigma)_k$ and $\overline{N}$ are $G(\overline{k})$-conjugate. We will address the $\mathcal{O}$-smoothness of $Z_G(N_\sigma)$ in § 4, especially Proposition 4.17. Then $(N'_\sigma)_k$ and $\overline{N}$ are conjugate by $\overline{g} \in G(k')$ for some finite extension k'/k. Lift $\overline{g}$ to an element $g \in G(\mathcal{O}')$ for a Henselian discrete valuation ring local over $\mathcal{O}$ and having residue field k'. The element $N_\sigma := gN'_\sigma g^{-1} \in \mathfrak{g}_{\mathcal{O}'}$ reduces to $\overline{N}_{k'}$ and has the required properties. So the above hypothesis is satisfied after a finite flat local extension of $\mathcal{O}$.

DEFINITION 3.9. Fix an $N_\sigma \in \mathfrak{g}$ lifting $\overline{N}$ such that $(N_\sigma)_{\overline{K}} \in O_{\overline{K},\sigma}$ and such that $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$. Define the functor $\mathrm{Nil}_{\overline{N}} : \mathcal{C}_{\mathcal{O}} \to \mathrm{Sets}$ by

$$\mathrm{Nil}_{\overline{N}}(R) = \{N \in \mathfrak{g}_R : \mathrm{Ad}_G(g)(N_\sigma) = N \text{ for some } g \in \widehat{G}(R)\}.$$

Call these $N \in \mathrm{Nil}_{\overline{N}}(R)$ the *pure nilpotents* lifting $\overline{N}$.

This is obviously a subfunctor of the formal neighborhood of $\overline{N}$ in the affine space $\underline{\mathfrak{g}}$ over $\mathcal{O}$ attached to $\mathfrak{g}$. The key to analyzing $\mathrm{Nil}_{\overline{N}}$ is that $Z_{G_R}(N)$ is smooth over R since $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth and N is in the G-orbit of $(N_\sigma)_R$. To ease notation below, we shall write gNg^{-1} rather than $\mathrm{Ad}_G(g)(N)$ for $g \in \widehat{G}(R)$.

LEMMA 3.10. *Assuming $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth, the functor $\mathrm{Nil}_{\overline{N}}$ is pro-representable.*

Proof. We will use Schlessinger's criterion to check pro-representability. As $\mathrm{Nil}_{\overline{N}}$ is a subfunctor of the formal neighborhood of the scheme $\underline{\mathfrak{g}}$ at $\overline{N}$, the only condition to check is the analogue of Definition 2.3(ii): given a Cartesian diagram in $\mathcal{C}_{\mathcal{O}}$

$$\begin{array}{ccc} R_1 \times_{R_0} R_2 & \xrightarrow{\pi_2} & R_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ R_1 & \longrightarrow & R_0 \end{array}$$

and $N_i \in \mathrm{Nil}_{\overline{N}}(R_i)$ such that N_1 and N_2 reduce to N_0, we want to check that $N_1 \times N_2 \in \mathrm{Nil}_{\overline{N}}(R_1 \times_{R_0} R_2)$. By definition, there exists $g_1 \in \widehat{G}(R_1)$ and $g_2 \in \widehat{G}(R_2)$ such that $N_1 = g_1 N_\sigma g_1^{-1}$ and $N_2 = g_2 N_\sigma g_2^{-1}$. Consider the element $g_1 g_2^{-1} \in \widehat{G}(R_0)$. Observe that

$$g_1 g_2^{-1} N_\sigma g_2 g_1^{-1} = g_1 N_0 g_1^{-1} = N_\sigma \in \mathfrak{g}_{R_0}.$$

In particular, $g_1 g_2^{-1} \in Z_G(N_\sigma)(R_0)$. The extension $R_2 \to R_0$ has nilpotent kernel, so as $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$ there exists $h \in Z_G(N_\sigma)(R_2)$ lifting $g_1 g_2^{-1}$. The element

$$(g_1, hg_2) \in R_1 \times_{R_0} R_2$$

conjugates $N_1 \times N_2$ to N_σ. Hence $N_1 \times N_2 \in \mathrm{Nil}_{\overline{N}}(R_1 \times_{R_0} R_2)$. □

LEMMA 3.11. *The functor $\mathrm{Nil}_{\overline{N}}$ is formally smooth, in the sense that for a small surjection $R_2 \to R_1$ of coefficient $\mathcal{O}$-algebras the map*

$$\mathrm{Nil}_{\overline{N}}(R_2) \to \mathrm{Nil}_{\overline{N}}(R_1)$$

is surjective. Moreover, when $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth and $\mathrm{Nil}_{\overline{N}}$ is representable, it has relative dimension $\dim G_k - \dim Z_{G_k}(N_k)$ over $\mathcal{O}$.

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Proof. Given $N \in \mathrm{Nil}_{\overline{N}}(R_1)$, there exists $g \in \widehat{G}(R_1)$ such that $gNg^{-1} = N_\sigma$. As G is smooth over $\mathcal{O}$, we may find $g' \in \widehat{G}(R_2)$ lifting g. Then $(g')^{-1}N_\sigma g'$ is a lift of N to R_2. From its definition, the tangent space to $\mathrm{Nil}_{\overline{N}}$ is $\mathfrak{g}_k/\mathfrak{z}_{\mathfrak{g}}(N_k)$, so the formally smooth $\mathrm{Nil}_{\overline{N}}$ has relative dimension $\dim G_k - \dim Z_{G_k}(N_k)$ since $Z_G(N)$ is $\mathcal{O}$-smooth. □

Lemma 3.12. *Suppose that A is a complete local Noetherian $\mathcal{O}$-algebra with residue field k. Under the assumption that $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth, the inverse limit $\varprojlim \mathrm{Nil}_{\overline{N}}(A/\mathfrak{m}_A^n)$ equals $\{N \in \mathfrak{g}_A : N = gN_\sigma g^{-1}$ for some $g \in G(A)\}$.*

Proof. This is immediate since A is complete and $\mathrm{Nil}_{\overline{N}}$ is representable by an affine scheme. □

Remark 3.13. If we had defined $\mathrm{Nil}_{\overline{N}}$ on the larger category $\widehat{\mathcal{C}}_{\mathcal{O}}$ in the obvious way, Lemma 3.12 would say that $\mathrm{Nil}_{\overline{N}}$ is continuous.

Remark 3.14. There is no problem generalizing Definition 3.9 to any almost-reductive group, using a construction of N_σ using root data as discussed in Remark 3.1. However, we only establish the smoothness of $Z_G(N_\sigma)$ for classical groups.

Remark 3.15. One can define a scheme-theoretic 'nilpotent cone' over $\mathcal{O}$ as the vanishing locus of the ideal of non-constant homogeneous G-invariant polynomials on $\mathfrak{g}$. The arguments in this section could be rephrased as constructing a formal scheme of pure nilpotents inside the formal neighborhood of $\overline{N}$ in $\underline{\mathfrak{g}}$. A natural question is whether there is a broader notion of pure nilpotents that gives a locally closed subscheme of the scheme-theoretic nilpotent cone. For instance, for $N, N' \in \mathfrak{g}$, if their images in $\mathfrak{g}_K$ and $\mathfrak{g}_k$ are nilpotent in orbits with the same combinatorial parameters, are N and N' conjugate under G over a discrete valuation ring local over $\mathcal{O}$?

When $G = \mathrm{GL}_n$, this has been explored by Taylor in the course of constructing local deformation conditions [Tay08, Lemma 2.5]. The method uses the explicit description of the orbit closures given by specifying the Jordan canonical form to define an analogue of the orbit closures over $\mathcal{O}$. It would be interesting to find a way to do so more generally.

4. Smoothness of centralizers of pure nilpotents

In order for the functor $\mathrm{Nil}_{\overline{N}}$ to be representable, we need that $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth. Recall that for $N \in \mathfrak{g}$, the scheme-theoretic centralizer $Z_G(N)$ represents the functor

$$R \mapsto \{g \in G(R) : \mathrm{Ad}_G(g)N_R = N_R\}$$

for $\mathcal{O}$-algebras R. We will study the centralizer $Z_G(N_\sigma)$ in more detail where $N_\sigma \in \mathfrak{g}$ is an element satisfying (3.1). In particular, this centralizer will be shown to be smooth when G is symplectic or orthogonal. We first review the known theory over fields, and then develop and apply a technique to deduce smoothness over $\mathcal{O}$ (i.e. $\mathcal{O}$-flatness) from the known smoothness in the field case.

4.1 Centralizers over fields

In this section, let k be an algebraically closed field, G be a connected reductive group over k, and N a nilpotent element of $\mathfrak{g} = \mathrm{Lie}\, G$. As the formation of the scheme-theoretic centralizer commutes with base change, smoothness results for $Z_G(N)$ over k will imply such results over general fields (not necessarily algebraically closed).

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The group scheme $Z_G(N)$ is the fiber over $0 \in \mathfrak{g}$ of the composition

$$G \xrightarrow{\mathrm{Ad}_G} \mathrm{GL}(\mathfrak{g}) \xrightarrow{T \mapsto TN-N} \mathfrak{g}.$$

Hence $\mathrm{Lie}\, Z_G(N)$ is the kernel of

$$\mathfrak{g} \xrightarrow{\mathrm{ad}_{\mathfrak{g}}} \mathrm{End}(\mathfrak{g}) \xrightarrow{T \mapsto TN} \mathfrak{g}$$

which is the Lie algebra centralizer $\mathfrak{z}_{\mathfrak{g}}(N)$.

Remark 4.1. In references using the language of varieties rather than schemes (such as [Jan04]), $Z_G(N)$ is usually *defined* via its geometric points and hence is reduced and smooth, so the condition that the scheme $Z_G(N)$ is smooth becomes the condition that the variety $Z_G(N)$ has Lie algebra $\mathfrak{z}_{\mathfrak{g}}(N)$.

In a wide range of situations, all nilpotent centralizers are smooth. A direct calculation shows that this holds for $G = \mathrm{GL}_n$ (see [Jan04, § 2.3]), and a criterion of Richardson [Jan04, Theorem 2.5] can be applied to show to following.

Fact 4.2. If G is an orthogonal or symplectic (similitude) group, any nilpotent centralizer is smooth over k.

Remark 4.3. Suppose $Z_G(N)$ is smooth over k and p is good for G. The classification of nilpotent orbits is independent of p, as are their dimensions, so the dimension of $Z_G(N)$ is independent of p as well.

4.2 Checking flatness over a Dedekind base

We want to analyze smoothness of centralizers in the relative setting (especially over $\mathrm{Spec}\,\mathcal{O}$). If $Z_G(N_\sigma) \to \mathrm{Spec}\,\mathcal{O}$ is flat and the special and generic fibers are smooth, then $Z_G(N)$ is smooth over $\mathcal{O}$. The following lemma gives a way to check that a morphism to a Dedekind scheme is flat.

LEMMA 4.4. *Let $f : X \to S$ be finite type for a connected Dedekind scheme S. Then f is flat provided the following all hold.*

(i) *For each $s \in S$, X_s is reduced and non-empty.*
(ii) *For each $s \in S$, X_s is equidimensional with dimension independent of s.*
(iii) *There are sections $\{\sigma_i \in X(S)\}$ to f such that for every irreducible component of a fiber above a closed point, there is a section σ_i which meets the fiber only in that component.*

Remark 4.5. This lemma is a modification of [GY03, Proposition 6.1] to allow multiple irreducible components in the fibers.

Proof. It suffices to prove the result when $S = \mathrm{Spec}(A)$ for A a discrete valuation ring with uniformizer π. Let X_η be the generic fiber and X_s the special fiber. Consider the schematic closure $\imath : X' \hookrightarrow X$ of the generic fiber. The scheme X' is flat over $\mathrm{Spec}(A)$ since flatness is equivalent to being torsion-free over a discrete valuation ring, and there is an exact sequence

$$0 \to J \to \mathcal{O}_X \to \imath_* \mathcal{O}_{X'} \to 0, \tag{4.1}$$

where J is a coherent sheaf killed by a power of π. We will show that $\imath$ is an isomorphism by analyzing the special fiber.

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First, we claim that the dimension of each irreducible component on the special fiber of X' is the same as the dimension of the equidimensional X_η. We will get this from flatness of X'. The generic fiber of X' is X_η, which is equidimensional and non-empty by hypothesis. Furthermore, X' is the union of the closures Z_i of the reduced irreducible components $X_{\eta,i}$ of X_η, and each Z_i is A-flat with integral η-fiber, hence integral. We just need to analyze the dimension of irreducible components of $(Z_i)_s$ when $(Z_i)_s \neq \emptyset$. Since Z_i is integral, we can apply [Mat89, Theorem 15.1, 15.5] to such Z_i to conclude that the dimension of each irreducible component of the special fiber of X' is the same as the dimension of the generic fiber.

Observe that the sections σ_i factor through the closed subscheme $X' \subset X$, as we can check this on the generic fiber since X' is A-flat. Thus X' meets every irreducible component of X_s away from the other irreducible components of X_s. We would have that $|X'_s| = |X_s|$ if X'_s is equidimensional of the same dimension as the equidimensional X_s. We have shown the dimension of any irreducible component in X'_s is the same dimension as the common dimension of irreducible components of the generic fiber X_η of X'. By hypothesis, the dimension of any irreducible component of the generic fiber of X is the same as the dimension of any irreducible component of the special fiber of X. Thus the dimension of any irreducible component of X'_s is the same as the dimension of each irreducible component of X_s, giving that $|X'_s| = |X_s|$. As X_s is reduced, this forces $\imath_s : X'_s \hookrightarrow X_s$ to be an isomorphism.

Now tensoring (4.1) with the residue field of A gives an exact sequence

$$0 \to J/\pi J \to \mathcal{O}_{X,s} \to \imath_* \mathcal{O}_{X',s} \to 0$$

because $\mathcal{O}_{X'}$ is A-flat. But $J/\pi J = 0$ as $\imath_s$ is an isomorphism. Hence $J = \pi J = \pi^2 J = \cdots = \pi^n J = 0$ for n large, so $X = X'$ is flat over A. □

Corollary 4.6. *In the situation of the lemma, if the fibers are also smooth, then X is smooth.*

Proof. For a flat morphism of finite type between Noetherian schemes, smoothness of all fibers is equivalent to smoothness of the morphism. □

4.3 Centralizers for orthogonal and symplectic groups

To apply Corollary 4.6, we need information about the component group of centralizers of nilpotents. For GL_n over a field, all such centralizers are connected. For symplectic and orthogonal groups, there is an explicit description of $Z(N_\sigma)$ where N_σ is the nilpotent constructed in § 3. We continue the notation of that section: G is Sp_m or O_m (with $m \geqslant 4$) over a discrete valuation ring $\mathcal{O}$ with a residue field k of good characteristic $p \neq 2$.

Let $\sigma : m_1 + \cdots + m_r$ be an admissible partition of m. We assume that $\mathcal{O}$ is large enough so that Proposition 3.7 holds, and take $N := N_\sigma$. Then there exists elements $v_1, \ldots, v_r \in M := \mathcal{O}^m$ such that

$$v_1, Nv_1, \ldots, N^{m_1-1}v_1, v_2, Nv_2, \ldots, N^{m_r-1}v_r$$

is a basis for M. Furthermore, $N^{m_i}v_i = 0$ for $i = 1, \ldots r$, and the pairing between basis elements is given by $\varphi := \varphi_\sigma$. In particular, each v_i pairs non-trivially with only one other basis element $X^{d_i-1}v_{i^*}$, for some $i^* \in \{1, \ldots, r\}$.

To understand the G-centralizer of N, we construct an associated grading of M as in [Jan04, § 3.3,3.4]. This is motivated by the Jacobson–Morosov theory of $\mathfrak{sl}_2$-triples over a field of sufficiently large characteristic, but for symplectic and orthogonal groups it is constructed by hand in characteristic $p \neq 2$ below.

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Remark 4.7. Let $M_k = M \otimes_{\mathcal{O}} k = k^m$. Every nilpotent $X \in \mathrm{End}(M_k)$ gives a filtration (and grading) of M_k defined by $\mathrm{Fil}^i = \ker(X^i)$. For GL_n, this is a nice filtration and is used in [CHT08] to define the minimally ramified deformation condition for GL_n. However, this filtration need not be isotropic with respect to the pairing, so we will construct a nicer grading associated to X.

DEFINITION 4.8. Let $M(s)$ be the span of $N^j v_i$ for all i and j such that $s = 2j + 1 - m_i$. We set $M^{(s)} = \bigoplus_{t \geqslant s} M(t)$, and also define $L(s)$ to be the span of $\{v_i : v_i \in M(s)\}$.

We now record some elementary properties of the preceding construction; all are routine to check, and the analogous proofs over a field may be found in [Jan04, § 3.4]. We have that $M = \bigoplus_s M(s)$, and

$$v_i \in M(-(d_i - 1)), N v_i \in M(-(d_i - 1) + 2), \ldots, N^{d_i - 1} v_r \in M(d_r - 1).$$

Furthermore, we know $N \cdot M(s) \subset M(s + 2)$ and $M(s) = N \cdot M(s - 2) \oplus L(s)$ for $s \leqslant 0$.

The dimension of $M(s)$ is $m_s(\sigma) := \#\{j : d_j - 1 \geqslant |s|\}$. The dimension of $L(s)$ equals $l_s(\sigma) := m_{s+1}(\sigma) - m_s(\sigma)$. Furthermore, the pairing φ interacts well with the grading: a computation with basis elements gives that $\varphi(M(s), M(t)) \neq 0$ implies $s + t = 0$.

The above grading on M corresponds to the one-parameter subgroup $\lambda : \mathbf{G}_m \to G$ for which the action of $t \in \mathbf{G}_m$ on $M(s)$ is given by scaling by t^s. The dynamic method (see [CGP15, § 2.1]) associates to λ a parabolic subgroup $P_G(\lambda)$ with Levi $Z_G(\lambda)$. Define C_N and U_N to be the scheme-theoretic intersections

$$\begin{aligned} C_N &= Z_G(N) \cap Z_G(\lambda) = \{g \in Z_G(N) : gM(i) = M(i) \text{ for all } i\}, \\ U_N &= Z_G(N) \cap U_G(\lambda) = \{g \in Z_G(N) : (g - 1)M^{(i)} \subset M^{(i+1)} \text{ for all } i\}. \end{aligned}$$

Fact 4.9. The group-scheme $Z_G(N)_k$ is a semidirect product of $(C_N)_k$ and the smooth connected unipotent subgroup $(U_N)_k$. In particular, the connected components of $Z_G(N)_k$ are the same as the connected components of $(C_N)_k$.

Remark 4.10. This is [Jan04, Proposition 3.12]. The existence of λ and this decomposition is not specific to symplectic and orthogonal groups [Jan04, Proposition 5.10].

We finally give a concrete description of C_N. We first define a pairing on $L(s)$. Recall that the space $L(s)$ of 'lowest weight vectors' in $M(s)$ has basis $\{v_i : 1 - d_i = s\}$. We define a pairing on $L(s)$ by

$$\psi_s(v, w) = \varphi(v, N^{-s} w).$$

A direct calculation shows that ψ_s is non-degenerate and that ψ_s is symmetric if $(-1)^s = \epsilon$ and is alternating if $(-1)^s = -\epsilon$ [Jan04, § 3.7].

A point of C_N preserves the grading on M, and since it commutes with the 'raising operator' N its action on M is determined by its action on the space $L(s)$ of 'lowest weight vectors' in $M(s)$. So the following fact is no surprise.

PROPOSITION 4.11. *There is an isomorphism of algebraic groups*

$$C_N \simeq \prod_{s \leqslant 0} \underline{\mathrm{Aut}}(L(s), \psi_s).$$

The corresponding statement over a field is [Jan04, § 3.8 Propositions 2, 3]: the proof is the same.

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Example 4.12. Let $G = \mathrm{Sp}_m$. Unraveling when ψ_s is symmetric or alternating, we see that

$$C_N \simeq \prod_{s\leqslant 0; s \text{ odd}} \mathrm{O}(L(s), \psi_s) \times \prod_{s\leqslant 0; s \text{ even}} \mathrm{Sp}(L(s), \psi_s).$$

The special fibers of the symplectic factors are connected, while the orthogonal factors have two connected components in the special fiber. Thus there are 2^t connected components, where t is the number of odd s for which $L(s) \neq 0$. For each component, there is a section $g \in C_N(\mathcal{O})$ meeting that component corresponding to a choice of $\pm\,\mathrm{Id} \in \mathrm{O}(L(s), \psi_s)$ for each odd s with $L(s) \neq 0$. The connected components of $Z_G(N)$ are the same as those for C_N by Fact 4.9.

Example 4.13. Let $G = \mathrm{O}_m$. We likewise see that

$$C_N \simeq \prod_{s\leqslant 0; s \text{ even}} \mathrm{O}(L(s), \psi_s) \times \prod_{s\leqslant 0; s \text{ odd}} \mathrm{Sp}(L(s), \psi_s).$$

Again there are 2^t connected components of $Z_G(N)$, where t is the number of even s for which $L(s) \neq 0$.

Now suppose that $G = \mathrm{SO}_m$. The elements N we considered in this section are representatives for some of the nilpotent orbits of SO_m. The group C_N has the same structure as for $G = \mathrm{O}_m$, except we require that the overall determinant be 1; this has 2^{t-1} connected components. Though SO_m has more nilpotent orbits than O_m, according to Remark 3.3 their representatives are conjugate by an element $\mathrm{O}_m(k)$ with determinant -1 to the representatives constructed in Proposition 3.7. This shows that there are sections $g \in C_N(\mathcal{O})$ meeting each component.

Remark 4.14. Suppose q is a square in $\mathcal{O}^\times$. For use in the proof of Proposition 5.6, we need the existence of an element $\Phi \in G(\mathcal{O})$ such that $\mathrm{ad}_G(\Phi)N_\sigma = qN_\sigma$. If $\alpha^2 = q$, taking $\Phi = \lambda(\alpha)$ would work: Φ would scale $N_\sigma^j v_i \in M(s)$ by α^s, and N_σ increases the degree by 2.

This Φ is a version for symplectic and orthogonal groups of the diagonal matrix denoted $\Phi(\sigma, a, q)$ whose diagonal entries are increasing powers of q used in [Tay08, § 2.3]. There it is checked that $\mathrm{ad}_G(\Phi(\sigma, a, q))N_\sigma = qN_\sigma$ where N_σ is the nilpotent representative in Jordan canonical form considered in Example 3.2 for the partition σ of m.

4.4 Smoothness of centralizers

We now return the case when G is an almost-reductive group over a discrete valuation ring $\mathcal{O}$ with residue field k of *very good* characteristic $p > 0$. Suppose we are given an integral representative $N = N_\sigma \in \mathfrak{g} := \mathrm{Lie}\, G$ for the nilpotent orbit on geometric fibers corresponding to $\sigma \in \mathcal{C}$ as in (3.1): that is, an element such that

$$N_k \in O_{\overline{k},\sigma} \quad \text{and} \quad N_K \in O_{\overline{K},\sigma}.$$

Proposition 3.7 provides such N in symplectic and orthogonal cases when $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. We wish to check that the $Z_G(N)$ is smooth over $\mathcal{O}$. This N satisfies

$$Z_{G_K}(N_K) \text{ and } Z_{G_k}(N_k) \text{ are smooth of the same dimension.} \tag{4.2}$$

Remark 4.15. Some assumption on N is essential. Otherwise $N_{\overline{K}}$ and $N_{\overline{k}}$ can lie in different nilpotent orbits (in terms of the combinatorial characteristic-free classification of geometric orbits), and so $Z_{G_K}(N_K)$ and $Z_{G_k}(N_k)$ could have different dimensions, in which case $Z_G(N)$ cannot be $\mathcal{O}$-flat. An example of this is the element N_2 in Example 1.4.

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Now we wish to check the conditions necessary to apply Corollary 4.6. We define

$$A(N) = Z_{G_k}(N_k)(k)/Z_{G_k}(N_k)^\circ(k),$$

and study when the following holds:

$$\text{each element of } A(N) \text{ arises from some } s \in Z_G(N)(\mathcal{O}). \tag{4.3}$$

Note that this checks the criterion in Corollary 4.6 as

$$Z_{G_k}(N_k)(k)/Z_{G_k}(N_k)^\circ(k) = (Z_{G_k}(N_k)/Z_{G_k}(N_k)^\circ)(k)$$

by Lang's theorem, and since the irreducible components of $Z_{G_k}(N_K)$ are the same as connected components by smoothness.

We are free to make a local flat extension of $\mathcal{O}$, as it suffices to check flatness after such an extension. In particular, it suffices to check (4.3) when $\mathcal{O}$ is Henselian and k is algebraically closed, and $\sqrt{-1}, \sqrt{2} \in \mathcal{O}$. Examples 4.12 and 4.13 give such sections when $G = \mathrm{Sp}_m$ or $G = \mathrm{SO}_m$. We will use these cases to get a result for similitude groups.

Let $\pi : \widetilde{G'} \to G'$ be the simply connected central cover of the derived group G' over $\mathcal{O}$. As p is very good, $\widetilde{G'}$ and G' have isomorphic Lie algebras via π and $\operatorname{Lie} G'$ is a direct factor of $\operatorname{Lie} G$ with complement $\operatorname{Lie}(Z_G)$, so we may abuse notation and view N as an element of all of these Lie algebras over $\mathcal{O}$.

Let S be a (split) maximal central torus in G. Consider the isogeny $S \times \widetilde{G'} \to G$. As S acts trivially on N, we see that $S \times Z_{\widetilde{G'}}(N)$ is the preimage of $Z_{G'}(N)$ under this isogeny. As p is very good for G, we obtain finite étale surjections

$$Z_{\widetilde{G'}}(N) \to Z_{G'}(N) \quad \text{and} \quad S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$$

over $\mathcal{O}$.

Lemma 4.16. *The condition (4.3) holds for $\widetilde{G'}$ if and only if (4.3) holds for G.*

Proof. Assume $\widetilde{G'}$ satisfies (4.3). Pick a connected component C of $Z_{G_k}(N_k)$. The preimage of C under $S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$ is a union of k-fiber components of the form $S_k \times C'$ where C' is a connected component of $Z_{\widetilde{G'_k}}(N_k)$. By assumption, there exists $s \in Z_{\widetilde{G'}}(N)(\mathcal{O})$ meeting any such C'. The image of $(1, s)$ is a point of $Z_G(N)(\mathcal{O})$ meeting C.

Conversely, assume G satisfies (4.3). Pick a connected component C' of $Z_{\widetilde{G'_k}}(N_k)$. Under $S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$, $S_k \times C'$ maps onto a connected component C of $Z_{G_k}(N_k)$. By assumption, there exists $s \in Z_G(N)(\mathcal{O})$ such that $s_k \in C$. As k is algebraically closed, there is $s'_k \in (S \times Z_{\widetilde{G'}}(N))(k)$ lifting s_k and lying in C'. As $S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$ is a finite étale cover and $\mathcal{O}$ is Henselian, there exists $s' \in (S \times Z_{\widetilde{G'}}(N))(\mathcal{O})$ lifting s and reducing to s'_k. □

For example, this lets us pass between Sp_{2n} and GSp_{2n} by way of the projective symplectic group.

Proposition 4.17. *For G a symplectic or orthogonal similitude group, and $N = N_\sigma \in \mathfrak{g}$ the element satisfying (3.1) given by Proposition 3.7 for an admissible partition σ, the centralizer $Z_G(N)$ is smooth over $\mathcal{O}$.*

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Proof. By Fact 4.2, $Z_{G_k}(N_k)$ and $Z_{G_K}(N_K)$ are smooth. By the classification of nilpotent orbits over algebraically closed fields, the dimension of the orbit only depends on the combinatorial classification for the orbit in very good characteristic and in characteristic 0, so these fibers are equidimensional of the same dimension. By Corollary 4.6, it suffices to find $s \in Z_G(N)(\mathcal{O})$ meeting any connected component of $Z_{G_k}(N_k)$.

Using Lemma 4.16, we reduce checking (4.3) to the cases of Sp_m and SO_m, covered by Examples 4.12 and 4.13. □

Remark 4.18. Consider a nilpotent orbit of GL_n with representative N given in Example 3.2. As $Z_{G_k}(N_k)$ is connected [Jan04, Proposition 3.10], the identity section shows (4.3) holds. This shows $Z_G(N)$ is smooth.

Remark 4.19. It is not hard to extend the above argument to work for groups such that all the irreducible factors of the root system are of classical type. For the exceptional groups, one could find N as in Remark 3.1 and attempt to check (4.3) holds by hand (there are finitely many cases). A conceptual approach would be preferable.

Remark 4.20. McNinch analyzes the centralizer of an 'equidimensional nilpotent' in [McN08]. An *equidimensional nilpotent* is an element $N \in \mathfrak{g}$ such that N_K is nilpotent and the dimension of the special and generic fibers of $Z_G(N)$ are the same. [McN08, § 5.2] claims that such $Z_G(N)$ are $\mathcal{O}$-smooth because the fibers are smooth of the same dimension. This deduction is incorrect: it relies on [McN08, 2.3.2] which uses the wrong definition of an equidimensional morphism and thereby incorrectly applies [SGA1, Exp. II, Proposition 2.3].

According to [SGA1, Exp. II, Proposition 2.3] (or [EGAIV$_3$, §§ 13.3, 14.4.6, 15.2.3]), for a Noetherian scheme Y, a morphism $f : X \to Y$ locally of finite type, and points $x \in X$ and $y = f(x)$ with $\mathscr{O}_y$ normal, f is smooth at x if and only if f is equidimensional at x and $f^{-1}(y)$ is smooth over $k(y)$ at x. But by definition in [EGAIV$_3$, 13.3.2], an *equidimensional* morphism is more than just a morphism all of whose fibers are of the same dimension (the condition checked in [McN08, 2.3.2]): a locally finite type morphism f is called equidimensional of dimension d at $x \in X$ when there exists an open neighborhood U of x such that for every irreducible component Z of U through x, $f(Z)$ is dense in some irreducible component of Y containing y and for all $x' \in U$ the fiber $f^{-1}(f(x')) \cap U$ has all irreducible components of dimension d.

This is much stronger than the fibers simply being of the same dimension. To see the importance of the extra conditions, consider a discrete valuation ring $\mathcal{O}$ with field of fractions K and residue field k, and the morphism from X, the disjoint union of $\operatorname{Spec} K$ and $\operatorname{Spec} k$, to $Y = \operatorname{Spec} \mathcal{O}$. The fibers are of the same dimension (zero) and smooth but the morphism is not flat. This morphism is also not equidimensional at $\operatorname{Spec} k$: the only irreducible component of X containing $\operatorname{Spec} k$ is the point itself, with image the closed point of $\operatorname{Spec} \mathcal{O}$. This is not dense in $\operatorname{Spec} \mathcal{O}$, the only irreducible component of the only open set containing the closed point of $\operatorname{Spec} \mathcal{O}$.

The smoothness of centralizers of an equidimensional pure nilpotent is important to proving the main results of [McN08]. In particular, the results in [McN08, §§ 6 and 7] crucially rely on the smoothness of the centralizers of such nilpotents, leaving a gap in the proof of Theorem B in [McN08] concerning the component group of centralizers. The method we have discussed here reverses this, understanding the geometric component group well enough to *produce* sufficiently many $\mathcal{O}$-valued points in order to deduce smoothness of the centralizer in classical cases in very good characteristic via Lemma 4.4.

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5. Minimally ramified deformations: tame case

In this section, we will generalize the tamely ramified case of the minimally ramified deformation condition introduced in [CHT08, § 2.4.4] for GL_n to symplectic and orthogonal groups. We also explain why another more immediate notion based on parabolic subgroups, giving the same deformation condition for GL_n, is *not liftable* in general (even for GSp_4).

5.1 Passing between unipotents and nilpotents

As before, G is either GSp_m or GO_m (or GL_m to recover the results of [CHT08, § 2.4.4]) over the ring of integers $\mathcal{O}$ in a p-adic field with residue field k of characteristic $p > 0$. As always, we assume that p is very good for G_k (i.e. $p \neq 2$). Let $\mathfrak{g} = \mathrm{Lie}(G)$.

As in § 3.2, we work with a pure nilpotent $N_\sigma \in \mathfrak{g}$ for which $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth, $(N_\sigma)_{\overline{K}} \in O_{\overline{K},\sigma}$, and $(N_\sigma)_{\overline{k}} \in O_{\overline{k},\sigma}$. Define $\overline{N} := (N_\sigma)_k$. We studied deformations of $\overline{N}$ in § 3.2, but will ultimately want to analyze deformations of Galois representations which take on unipotent values at certain elements of a local Galois group. Thus, we need a way to pass between unipotent and nilpotent elements. For classical groups, we can use a truncated version of the exponential and logarithm maps as follows.

Fact 5.1. Suppose that $p \geqslant m$ and that R is an $\mathcal{O}$-algebra. If $A \in \mathrm{Mat}_m(R)$ has characteristic polynomial x^m, then

$$\exp(A) := 1 + A + A^2/2 + \cdots + A^{m-1}/(m-1)!$$

has characteristic polynomial $(x-1)^m$. If $B \in \mathrm{Mat}_m(R)$ has characteristic polynomial $(x-1)^m$, then

$$\log(B) := (B-1) - (B-1)^2/2 + \cdots + (-1)^m (B-1)^{m-1}/(m-1)$$

has characteristic polynomial x^m. Furthermore for $C \in \mathrm{GL}_m(R)$ and an integer q, we have

- $\exp(CAC^{-1}) = C \exp(A) C^{-1}$
- $\log(CBC^{-1}) = C \log(B) C^{-1}$
- $\log(\exp(A)) = A$
- $\exp(\log(B)) = B$
- $\exp(qA) = \exp(A)^q$
- $\log(B^q) = q \log(B)$.

This is [Tay08, Lemma 2.4]. The key idea is that because all the higher powers of A and $B-1$ vanish and all of the denominators appearing are invertible as $p \geqslant m$, we can deduce these facts from results about the exponential and logarithm in characteristic zero.

Suppose J is the matrix for a perfect symmetric or alternating pairing over R.

Corollary 5.2. *For A and B as in Fact 5.1 with $\exp(A) = B$, $A^T J + JA = 0$ if and only if $B^T JB = J$.*

Proof. Directly from the definitions we see that $\exp(A^T) = \exp(A)^T$. Observe that

$$\exp(JAJ^{-1}) = JBJ^{-1} \quad \text{and} \quad \exp(-A^T) = (B^T)^{-1}.$$

Thus $JAJ^{-1} = -A^T$ if and only if $(B^T)^{-1} = JBJ^{-1}$. □

We shall use this exponential map to convert pure nilpotents into unipotent elements. Let R be a coefficient ring over $\mathcal{O}$. By Definition 3.9, any pure nilpotent $N \in \mathrm{Nil}_{\overline{N}}(R)$ is $G(R)$-conjugate to N_σ, so it has characteristic polynomial x^m. Denoting the derived group of G by G', any

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nilpotent element of $\mathfrak{g}$ lies in $(\mathfrak{g}') = (\operatorname{Lie} G')$, so $NJ + JN = 0$ (and not just $NJ + JN = \lambda J$ for some $\lambda \in \mathcal{O}$). Thus, Corollary 5.2 shows that $\exp(N) \in G(R)$. This gives an exponential map

$$\exp : \mathrm{Nil}_{\overline{N}} \to G \tag{5.1}$$

such that for $g \in \widehat{G}(R)$, $N \in \mathrm{Nil}_{\overline{N}}(R)$, and $q \in \mathbf{Z}$ we have $\exp(qN) = \exp(N)^q$ and $g \exp(N) g^{-1} = \exp(\mathrm{Ad}_G(g)N)$.

Remark 5.3. This is a realization over $\mathcal{O}$ of a special case of the Springer isomorphism identifying the nilpotent and unipotent varieties in very good characteristic. For later purposes, we will need that the identification is compatible with the multiplication in the sense that $\exp(qA) = \exp(A)^q$. In the case of GL_m, a Springer isomorphism that works in any characteristic is given by $X \to 1 + X$ for nilpotent X, but this is not compatible with multiplication.

5.2 Minimally ramified deformations

As before, G is GSp_m, GO_m or GL_m over the ring of integers $\mathcal{O}$ of a p-adic field with residue field k with $p > m$. Let L be a finite extension of $\mathbf{Q}_\ell$ (with $\ell \neq p$), and denote its absolute Galois group by Γ_L. Consider a representation $\overline{\rho} : \Gamma_L \to G(k)$. We wish to define a (large) smooth deformation condition for $\overline{\rho}$ generalizing the minimally ramified deformation condition for GL_n defined in [CHT08, § 2.4.4]. In this section we do so for a special class of tamely ramified representations. This requires making an étale local extension of $\mathcal{O}$, which will be harmless for our purposes.

Recall that Γ_L^{t}, the Galois group of the maximal tamely ramified extension of L, is isomorphic to the semidirect product

$$\widehat{\mathbf{Z}} \ltimes \prod_{p' \neq \ell} \mathbf{Z}_{p'},$$

where $\widehat{\mathbf{Z}}$ is generated by a Frobenius ϕ and the conjugation action by ϕ on the direct product is given by the cyclotomic character. We consider representations of Γ_L^{t} which factor through the quotient $\widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$. Picking a topological generator τ for $\mathbf{Z}_p$, the action is explicitly given by

$$\phi \tau \phi^{-1} = q\tau,$$

where q is the size of the residue field of L. Note q is a power of ℓ, so it is relatively prime to p. This leads us to study representations of the group $T_q := \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$.

Let $\overline{\rho} : T_q \to G(k)$ be such a representation. We first claim that $\overline{\rho}(\tau) \in G(k)$ is unipotent. This element decomposes as a commuting product of semisimple and unipotent elements of $G(k)$. The order of a semisimple element in $G(k)$ is prime to p, while by continuity there is an $r \geqslant 0$ such that $\tau^{p^r} \in \ker(\overline{\rho})$. Thus $\overline{\rho}(\tau)$ is unipotent.

Informally, a deformation $\rho : T_q \to G(R)$ will be minimally ramified if $\rho(\tau)$ lies in the 'same' unipotent orbit as $\overline{\rho}(\tau)$. To make this meaningful over an infinitesimal thickening of k, we shall use the notion of pure nilpotents as in Definition 3.9 since unipotence and unipotent orbits are not good notions when not over a field. As $\overline{N} := \log(\overline{\rho}(\tau))$ is nilpotent, by Remark 3.8 after making an étale local extension of $\mathcal{O}$ we may assume that there exists a pure nilpotent $N_\sigma \in \mathfrak{g}$ lifting $\overline{N}$ for which $Z_G(N_\sigma)$ is smooth. Making a further extension if necessary, we may also assume that the unit $q \in \mathcal{O}^\times$ is a square. We obtain an exponential map $\exp : \mathrm{Nil}_{\overline{N}} \to G$ as in (5.1).

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Definition 5.4. Under our standing assumptions (collected as (A1)–(A4) in § 1.2), for a coefficient ring R over $\mathcal{O}$, a continuous lift $\rho : T_q \to G(R)$ of $\overline{\rho}$ is *minimally ramified* if $\rho(\tau) = \exp(N)$ for some $N \in \mathrm{Nil}_{\overline{N}}(R)$.

Example 5.5. Take $G = \mathrm{GL}_m$. Then $X \mapsto 1_m + X$ gives an identification of nilpotents and unipotents. Up to conjugacy, over algebraically closed fields parabolic subgroups correspond to partitions of m and every nilpotent orbit is the Richardson orbit of such a parabolic. Let $\overline{\rho}(\tau) - 1_m =: \overline{N}$ correspond to the partition $\sigma = n_1 + n_2 + \cdots + n_r$. By Example 3.2, the lift N_σ of $\overline{N}$ is conjugate to a block nilpotent matrix with blocks of size $n_1, n_2, \ldots, n_r$. The points $N \in \mathrm{Nil}_{\overline{N}}(R)$ are the $\widehat{G}(R)$-conjugates of N_σ. It is clear (since $p > m$) that if $\rho(\tau) \in \mathrm{Nil}_{\overline{N}}(R)$, then

$$\ker(\rho(\tau) - 1_m)^i \otimes_R k \to \ker(\overline{\rho}(\tau) - 1_m)^i \tag{5.2}$$

is an isomorphism for all i. Conversely, repeated applications of [CHT08, Lemma 2.4.15] show that any $\rho(\tau)$ satisfying this collection of isomorphism conditions is $\widehat{G}(R)$-conjugate to N_σ. Thus the minimally ramified deformation condition for GL_m defined in [CHT08] agrees with our definition. Note that the identification $X \mapsto 1_m + X$ does not satisfy $qX \to (1 + X)^q$, so it will not work in our argument. The proof of [CHT08, Lemma 2.4.19] uses a different method for which this non-homomorphic identification suffices.

The functor of minimally ramified lifts is pro-representable by a ring $R^{\mathrm{m.r.}\,\square}_{\overline{\rho}}$ as it suffices to specify images of τ and ϕ subject to constraints that $\rho(\tau) = \exp(N)$ and $\rho(\phi)\rho(\tau)\rho(\phi)^{-1} = \rho(\tau)^q$.

Proposition 5.6. *Under assumptions (A1)–(A4), the lifting ring $R^{\mathrm{m.r.}\,\square}_{\overline{\rho}}$ is formally smooth over $\mathcal{O}$ of relative dimension $\dim G_k$.*

Proof. Let $\overline{\Phi} = \overline{\rho}(\phi) \in G(k)$ and let $\widehat{G}_{\overline{\Phi}}$ be the formal completion of G at $\overline{\Phi}$. Using the relation

$$\overline{\rho}(\phi)\overline{\rho}(\tau)\overline{\rho}(\phi)^{-1} = \overline{\rho}(\tau)^q,$$

we deduce that $\overline{\Phi}\,\overline{N}\,\overline{\Phi}^{-1} = q\overline{N}$. Therefore we study the functor $M_{\overline{N}}$ on $\widehat{\mathcal{C}}_{\mathcal{O}}$ defined by

$$M_{\overline{N}}(R) = \{(\Phi, N) : N \in \mathrm{Nil}_{\overline{N}}(R),\ \Phi \in \widehat{G}_{\overline{\Phi}}(R),\ \Phi N \Phi^{-1} = qN\} \subset \mathrm{Nil}_{\overline{N}}(R) \times \widehat{G}_{\overline{\Phi}}(R).$$

Any such lift (Φ, N) to a coefficient ring R determines a homomorphism $T_q \to G(R)$ lifting $\overline{\rho}$ via $\phi \mapsto \Phi$ and $\tau \mapsto \exp(N)$: it is continuous because $\exp(\overline{N})$ is unipotent. We will analyze $M_{\overline{N}}$ through the composition

$$M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}} \to \mathrm{Spf}\,\mathcal{O}.$$

First, observe that $M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}}$ is relatively representable as '$\Phi N = qN\Phi$' is a formal closed condition on points Φ of $(\widehat{G}_{\overline{\Phi}})_R$ for each $N \in \mathrm{Nil}_{\overline{N}}(R)$.

From Lemma 3.11, we know that $\mathrm{Nil}_{\overline{N}}$ is formally smooth over $\mathcal{O}$, and the universal nilpotent is $gN_\sigma g^{-1}$ for some $g \in \widehat{G}(\mathrm{Nil}_{\overline{N}})$. To check formal smoothness of the map $M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}}$, it therefore suffices to check the formal smoothness of the fiber of $M_{\overline{N}}$ over the $\mathcal{O}$-point N_σ of $\mathrm{Nil}_{\overline{N}}$.

We have written down $\Phi_\sigma \in G(\mathcal{O})$ satisfying $\Phi_\sigma N_\sigma \Phi_\sigma^{-1} = qN_\sigma$ in Remark 4.14. Observe that $\overline{\Phi}\,\overline{\Phi}_\sigma^{-1} \in Z_G(N_\sigma)(k)$. By smoothness, we may lift $\overline{\Phi}\,\overline{\Phi}_\sigma^{-1}$ to an element $s \in Z_G(N_\sigma)(\mathcal{O})$. Then $s\Phi_\sigma$ reduces to $\overline{\Phi}$ and satisfies $(s\Phi_\sigma)N_\sigma(s\Phi_\sigma)^{-1} = qN_\sigma$, so the fiber of $M_{\overline{N}}$ over N_σ has an $\mathcal{O}$-point. The relative dimension of the formally smooth $\mathrm{Nil}_{\overline{N}}$ is $\dim G_k - \dim Z_{G_k}(\overline{N})$ by Lemma 3.11, and $M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}}$ is a $\widehat{Z_G(N_\sigma)}$-torsor since it has an $\mathcal{O}$-point over N_σ. As $Z_G(N_\sigma)$ is smooth it follows that $M_{\overline{N}}$ is formally smooth over $\mathrm{Spf}\,\mathcal{O}$ of relative dimension $\dim G_k$. $\square$

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Example 5.7. This recovers [CHT08, Lemma 2.4.19] in the case $G = \mathrm{GL}_m$.

Let S be the (torus) quotient of G by its derived group G', and $\mu : G \to S$ the quotient map. For use later, we now study a variant where we fix a lift $\nu : T_q \to S(\mathcal{O})$ of $\mu \circ \overline{\rho} : T_q \to S(k)$, as follows.

Corollary 5.8. *Under assumptions (A1)–(A4), the deformation condition of minimally ramified lifts $\rho : T_q \to G(R)$ satisfying $\mu \circ \rho = \nu$ is a liftable deformation condition. The lifting ring $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square}$ is of relative dimension $\dim G_k - 1$, and the tangent space to the deformation functor has dimension $\dim_k H^0(T_q, \mathrm{ad}(\overline{\rho})) - 1$.*

Proof. The last claim about the dimension of the tangent space to the deformation functor follows from the claim about the dimension of the lifting ring and Remark 2.2.

The quotient torus $S = G/G'$ is split of rank 1, so the subscheme $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square} \subset R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ is the vanishing of locus of a single function. As $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ is formally smooth over $\mathcal{O}$ with relative dimension $\dim G_k$, it suffices to check that the tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square}$ over k (in the sense of [Maz97, § 15]) is a proper subspace of the tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$: this will establish formal smoothness and the dimension claim.

Let Z be the maximal central torus of G. On the level of Lie algebras, we know that $\mathrm{Lie}\, G$ splits over $\mathcal{O}$ as a direct sum of $\mathrm{Lie}\, G'$ and $\mathrm{Lie}\, S \simeq \mathrm{Lie}\, Z$ as p is very good for G. We can modify a lift ρ_0 over $R = k[\epsilon]/(\epsilon^2)$ by multiplying against an unramified non-trivial character $T_q \to Z(R)$ with trivial reduction, changing $\mu \circ \rho_0$. Thus the tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square}$ is a proper subspace of that of $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$. □

5.3 Deformation conditions based on parabolic subgroups

The use of nilpotent orbits is not the only approach to defining a deformation condition at ramified places not above p. As discussed in the introduction, the method used to prove [CHT08, Lemma 2.4.19] suggests a generalization from GL_n to other groups G based on deformations lying in certain parabolic subgroups of G. This deformation condition is not smooth for algebraic groups beyond GL_n, so it does not work in Ramakrishna's method. In this section we give a conceptual explanation for this phenomenon.

Let $P \subset G$ be a parabolic $\mathcal{O}$-subgroup. The Richardson orbit for P_k intersects $(\mathrm{Lie}\, R_u P)_k$ in a dense open set which is a single geometric orbit under P_k. Suppose that $\overline{\rho}(\tau)$ is the exponential of a k-point $\overline{N}$ in the Richardson orbit, and consider deformations $\rho : T_q \to G(\mathcal{O})$ of $\overline{\rho}$ ramified with respect to P in the sense that $\rho(\tau) \in P$ (compare with Definition 1.2). This requires specifying a lift of $\overline{N}$ that lies in $\mathrm{Lie}\, P$. One could hope that such lifts would automatically be $G(\mathcal{O})$-conjugate to the fixed lift N_σ defined in Proposition 3.7, reminiscent of the definition we gave for $\mathrm{Nil}_{\overline{N}}$, a situation in which the associated deformation (or lifting) ring is smooth.

We now show that often smoothness fails if $\overline{N}$ does *not* lie in the Richardson orbit of P_k. Lifts of $\overline{N}$ can 'change nilpotent type' yet still lie in a parabolic lifting P_k, such as the example of the standard Borel subgroup in GL_3 with

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \quad \text{lifting} \quad \overline{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, we easily obtain non-pure nilpotents. This is very bad: the nilpotent orbits over a field are smooth but the nilpotent cone is not smooth, so the deformation problem of deforming

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with respect to P should not be smooth because 'it sees multiple orbits'. Furthermore, even if we could lift $\overline{\rho}(\tau)$ appropriately, there would still be problems lifting $\overline{\rho}(\phi)$ because the centralizer of a non-pure nilpotent is not smooth over $\mathcal{O}$ (the special and generic fiber typically have different dimensions). So it is crucial to choose a parabolic such that $\overline{N}$ lies in the Richardson orbit of P_k.

For GL_n, *all* nilpotent orbits are Richardson orbits. This is not true in general. In particular, we should not expect the deformation condition of being ramified with respect to a parabolic to be liftable. Example 1.3 illustrates this phenomenon for GSp_4, which we now revisit in a more conceptual manner.

Example 5.9. Take $G = \mathrm{GSp}_4$. Parabolic subgroups correspond to isotropic flags. Up to conjugacy, these subgroups are stabilizers of the flags

$$0 \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2) \subset \mathrm{Span}(v_1, v_2, v_3) \subset k^4, \quad 0 \subset k^4$$
$$0 \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2, v_3) \subset k^4, \quad 0 \subset \mathrm{Span}(v_1, v_2) \subset k^4,$$

where $\{v_1, v_2, v_2, v_4\}$ is a basis of k^4. Their Richardson orbits correspond to the nilpotent orbits indexed respectively by the partitions 1+1+1+1, 4, 4, and 2+2. In particular, the same nilpotent orbit is associated with two flags, and the nilpotent orbit corresponding to the partition $2+1+1$ does not appear. This corresponds to a nilpotent orbit that is not a Richardson orbit; for the representation in Example 1.3, $\log(\overline{\rho}(\tau))$ is in this nilpotent orbit.

6. Minimally ramified deformations in general

We continue the notation of the previous section. We have defined the minimally ramified deformation condition for representations factoring through the quotient $T_q = \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$ of the tame Galois group Γ_L^{t} at a place away from p. In this section, we will adapt the matrix-theoretic methods in [CHT08, § 2.4.4], making use of more conceptual module-theoretic arguments, to define the minimally ramified deformation condition for any representation when $G = \mathrm{GSp}_m$ or $G = \mathrm{GO}_m$. (Minor variants of this method work for Sp_m and SO_m, and the original method of [CHT08, § 2.4.4] works for GL_m.) We naturally embed G into $\mathrm{GL}(M)$ for a free $\mathcal{O}$-module M of rank m, and let V denote the reduction of M, a vector space over the residue field k.

We consider a representation $\overline{\rho} : \Gamma_L \to G(k) \subset \mathrm{GL}(V)(k)$ which may be wildly ramified (with L an ℓ-adic field for $\ell \neq p$). We will define a deformation condition for $\overline{\rho}$ in terms of the minimally ramified deformation condition for certain associated tamely ramified representations, after possibly extending $\mathcal{O}$. In § 6.1, we analyze $\overline{\rho}$ as being built out of two pieces of data: representations of a closed normal subgroup Λ_L of Γ_L whose pro-order is prime to p, and tamely ramified representations of Γ_L/Λ_L. The representation theory of Λ_L is manageable since its pro-order is prime to p, and representations of Γ_L/Λ_L can be understood using the results of the previous section.

6.1 Decomposing representations

We begin with a few preliminaries about representations over rings. Let Λ' be a profinite group and R be an Artinian coefficient ring with residue field k. If Λ' has pro-order prime to p, the representation theory of Λ' over k is nice: every finite-dimensional continuous representation is a direct sum of irreducibles, and every such representation is projective over $k[\Lambda]$ for any finite discrete quotient Λ of Λ' through which the representation factors. We are also interested in corresponding statements over an Artinian coefficient ring R.

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Fact 6.1. Let R be an Artinian coefficient ring with residue field k. Suppose the pro-order of Λ' is prime to p. Let P and P' be $R[\Lambda']$-modules that are finitely generated over R with continuous action of Λ', and F be a $k[\Lambda']$-module that is finite dimensional over k with continuous action of Λ'. Let Λ be a finite discrete quotient of Λ' through which the Λ'-actions on P, P', and F factor.

(i) If P is free as an R-module, it is projective as a $R[\Lambda]$-module.
(ii) If P and P' are projective over $R[\Lambda]$, they are isomorphic if and only if $\overline{P}$ and $\overline{P'}$ are isomorphic.
(iii) There exists a projective $R[\Lambda]$-module (unique up to isomorphism) whose reduction is F.

These statements are special cases of results in [Ser77, § 14.4]. We now record two lemmas which do not need the assumption that the pro-order of Λ' is prime to p. Here and elsewhere, we use $\mathrm{Hom}_{\Lambda'}$ to denote homomorphisms as representations of Λ' (equivalently as $R[\Lambda']$-modules).

Lemma 6.2. *Let P and P' be $R[\Lambda']$-modules, finitely generated over R with continuous action of Λ' factoring through a finite discrete quotient Λ of Λ'. Assume P and P' are $R[\Lambda]$-projective. The natural map gives an isomorphism*

$$\mathrm{Hom}_{\Lambda'}(P, P') \otimes_R k \to \mathrm{Hom}_{\Lambda'}(\overline{P}, \overline{P'}).$$

Proof. We may replace $\mathrm{Hom}_{\Lambda'}$ with Hom_{Λ}. Note that $\mathfrak{m}P' = \mathfrak{m} \otimes_R P'$, so $\mathrm{Hom}_\Lambda(P, \mathfrak{m}P') = \mathrm{Hom}_\Lambda(P, P') \otimes_R \mathfrak{m}$ as P and P' are $R[\Lambda]$-projective. Then apply $\mathrm{Hom}_\Lambda(P, -)$ to the exact sequence $0 \to \mathfrak{m}P' \to P' \to P'/\mathfrak{m}P' \to 0$. □

Lemma 6.3. *Let Λ be a finite group and let M and M' be finite $R[\Lambda]$-modules whose reductions $\overline{M}$ and $\overline{M}'$ are non-isomorphic irreducible $k[\Lambda]$-modules. Then $\mathrm{Hom}_{R[\Lambda]}(M, M') = 0$.*

Proof. Filter M' by the composition series $\{\mathfrak{m}^i M'\}$, and consider the surjection

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes \overline{M}' \twoheadrightarrow \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M'.$$

Now Λ acts on $\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes M'$ via its action on $\overline{M}'$, so as a $k[\Lambda]$-module $\mathfrak{m}^i M'/\mathfrak{m}^{i+1} M'$ is isomorphic to a direct sum of copies of $\overline{M}'$. Thus

$$\mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M') = \mathrm{Hom}_{k[\Lambda]}(\overline{M}, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M') = 0$$

as $\overline{M}$ and $\overline{M}'$ are non-isomorphic $k[\Lambda]$-modules.

By descending induction on i, we shall show that

$$\mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M') = 0.$$

For large i, $\mathfrak{m}^i M' = 0$. Consider the exact sequence

$$0 \to \mathfrak{m}^{i+1} M' \to \mathfrak{m}^i M' \to \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M' \to 0.$$

Applying $\mathrm{Hom}_{R[\Lambda]}(M, -)$, we obtain a left exact sequence

$$0 \to \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^{i+1} M') \to \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M') \to \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M').$$

The left term is 0 by induction, and the right term is 0 by the above calculation. This completes the induction. □

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Given $\overline{\rho}: \Gamma_L \to G(k) \subset \mathrm{GL}(V)(k)$ and a lift $\rho: \Gamma_L \to G(R) \subset \mathrm{GL}(M)(R)$ for some $R \in \mathcal{C}_{\mathcal{O}}$, we now turn to decomposing the $R[\Gamma_L]$-module M. Let $I_L \subset \Gamma_L$ be the inertia group, and pick a surjection $I_L \to \mathbf{Z}_p$. Define Λ_L to be the kernel of this surjection (normal in Γ_L). This is a pro-finite group with pro-order prime to p, and is independent of the choice of surjection. Define the quotient

$$T_L := \Gamma_L/\Lambda_L,$$

which is a quotient of the tamely ramified Galois group Γ_L^{t} and of the form $T_q = \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$ as in § 5. We wish to compatibly decompose V and M as Λ_L-modules and then understand the action of Γ_L on the decomposition.

We first make a finite extension of k (and of $\mathcal{O}$) so that all of the (finitely many) irreducible representations of Λ_L over k occurring in V are absolutely irreducible over k.

Because Λ_L has order prime to p, $\mathrm{Res}^{\Gamma_L}_{\Lambda_L}(V)$ is completely reducible and we can write

$$\mathrm{Res}^{\Gamma_L}_{\Lambda_L}(V) = \bigoplus_{\tau} V_\tau,$$

where τ runs through the set of isomorphism classes of irreducible representations of Λ_L over k occurring in V, and each V_τ is the τ-isotypic component. We will obtain an analogous decomposition for M.

Let Γ be a finite discrete quotient of Γ_L through which ρ factors, and let Λ be the image of Λ_L in Γ. Using Fact 6.1(iii) we can lift τ to a projective $R[\Lambda]$-module $\widetilde{\tau}$ unique up to isomorphism. We will eventually want this lift to have additional properties (see § 6.2), but this is not yet necessary. We set $W_\tau := \mathrm{Hom}_{\Lambda_L}(\widetilde{\tau}, M)$ and consider the natural morphism

$$\bigoplus_{\tau} \widetilde{\tau} \otimes_R W_\tau \to M.$$

Note that M is $R[\Lambda]$-projective by Fact 6.1(i).

Lemma 6.4. *This map is an isomorphism of $R[\Lambda_L]$-modules.*

Proof. It suffices to check the map is an isomorphism of $R[\Lambda]$-modules. When $R = k$, $\mathrm{End}_\Lambda(\tau) = k$ as we extended k so that all of the irreducible representations of Λ over k occurring inside V are absolutely irreducible. Splitting up V as a direct sum of irreducibles, we obtain the desired isomorphism.

In the general case, the map is an isomorphism after reducing modulo $\mathfrak{m}$ (use Lemma 6.2). Thus by Nakayama's lemma it is surjective. Since M is R-projective, the formation of the kernel commutes with reduction modulo $\mathfrak{m}$. Thus, again using Nakayama's lemma, the kernel is zero. □

We define M_τ to be the image of $\widetilde{\tau} \otimes_R \mathrm{Hom}_{\Lambda_L}(\widetilde{\tau}, M)$ in M. It is the largest $R[\Lambda_L]$-direct summand whose reduction is a direct sum of copies of τ.

We next seek to understand the action of Γ_L on this canonical decomposition of M. For $g \in \Gamma_L$, consider the $R[\Lambda_L]$-module gM_τ: it is a direct summand of M over R whose reduction is a direct sum of copies of the representation τ^g defined by $\tau^g(h) = \tau(g^{-1}hg)$ for $h \in \Lambda_L$. Thus we see that $gM_\tau = M_{\tau^g}$ inside M, and Γ_L permutes the modules M_τ. The orbits corresponds to sets of conjugate representations.

Consider the stabilizer of V_τ:

$$\Gamma_{L,\tau} = \{g \in \Gamma_L : gV_\tau = V_\tau \text{ inside } V\} = \{g \in \Gamma_L : \tau^g \simeq \tau\} \subset \Gamma_L$$

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with corresponding image

$$\Gamma_\tau = \{g \in \Gamma : gV_\tau = V_\tau \text{ inside } V\} = \{g \in \Gamma : \tau^g \simeq \tau\} \subset \Gamma.$$

Then the k-span of the Γ_L-orbit of V_τ is exactly the representation $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} V_\tau = \mathrm{Ind}_{\Gamma_\tau}^{\Gamma} V_\tau$. Letting $[\tau]$ denote the set of $R[\Lambda]$-isomorphism classes of Λ-representations Γ-conjugate to τ, by taking into account the action of Γ_τ the isomorphism in Lemma 6.4 becomes an isomorphism of $R[\Gamma_L]$-modules

$$\bigoplus_{[\tau]} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \xrightarrow{\sim} M \tag{6.1}$$

using one representative τ per Γ_L-conjugacy class $[\tau]$.

For orthogonal or symplectic representations, we will make precise the notion that this decomposition is 'compatible with duality'. Denote the similitude character by μ, and let $\overline{\nu} := \mu \circ \overline{\rho} : \Gamma_L \to k^\times$. Let N be a free $\mathcal{O}$-module of rank 1 on which Γ_L acts by a specified continuous $\mathcal{O}^\times$-valued lift ν of $\overline{\nu}$, and let $\overline{N}$ be its reduction modulo $\mathfrak{m}$. For an R-module M, define $M^\vee = \mathrm{Hom}_R(M, N_R)$ with the evident Γ_L-action.

Now suppose we have chosen ν so that $\nu = \mu \circ \rho$ (viewed as maps $\Gamma_L \to R^\times$). The perfect pairing on the M corresponding to ρ gives an isomorphism of $R[\Gamma_L]$-modules $\psi : M \simeq M^\vee$. In particular, using Lemma 6.4 we see that

$$\bigoplus_\tau M_\tau = M \overset{\psi}{\simeq} M^\vee = \bigoplus_\tau (M_\tau)^\vee.$$

To simplify notation, we will write $M_\tau^\vee$ for $(M_\tau)^\vee$. Note that the right side is also an isotypic decomposition, with $M_\tau^\vee$ the maximal direct whose direct sum is a direct sum of copies of $\tau^\vee$. By comparing isotypic pieces, we obtain a natural isomorphism (of $R[\Lambda_L]$-modules)

$$\psi_\tau : M_\tau^\vee \simeq M_{\tau^*}$$

for some irreducible representation τ^* of Λ_L occurring in V. Note that $\tau^* \simeq \tau^\vee$ as $k[\Lambda_L]$-modules, but that τ^* is not necessarily the dual of τ as $k[\Gamma_L]$-modules. There are three cases.

- *Case 1:* τ is not Γ_L-conjugate to τ^*.
- *Case 2:* τ is isomorphic to τ^* as Γ_L-modules.
- *Case 3:* τ is Γ_L-conjugate to τ^* but not isomorphic.

In the second case, we claim that the isomorphism of $k[\Lambda_L]$-modules $\imath : \tau \simeq \tau^\vee$ gives a sign-symmetric (for some fixed sign ϵ_τ) perfect pairing on τ. Note that $\overline{W}_\tau = \mathrm{Hom}_\Lambda(\tau, V)$ by Lemma 6.2, and that

$$\overline{W}_\tau = \mathrm{Hom}_\Lambda(\tau, V) \overset{\overline{\psi}}{\simeq} \mathrm{Hom}_\Lambda(\tau, V^\vee) \overset{\imath}{\simeq} \mathrm{Hom}_\Lambda(\tau^\vee, V^\vee) \simeq \overline{W}_\tau^\vee.$$

Denote this isomorphism by φ_τ: it defines a pairing $\langle\, ,\rangle_{\overline{W}_\tau}$ on $\overline{W}_\tau$ via

$$\langle w_1, w_2\rangle_{\overline{W}_\tau} := \varphi_\tau(w_1)(w_2).$$

We can also define $\langle v_1, v_2\rangle_\tau := \imath(v_1)(v_2)$ for $v_1, v_2 \in \tau$. We have a commutative diagram of isomorphisms.

$$\begin{array}{ccccc}
\tau \otimes \overline{W}_\tau & \xrightarrow{\mathrm{id} \otimes \varphi_\tau} & \tau \otimes \overline{W}_\tau^\vee & \xrightarrow{\imath \otimes \mathrm{id}} & \tau^\vee \otimes \overline{W}_\tau^\vee \\
\Big\downarrow{\scriptstyle \mathrm{eval}} & & & & \Big\downarrow{\scriptstyle \mathrm{eval}^\vee} \\
V_\tau & & \xrightarrow{\overline{\psi_\tau}} & & V_\tau^\vee
\end{array}$$

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The commutativity says that for elementary tensors $m_i = v_i \otimes w_i \in V_\tau = \tau \otimes \overline{W}_\tau$ we have

$$\begin{aligned}\langle m_1, m_2\rangle_M = \overline{\psi}(m_1)(m_2) &= (\imath(v_1) \otimes \varphi_\tau(w_1))(v_2 \otimes w_2) \\ &= \imath(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \langle v_1, v_2\rangle_\tau \langle w_1, w_2\rangle_{\overline{W}_\tau}. \end{aligned} \tag{6.2}$$

Remember that the pairing on V is ϵ-symmetric.

LEMMA 6.5. *The pairing $\langle\, ,\rangle_\tau$ is a sign-symmetric (for fixed sign ϵ_τ).*

Proof. Suppose there exists $v \in \tau$ such that $\imath(v)(v) \neq 0$. For $w_1, w_2 \in \overline{W}_\tau$, (6.2) gives

$$\imath(v)(v)\varphi_\tau(w_1)(w_2) = \langle v \otimes w_1, v \otimes w_2\rangle_V = \epsilon\langle v \otimes w_2, v \otimes w_1\rangle_V = \epsilon\imath(v)(v)\varphi_\tau(w_2)(w_1).$$

Canceling $\imath(v)(v)$, we conclude that $\langle w_1, w_2\rangle_{\overline{W}_\tau} = \epsilon\langle w_2, w_1\rangle_{\overline{W}_\tau}$. Using (6.2), we conclude that

$$\begin{aligned}\epsilon\imath(v_2)(v_1) \cdot \varphi_\tau(w_2)(w_1) &= \epsilon\langle m_2, m_1\rangle_V = \langle m_1, m_2\rangle_V \\ &= \imath(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \epsilon\imath(v_1)(v_2) \cdot \varphi_\tau(w_2)(w_1).\end{aligned}$$

Choosing w_1 and w_2 with $\langle w_2, w_1\rangle_{\overline{W}_\tau} \neq 0$ (possible as $\langle\, ,\rangle_V$ is perfect), we then conclude that $\langle v_1, v_2\rangle_{\widetilde{\tau}} = \langle v_2, v_1\rangle_{\widetilde{\tau}}$.

Otherwise $\imath(v)(v) = 0$ for all $v \in \tau$, in which case $\langle\, ,\rangle_{\widetilde{\tau}}$ is alternating. □

In § 6.2 we will see that the action of Λ_L on the module underlying $\widetilde{\tau}$ can be extended to an action of $\Gamma_{L,\tau}$ factoring through Γ_τ. Therefore, $W_\tau = \operatorname{Hom}_{\Lambda_L}(\widetilde{\tau}, M)$ is naturally a representation of $T_{L,\tau} := \Gamma_{L,\tau}/\Lambda_L$, and of $T_\tau := \Gamma_\tau/\Lambda$ (a finite quotient of $T_{L,\tau}$). In § 6.4, we will use the minimally ramified deformation condition of § 5 to specify which deformations W_τ are allowed. Together with the decomposition (6.1)

$$\bigoplus_{[\tau]} \operatorname{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W_\tau) \to M$$

this defines a deformation condition for $\overline{\rho}$. Some care is needed to ensure compatibility of the lifts with the pairing on M, which will require breaking into cases in the next sections based on the relationship between τ and τ^*.

6.2 Extension of representations

We continue the notation of the previous section, where τ is an absolutely irreducible representation of Λ_L over k. We need to lift this to a representation over $\mathcal{O}$ and extend it to a representation of $\Gamma_{L,\tau}$. We will have to do something extra for the representation to be compatible with a pairing, depending on how τ and τ^* are related.

In Case 1, we ignore the pairing. Lemma 2.4.11 of [CHT08] lets us pick a $\mathcal{O}[\Gamma_{L,\tau}]$-module $\widetilde{\tau}$ that is a free $\mathcal{O}$-module and reduces to τ. In this case, $\widetilde{\tau}^\vee$ is a free $\mathcal{O}$-module reducing to τ^*.

In Case 2, from Lemma 6.5 it follows that τ is a symplectic or orthogonal representation. We will adapt the GL_n-technique of [CHT08] to produce a symplectic or orthogonal extension $\widetilde{\tau}$. Letting $n = \dim \tau$, the representation τ gives a homomorphism $\tau : \Lambda_L \to G(k)$ where G is GSp_n or GO_n.

First, we claim that there is a continuous lift $\widetilde{\tau} : \Lambda_L \to G(W(k))$: without the pairing, this would be Fact 6.1(iii). To also take into account the pairing, consider deformation theory for the residual representation τ. This is a smooth deformation condition as $H^2(\Lambda_L, \operatorname{ad}\tau) = 0$: Λ_L has pro-order prime to p and $\operatorname{ad}\tau$ has order a power of p. Therefore the desired lift exists. It is unique (up to conjugation by an element of $\widehat{G}(\mathcal{O})$) because the tangent space is zero dimensional as $H^1(\Lambda_L, \operatorname{ad}\tau) = 0$. By considering representations of the group $\Lambda_L/\ker(\tau)$, we may and do assume that $\ker(\widetilde{\tau}) = \ker(\tau)$ as subgroups of Λ_L.

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Remark 6.6. For $g \in \Gamma_{L,\tau}$, the $k[\Lambda_L]$-modules $\tau^g \simeq \tau$ are isomorphic. By uniqueness of the lift, this means that there is $A \in G(\mathcal{O})$ such that $\widetilde{\tau}^g(\gamma) = A\widetilde{\tau}(\gamma)A^{-1}$ for all $\gamma \in \Gamma_L$. Furthermore, $\Gamma_{L,\tau} = \{g \in \Gamma_L : \widetilde{\tau}^g \simeq \widetilde{\tau}\}$.

We will now show how to continuously extend $\widetilde{\tau}$ to $\Gamma_{L,\tau}$. The first step in constructing the extension is to understand the structure of $\Gamma_{L,\tau}$ and $I_L \cap \Gamma_{L,\tau}$, where I_L is the inertia group.

Recall that $T_L = \Gamma_L/\Lambda_L$ is the semidirect product of $\widehat{\mathbf{Z}}$ and $\mathbf{Z}_p$, where $\widehat{\mathbf{Z}}$ is generated by a lift of Frobenius ϕ and $\mathbf{Z}_p$ is generated by an element σ, with $\phi\sigma\phi^{-1} = \sigma^q$ where $q = \ell^a$ is the size of the residue field of L.

Fact 6.7. The exact sequence

$$1 \to \Lambda_L \to \Gamma_L \to T_L \to 1$$

is topologically split, so Γ_L is a semidirect product.

Proof. This is [CHT08, 2.4.10]. □

For $T_{L,\tau} := \Gamma_{L,\tau}/\Lambda_L$, this gives a topological splitting of

$$1 \to \Lambda_L \to \Gamma_{L,\tau} \to T_{L,\tau} \to 1.$$

As $\Gamma_{L,\tau}$ is an open subgroup of Γ_L, we observe that $T_{L,\tau}$ is an open subgroup of T_L. Note that $T_{L,\tau}$ is normal and topologically generated by some powers of ϕ and σ which will be denoted by ϕ_τ and σ_τ (since any open subgroup of a semidirect product $C \ltimes C'$ for pro-cyclic C and C' is of the form $C_0 \ltimes C'_0$ for open subgroups $C_0 \subset C$ and $C'_0 \subset C'$). In particular, using the notation of § 5.2 $T_{L,\tau}$ is itself isomorphic to $T_{q'}$ for some q'. The element σ_τ and Λ_L together topologically generate $\Gamma_{L,\tau} \cap I_L$.

Before extending $\widetilde{\tau}$, we need several technical lemmas.

Lemma 6.8. *We have that* $\mathrm{End}_{\Lambda_L}(\widetilde{\tau}) = \mathcal{O}$.

Proof. As τ is absolutely irreducible, $\mathrm{End}_{\Lambda_L}(\tau) = k$. By Lemma 6.2, we see that the reduction of $\mathrm{End}_{\Lambda_L}(\widetilde{\tau})$ modulo the maximal ideal of $\mathcal{O}$ is k, so the map $\mathcal{O} \hookrightarrow \mathrm{End}_{\Lambda_L}(\widetilde{\tau})$ is surjective by Nakayama's lemma. □

Lemma 6.9. *The dimension of τ is not divisible by p.*

Proof. As τ is continuous and Λ_L has pro-order prime to p, the representation τ factors through a finite discrete quotient Λ of Λ_L whose order is prime to p. Such a representation is the reduction of a projective $\mathcal{O}[\Lambda]$-module by Fact 6.1(iii). Inverting p, we obtain a representation of Λ in characteristic zero that is absolutely irreducible since the 'reduction' τ is absolutely irreducible over k. By [Ser77, § 6.5 Corollary 2], the dimension of this representation (equal to the dimension of τ) divides the order of Λ. □

We will now extend $\widetilde{\tau}$ from $\Lambda_L \subset I_L$ to $\Gamma_{L,\tau}$ by defining it on the topological generators σ_τ and ϕ_τ. We say that such an extension has *tame determinant* if $\det(\widetilde{\tau}(\sigma_\tau))$ has finite order which is prime to p. Lemmas 6.10 and 6.11 adapt [CHT08, Lemma 2.4.11] and fill in some details.

Lemma 6.10. *There is a unique continuous extension $\widetilde{\tau} : \Gamma_{L,\tau} \cap I_L \to G(\mathcal{O})$ with tame determinant.*

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Proof. A continuous extension of $\widetilde{\tau}$ to $\Gamma_{L,\tau} \cap I_L$ is determined by its value on σ_τ. As $\sigma_\tau \in \Gamma_{L,\tau}$, in light of Remark 6.6 there is an $A \in G(\mathcal{O})$ such that for $g \in \Lambda_L$ we have

$$\widetilde{\tau}(\sigma_\tau g \sigma_\tau^{-1}) = A\widetilde{\tau}(g)A^{-1}.$$

We would like to send σ_τ to the element A. For an appropriate modification of A (still lying in $G(\mathcal{O})$), this will produce a continuous extension with tame determinant. As σ_τ is a topological generator for a group isomorphic to $\mathbf{Z}_p$, the continuity of the extension with $\sigma_\tau \mapsto A$ is equivalent to some p-power of A having trivial reduction. We wish to show that there is a unique choice of such A that also makes the extension have tame determinant.

We will first show that some power A^{p^b} lies in the centralizer of the image $\widetilde{\tau}(\Lambda_L)$. Consider the conjugation action of $\langle\sigma_\tau\rangle$ on Λ_L. As $\ker\widetilde{\tau} = \ker\tau$ is a normal subgroup of $\Gamma_{L,\tau}$ (if $g \in \Gamma_{L,\tau}$ and $\tau(h) = 1$, then $\tau^g(h)$ is conjugate to $\tau(h) = 1$ by Remark 6.6) we get an action of $\langle\sigma_\tau\rangle$ on $\Lambda_L/\ker\tau \simeq \tau(\Lambda_L)$. The action is continuous, so there is a power p^b such that for all $g \in \Lambda_L$ we have

$$\tau(\sigma_\tau^{p^b} g \sigma_\tau^{-p^b}) = \tau(g).$$

As $\ker\widetilde{\tau} = \ker\tau$, we see that

$$A^{p^b}\widetilde{\tau}(g)A^{-p^b} = \widetilde{\tau}(\sigma_\tau^{p^b} g \sigma_\tau^{-p^b}) = \widetilde{\tau}(g).$$

Therefore A^{p^b} lies in the centralizer of $\widetilde{\tau}(\Lambda_L)$ in $G(\mathcal{O})$.

By Lemma 6.8, this centralizer is isomorphic to $\mathcal{O}^\times$. We claim that by multiplying A by some unit in $\mathcal{O}$, we can arrange for the continuous extension $\widetilde{\tau}$ to exist and have tame determinant. We will use the fact that an element of $\mathcal{O}^\times$ is the product of a 1-unit and a Teichmuller lift of an element of $k^\times$. As $A^{p^b} \in \mathcal{O}^\times$ and the pth power map is an automorphism of $k^\times$, we may multiply A by a unit scalar so that A^{p^b} reduces to the identity matrix. By Lemma 6.9, the dimension n of τ is prime to p so we may multiply A by a 1-unit so that $\det(A)$ has finite order prime to p. Sending σ_τ to this particular A gives a continuous extension with tame determinant.

Let us show this extension is unique. Any extension must send σ_τ to an element of the form wA for $w \in \mathcal{O}^\times$ (the centralizer of the image $\widetilde{\tau}(\Lambda_L)$). By continuity, there is a power p^b such that $(wA)^{p^b}$ reduces to the identity. This means that w^{p^b} reduces to the identity, and hence that w reduces to the identity. On the other hand, $\det(wA)\det(A)^{-1} = w^n$. The left side has finite order that is relatively prime to p, so w^n does too. This forces $w^n = 1$ since its reduction is 1. But as n is prime to p (Lemma 6.9), the only nth roots of unity in $\mathcal{O}^\times$ are Teichmuller lifts. Therefore $w = 1$. □

LEMMA 6.11. *There is a continuous extension $\widetilde{\tau} : \Gamma_{L,\tau} \to G(\mathcal{O})$.*

Proof. We extend $\widetilde{\tau}$ in Lemma 6.10 continuously to $\Gamma_{L,\tau}$ by defining it on ϕ_τ. As $\phi_\tau \in \Gamma_{L,\tau}$, by Remark 6.6 there is an element $A \in G(\mathcal{O})$ conjugating $\widetilde{\tau} : \Lambda_L \to G(\mathcal{O})$ to $\widetilde{\tau}^{\phi_\tau} : \Lambda_L \to G(\mathcal{O})$. Each has a unique extension to a continuous morphism from $I_L \cap \Gamma_{L,\tau}$ to $G(\mathcal{O})$ with tame determinant. Therefore for $g \in I_L \cap \Gamma_{L,\tau}$ we have

$$\widetilde{\tau}(\phi_\tau g \phi_\tau^{-1}) = A\widetilde{\tau}(g)A^{-1}$$

since the right side has the same (tame) determinant as $\widetilde{\tau}$ on $I_L \cap T_\tau$. We can continuously extend $\widetilde{\tau} : I_L \cap \Gamma_{L,\tau} \to G(\mathcal{O})$ by sending ϕ_τ to A since A has reduction with finite order. □

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This gives the desired lift and extension of τ in the case that $\tau \simeq \tau^*$.

In Case 3, τ is conjugate to τ^* but not isomorphic. The argument follows the same structure as the previous case, but we make a few modifications to treat $\tau \oplus \tau^*$ together. In particular, we can pick a copy of the $k[\Lambda_L]$-module τ inside V and a copy of $\tau^* \simeq \tau^\vee$ inside V such that the pairing restricted to $\tau \oplus \tau^*$ is perfect. It is sign-symmetric with sign $\epsilon_{\tau\oplus\tau^*}$.

Define $\Gamma_{L,\tau\oplus\tau^*} = \{g \in \Gamma_L : (\tau \oplus \tau^*)^g \simeq \tau \oplus \tau^*\}$. It contains $\Gamma_{L,\tau}$ with index 2, as conjugation either preserves τ and τ^* or swaps them. Arguing as in the paragraph after Fact 6.7, we obtain a split exact sequence

$$0 \to \Lambda_L \to \Gamma_{L,\tau\oplus\tau^*} \to T_{L,\tau\oplus\tau^*} \to 1,$$

where $T_{L,\tau\oplus\tau^*}$ is an open normal subgroup of T_L topologically generated by some powers of ϕ and σ which we denote by $\phi_{\tau\oplus\tau^*}$ and $\sigma_{\tau\oplus\tau^*}$. We may arrange that either:

- *Case 3a:* $\phi^2_{\tau\oplus\tau^*} = \phi_\tau$ and $\sigma_{\tau\oplus\tau^*} = \sigma_\tau$; or
- *Case 3b:* $\phi_{\tau\oplus\tau^*} = \phi_\tau$ and $\sigma^2_{\tau\oplus\tau^*} = \sigma_\tau$.

In Case 3a, we begin by lifting τ to $\mathcal{O}$ as a representation of Λ_L: as before, we do this using the fact that the pro-order of Λ_L is prime to p, and obtain a lift $\widetilde{\tau}$ unique up to isomorphism. We extend $\widetilde{\tau}$ to be a representation of $\Gamma_{L,\tau} \cap I_L$ by defining it on σ_τ using the GL_n-version of Lemma 6.10, [CHT08, Lemma 2.4.11]. There it is shown all such extensions are unique up to equivalence. In particular, $\widetilde{\tau}$ and $(\widetilde{\tau}^{\phi_{\tau\oplus\tau^*}})^\vee$ are isomorphic $\mathcal{O}[\Gamma_{L,\tau} \cap I_L]$-modules. We can use this to define a sign-symmetric perfect pairing on $\widetilde{\tau} \oplus \widetilde{\tau}^{\phi_{\tau\oplus\tau^*}}$ that is compatible with the action of $\Gamma_{L,\tau} \cap I_L$ and $\phi_{\tau\oplus\tau^*}$, hence of $\Gamma_{L,\tau\oplus\tau^*}$.

In Case 3b, as $\tau^\vee$ and $\tau^{\sigma_{\tau\oplus\tau^*}}$ are isomorphic $k[\Lambda_L]$-modules it follows that $\widetilde{\tau}^\vee$ and $\widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$ are isomorphic $\mathcal{O}[\Lambda_L]$-modules. In particular, this isomorphism gives a natural way to define a sign-symmetric perfect pairing on $M = \widetilde{\tau} \oplus \widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$ lifting the residual one. This pairing is compatible with the action of $\Gamma_{L,\tau\oplus\tau^*} \cap I_L$ (which is generated by Λ_L and $\sigma_{\tau\oplus\tau^*}$). Finally, we claim that M and M^{ϕ_τ} are isomorphic. As $\phi_\tau \in \Gamma_{L,\tau}$ preserves τ, acting by ϕ_τ gives an isomorphism $\overline{\psi} : \overline{M} \simeq \overline{M}^{\phi_\tau}$ of $k[\Lambda_L]$-modules such that $\overline{\psi}(\tau) = \tau^{\phi_\tau}$. By uniqueness of the lift of τ as a $\mathcal{O}[\Lambda_L]$-module, we obtain an isomorphism ψ_τ of $\widetilde{\tau}$ and $\widetilde{\tau}^{\phi_\tau}$ and hence an isomorphism $\psi : M \simeq M^{\phi_\tau}$ via the identifications

$$\widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}} \simeq \widetilde{\tau}^\vee \overset{\psi_\tau^{-1}}{\simeq} (\widetilde{\tau}^{\phi_\tau^{-1}})^\vee \simeq (\widetilde{\tau}^\vee)^{\phi_\tau} \simeq (\widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}})^{\phi_\tau}.$$

This isomorphism is compatible with the pairing. The key observation is that for $m \in \widetilde{\tau}$ and $f \in \widetilde{\tau}^\vee \simeq \widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$, we have that

$$\langle \psi(m), \psi(f) \rangle_M = \langle \psi_\tau(m), f \circ \psi_\tau^{-1} \rangle_M = f(\psi_\tau^{-1}(\psi_\tau(m))) = f(m) = \langle m, f \rangle_M.$$

Then proceed as in the proof of Lemma 6.11, defining an image of ϕ_τ using the isomorphism ψ.

In conclusion, we have shown the following.

Lemma 6.12. *In Case 3, there exists an $\mathcal{O}[\Gamma_{L,\tau\oplus\tau^*}]$-module $\widetilde{\tau \oplus \tau^*}$ with pairing lifting $\tau \oplus \tau^*$ together with its pairing.*

6.3 Lifts with pairings

We continue the notation of § 6.1, and analyze how the duality pairing interacts with the decomposition (6.1). Recall that we obtained an isomorphism $M \simeq M^\vee$ of $R[\Gamma_L]$-modules which gave isomorphisms $M_\tau \simeq M^\vee_{\tau^*}$ of $R[\Gamma_{L,\tau}]$-modules. The key point is that for any lift and extension τ' of τ, the isomorphism of $R[\Lambda_L]$-modules

$$\tau' \otimes \mathrm{Hom}_{\Lambda_L}(\tau', M) \to M_\tau$$

is compatible with the $\Gamma_{L,\tau}$-action.

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To do this, it is convenient to break into the cases introduced at the end of § 6.1. For an irreducible $k[\Lambda]$-module τ occurring in V, note that $(\tau^g)^\vee = (\tau^\vee)^g$ for any $g \in \Gamma_L$, so if $\tau \simeq \tau^*$ then $\tau^g \simeq (\tau^g)^*$. We let

- Σ_n denote the set of Γ_L-conjugacy classes of such τ for which τ is not conjugate to τ^*;
- Σ_e denote the set of Γ_L-conjugacy classes of such τ for which $\tau \simeq \tau^*$;
- Σ_c denote the set of Γ_L-conjugacy classes of such τ for which τ^* is conjugate to τ but $\tau \not\simeq \tau^*$.

From (6.1), we obtain a decomposition

$$M = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tau' \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tau' \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tau' \otimes W_\tau) \tag{6.3}$$

where τ' is any lift and extension of τ to Γ_τ and $W_\tau = \mathrm{Hom}_\Lambda(\tau', M)$ is a representation of $T_{L,\tau}$. Note that W_τ is free as an R-module (since M and τ' are, with $\tau' \neq 0$ and R local), and hence that W_τ is tamely ramified of the type considered in § 5.

We may rewrite this to make use of the special extensions constructed in § 6.2. In particular, for $\tau \in \Sigma_c$ we rewrite

$$\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tau' \otimes W_\tau) = \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*}).$$

This uses the notation and results from Case 3 in § 6.2, in particular the fact that $\tau \oplus \tau^*$ is an irreducible representation of the group Λ_L' generated by Λ_L and a $g \in \Gamma_L$ with $\tau^* \simeq \tau^g$, and the definition $W_{\tau\oplus\tau^*} := \mathrm{Hom}_{\Lambda_L'}(\widetilde{\tau \oplus \tau^*}, M)$. Note that $W_{\tau\oplus\tau^*}$ is a representation of $T_{L,\tau\oplus\tau^*}$, which is a subgroup of $T_L = \Gamma_L/\Lambda_L$, hence of the form T_q as considered in § 5. Using the extensions $\widetilde{\tau}$ and $\widetilde{\tau \oplus \tau^*}$ from Cases 1 and 2 from § 6.2, we obtain a decomposition

$$M = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*}). \tag{6.4}$$

Now let M' be another $R[\Gamma_L]$-module that is finite free over R such that the irreducible representations of Λ_L occurring in $V' := M'/\mathfrak{m}M'$ are among the irreducible representations occurring in $V = M/\mathfrak{m}M$, so

$$M' = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}(\widetilde{\tau \oplus \tau^*} \otimes W'_{\tau\oplus\tau^*}).$$

LEMMA 6.13. *The natural map*

$$\bigoplus_{\tau \in \Sigma_n} \mathrm{Hom}_{T_{L,\tau}}(W_\tau, W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Hom}_{T_{L,\tau}}(W_\tau, W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Hom}_{T_{L,\tau\oplus\tau^*}}(W_{\tau\oplus\tau^*}, W'_{\tau\oplus\tau^*}) \\ \to \mathrm{Hom}_{\Gamma_L}(M, M')$$

is an isomorphism.

Proof. We may immediately pass to working with representations of the finite discrete groups Γ and Λ. Notice that

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_\tau}^\Gamma(M_\tau), \mathrm{Ind}_{\Gamma_\tau}^\Gamma(M'_\tau)) \simeq \mathrm{Hom}_{\Gamma_\tau}(\mathrm{Ind}_{\Gamma_\tau}^\Gamma(M_\tau), M'_\tau) \simeq \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau),$$

where the second isomorphism uses that $\mathrm{Hom}_{\Gamma_\tau}(M_{\tau^g}, M'_\tau) = 0$ by Lemma 6.3 when τ and τ^g are non-isomorphic. Furthermore, if τ_1 and τ_2 are not Γ-conjugate, then

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tau_1}}^\Gamma(M_{\tau_1}), \mathrm{Ind}_{\Gamma_{\tau_2}}^\Gamma(M'_{\tau_2})) \simeq \mathrm{Hom}_{\Gamma_\tau}(\mathrm{Ind}_{\Gamma_{\tau_1}}^\Gamma(M_{\tau_1}), M_{\tau_2}) = 0$$

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using Lemma 6.3 as τ_1^g is not isomorphic to τ_2 for any $g \in \Gamma$. Then using (6.1) we see that

$$\mathrm{Hom}_\Gamma(M, M') = \bigoplus_{[\tau_1],[\tau_2]} \mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tau_1}}^{\Gamma}(M_{\tau_1}), \mathrm{Ind}_{\Gamma_{\tau_2}}^{\Gamma}(M'_{\tau_2})) = \bigoplus_{[\tau]} \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau).$$

All the irreducible finite-dimensional representations of Λ occurring in V and V' are absolutely irreducible over k by design. For $\tau \in \Sigma_n \cup \Sigma_e$, consider the natural inclusion

$$\mathrm{Hom}_R(W_\tau, W'_\tau) \hookrightarrow \mathrm{Hom}_\Lambda(\widetilde{\tau} \otimes W_\tau, \widetilde{\tau} \otimes W'_\tau) = \mathrm{Hom}_\Lambda(\widetilde{\tau}, \widetilde{\tau}) \otimes_R \mathrm{Hom}_R(W_\tau, W'_\tau), \tag{6.5}$$

using that W_τ and W'_τ are R-free of finite rank and Λ acts trivially. But $R \hookrightarrow \mathrm{Hom}_\Lambda(\widetilde{\tau}, \widetilde{\tau})$ is an isomorphism because $\mathrm{End}_\Lambda(\tau) = k$ and because surjectivity can be checked modulo $\mathfrak{m}_R$ using Lemma 6.2. As $M_\tau \simeq \widetilde{\tau} \otimes W_\tau$, this implies that

$$\begin{aligned} \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau) = \mathrm{Hom}_\Lambda(M_\tau, M'_\tau)^{T_\tau} &= \mathrm{Hom}_\Lambda(\widetilde{\tau} \otimes W_\tau, \widetilde{\tau} \otimes W'_\tau)^{T_\tau} \\ &= \mathrm{Hom}_R(W_\tau, W_\tau)^{T_\tau} = \mathrm{Hom}_{T_\tau}(W_\tau, W'_\tau), \end{aligned}$$

where T_τ is the image of $T_{L,\tau}$ in Γ_τ. An analogous computation in the case $\tau \in \Sigma_c$ completes the proof. $\square$

We can now consider the duality isomorphism $M \simeq M^\vee$. By Lemma 6.13, this is equivalent to a collection of isomorphisms of $R[T_{L,\tau}]$-modules $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$ for $\tau \in \Sigma_e \cup \Sigma_n$ and an isomorphism of $R[T_{L,\tau\oplus\tau^*}]$-modules $\varphi_\tau : W_{\tau\oplus\tau^*} \simeq W_{\tau\oplus\tau^*}^\vee$ for $\tau \in \Sigma_c$. We analyze the cases separately.

In Case 1 (when τ is not conjugate to τ^*), note that $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau$ is an isotropic subspace of M. In particular, the perfect sign-symmetric pairing on $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \oplus \mathrm{Ind}_{\Gamma_{L,\tau^*}}^{\Gamma_L} M_{\tau^*}$ is equivalent to an isomorphism of $R[\Gamma_L]$-modules

$$\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \simeq (\mathrm{Ind}_{\Gamma_{L,\tau^*}}^{\Gamma_L} M_{\tau^*})^\vee,$$

which is equivalent to the isomorphism of $R[T_{L,\tau}]$-modules $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$. (Note that the similitude character ν is present in the use of the dual.)

In Case 2 (when τ is isomorphic to τ^*), the perfect sign-symmetric pairing on $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau$ is equivalent to an isomorphism $W_\tau \simeq W_\tau^\vee$ of $R[T_{L,\tau}]$-modules. Thus it gives a pairing $\langle\, ,\rangle_{W_\tau}$ on W_τ via

$$\langle w_1, w_2 \rangle_{W_\tau} := \varphi_\tau(w_1)(w_2).$$

We claim this pairing is sign-symmetric.

From § 6.2 we have an isomorphism $\imath : \widetilde{\tau} \simeq \widetilde{\tau}^\vee$ of $R[\Gamma_{L,\tau}]$-modules. As at the end of § 6.1, let $\psi : M \to M^\vee$ be the isomorphism of $R[\Gamma_L]$-modules given by $m \mapsto \langle m, - \rangle_M$, and define $\langle v_1, v_2 \rangle_{\widetilde{\tau}} := \imath(v_1)(v_2)$. We have a commutative diagram as follows.

$$\begin{array}{ccccc} \widetilde{\tau} \otimes W_\tau & \xrightarrow{\mathrm{id} \otimes \varphi_\tau} & \widetilde{\tau} \otimes W_\tau^\vee & \xrightarrow{\imath \otimes \mathrm{id}} & \widetilde{\tau}^\vee \otimes W_\tau^\vee \\ \downarrow & & & & \downarrow \\ M_\tau & & \xrightarrow{\qquad\psi\qquad} & & M_\tau^\vee \end{array}$$

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The commutativity says that for elementary tensors $m_i = v_i \otimes w_i \in M_\tau = \widetilde{\tau} \otimes W_\tau$ we have

$$\begin{aligned}\langle m_1, m_2\rangle_M = \psi(m_1)(m_2) &= (\imath(v_1) \otimes \varphi_\tau(w_1))(v_2 \otimes w_2)\\ &= \imath(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \langle v_1, v_2\rangle_\tau \langle w_1, w_2\rangle_{W_\tau}. \end{aligned} \tag{6.6}$$

The pairings are perfect and $\langle\cdot,\cdot\rangle_\tau$ is ϵ_τ-symmetric, so the pairing on W_τ is ϵ_{W_τ}-symmetric if and only if the pairing on M_τ is ϵ-symmetric. We have that $\epsilon = \epsilon_{W_\tau}\epsilon_\tau$.

In Case 3 ($\tau \in \Sigma_c$), an analogous argument using the isomorphism $\widetilde{\tau \oplus \tau^*} \simeq \widetilde{\tau \oplus \tau^*}^\vee$ of $R[\Gamma_{L,\tau\oplus\tau^*}]$-modules (which define the $\epsilon_{W_{\tau\oplus\tau^*}}$-symmetric pairing on $\widetilde{\tau \oplus \tau^*}$) shows that the pairing induced by $\varphi_\tau : W_{\tau\oplus\tau^*} \simeq W_{\tau\oplus\tau^*}^\vee$ is $\epsilon_{\tau\oplus\tau^*}$-symmetric if and only if the pairing on

$$\mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}(\widetilde{\tau\oplus\tau^*} \otimes W'_{\tau\oplus\tau^*})$$

induced from the pairing on M is sign-symmetric with sign $\epsilon = \epsilon_{\tau\oplus\tau^*}\epsilon_{W_{\tau\oplus\tau^*}}$.

6.4 Minimally ramified deformations

We can now define the minimally ramified deformation condition for $\overline{\rho} : \Gamma_L \to G(k)$, under the continuing assumption that we have extended k so all irreducible representations of Λ_L occurring in V are absolutely irreducible over k. From (6.4), we obtain a decomposition

$$V = \bigoplus_{\tau\in\Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes \overline{W}_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes \overline{W}_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}(\widetilde{\tau\oplus\tau^*} \otimes \overline{W}_{\tau\oplus\tau^*}), \tag{6.7}$$

where $\overline{W}_\tau$ is a representation of $T_{L,\tau}$ over k and $\overline{W}_{\tau\oplus\tau^*}$ is a representation of $T_{L,\tau\oplus\tau^*}$.

If $\tau \in \Sigma_n$, define $\overline{G}_\tau := \underline{\mathrm{Aut}}(\overline{W}_\tau)$. If $\tau \in \Sigma_e$, there is a sign-symmetric perfect pairing $\langle\cdot,\cdot\rangle_{\overline{W}_\tau}$ on $\overline{W}_\tau$; in that case define $\overline{G}_\tau := \underline{\mathrm{GAut}}(\overline{W}_\tau, \langle\cdot,\cdot\rangle_{\overline{W}_\tau})$. (The notation $\underline{\mathrm{GAut}}$ means automorphisms preserving the pairing up to scalar.) If $\tau \in \Sigma_c$, there is a sign-symmetric perfect pairing on $\overline{W}_{\tau\oplus\tau^*}$; in that case define $\overline{G}_\tau := \underline{\mathrm{GAut}}(\overline{W}_{\tau\oplus\tau^*}, \langle\cdot,\cdot\rangle_{\overline{W}_{\tau\oplus\tau^*}})$. Make a finite extension of k so that all the pairings are split. Lift $\overline{G}_\tau$ to a split reductive group G_τ over $\mathcal{O}$ by lifting the split linear algebra data.

DEFINITION 6.14. Let $\rho : \Gamma_L \to G(R)$ be a continuous Galois representation lifting $\overline{\rho}$ as above, with associated $R[\Gamma]$-module

$$M = \bigoplus_{\tau\in\Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}(\widetilde{\tau\oplus\tau^*} \otimes W_{\tau\oplus\tau^*}).$$

We say that ρ is *minimally ramified* with similitude character ν if each W_τ and $W_{\tau\oplus\tau^*}$ is minimally ramified in the sense of Definition 5.4 as a representation of $T_{L,\tau}$ or $T_{L,\tau\oplus\tau^*}$ valued in the group G_τ with specified similitude character. (Note that defining the minimally ramified deformation condition as in § 5 may require an additional étale local extension of $\mathcal{O}$, which as always is harmless for applications.)

Let $D_{\overline{\rho}}^{\mathrm{m.r.},\nu}$ denote the deformation functor for $\overline{\rho}$ with specified similitude character ν, and $\mathcal{D}_{G_\tau}^{\mathrm{m.r.}}$ (respectively $\mathcal{D}_{G_\tau}^{\mathrm{m.r.},\nu}$) denote the deformation functor for $\overline{W}_\tau$ or $\overline{W}_{\tau\oplus\tau^*}$ viewed as a representation valued in G_τ (respectively with specified similitude character ν). In particular, letting $r = \dim \overline{W}_\tau$ (or $\dim \overline{W}_{\tau\oplus\tau^*}$ when $\tau \in \Sigma_c$), we have that the adjoint representation $\mathrm{ad}\, \overline{W}_\tau$ is the Lie algebra of $\overline{G}_\tau$, which is the Lie algebra of GSp_r or GO_r when $\tau \in \Sigma_e$ or Σ_c, and the Lie algebra of GL_r when $\tau \in \Sigma_n$. Let Σ'_n consist of one representative for each pair of representations $\tau, \tau^* \in \Sigma_n$.

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Proposition 6.15. *There is a natural isomorphism of functors*

$$D^{\mathrm{m.r.},\nu}_{\overline{\rho}} \to \prod_{\tau \in \Sigma'_n} D^{\mathrm{m.r.}}_{G_\tau} \times \prod_{\tau \in \Sigma_e} D^{\mathrm{m.r.},\nu}_{G_\tau} \times \prod_{\tau \in \Sigma_c} D^{\mathrm{m.r.},\nu}_{G_\tau}.$$

Proof. This expresses the decomposition obtained in this section: given a lift ρ of $\overline{\rho}$, we obtain a decomposition of M as in Definition 6.14. Our analysis with pairings shows that when $\tau \in \Sigma_e$, W_τ is a deformation of $\overline{W}_\tau$ together with its ϵ_{W_τ}-symmetric perfect pairing. Likewise, when $\tau \in \Sigma_c$ we know that $W_{\tau\oplus\tau^*}$ is a deformation of $\overline{W}_{\tau\oplus\tau^*}$ together with its $\epsilon_{W_{\tau\oplus\tau^*}}$-symmetric pairing. When $\tau \in \Sigma_n$, we know $W_\tau \simeq W^\vee_{\tau^*}$. This gives the natural map: to $\rho \in D^{\mathrm{m.r.},\nu}_{\overline{\rho}}(R)$ associate the collection of the W_τ for $\tau \in \Sigma_e \cup \Sigma_c \cup \Sigma'_n$.

Conversely, given W_τ for $\tau \in \Sigma_e \cup \Sigma_c \cup \Sigma'_n$, and defining $W_{\tau^*} := W^\vee_\tau$ for $\tau \in \Sigma'_n$ we can define a lift

$$M := \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}}(\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}}(\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau\oplus\tau^*}}(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*})$$

as in (6.1). (Note that the groups $\Gamma_{L,\tau}$ depend only on the fixed residual representation V.) For $\tau \in \Sigma_e$, the perfect pairing on the lift W_τ gives an isomorphism $\varphi_\tau : W_\tau \simeq W^\vee_\tau$ of $R[T_{L,\tau}]$-modules, which gives a sign-symmetric pairing (with sign $\epsilon_{W_\tau}\epsilon_\tau = \epsilon$) on $\mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}}(\widetilde{\tau} \otimes W_\tau)$ (using equation (6.6)). Likewise, for $\tau \in \Sigma_c$ the sign-symmetric pairing on $W_{\tau\oplus\tau^*}$ gives one on $\mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau\oplus\tau^*}}(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*})$. For $\tau \in \Sigma_n$, we obtain an isomorphism $\varphi_\tau : W_\tau \simeq W^\vee_{\tau^*}$ of $R[T_{L,\tau}]$-modules and hence an ϵ-symmetric perfect pairing on $(\widetilde{\tau} \otimes W_\tau) \oplus (\widetilde{\tau}^\vee \otimes W_{\tau^*})$ which gives one on $\mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}}(\widetilde{\tau} \otimes W_\tau) \oplus \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}}(\widetilde{\tau}^\vee \otimes W_{\tau^*})$. Putting these together, we obtain a sign-symmetric pairing on M; the action of Γ_L preserves it up to scalar, giving a continuous homomorphism $\rho : \Gamma_L \to G(R)$.

Finally, we claim that these constructions are compatible with strict equivalence of lifts, giving an identification of the deformation functors. For $g \in \widehat{G}(R)$, decompose the g-conjugate Γ_L-representation M^g according to (6.4). As g reduces to the identity, it must respect the decomposition into τ-isotypic pieces, so gives automorphisms $g_\tau \in \mathrm{Aut}(W_\tau)$ and $g_\tau \in \mathrm{Aut}(W_{\tau\oplus\tau^*})$. If $\tau \in \Sigma_e$ or Σ_c, as g is compatible with the pairing on M we see g_τ is compatible with the pairing as well. For $\tau \in \Sigma_e$, the g_τ-conjugate $T_{L,\tau}$-representation $W^{g_\tau}_\tau$ is minimally ramified as minimally ramified lifts of $\overline{W}_\tau$ for the group $T_{L,\tau}$ are a deformation condition, and likewise for $\tau \in \Sigma_c$ and $\tau \in \Sigma'_n$.

Conversely, given $g_\tau \in \mathrm{Aut}(W_\tau)$ reducing to the identity (compatible with the pairing on W_τ or $W_{\tau\oplus\tau^*}$ if there is one), using (6.1) and acting on each piece we obtain a lift of the form M^g for $g \in \widehat{G}(R)$. Thus the identification is compatible with strict equivalence. □

Corollary 6.16. *Under the assumptions (A1)–(A4) needed to analyze the tame case, and our standing assumption that all the irreducible representations of Λ_L appearing in V are absolutely irreducible over k, the minimally ramified deformation condition with fixed similitude character is liftable. The dimension of the tangent space is $h^0(\Gamma_L, \mathrm{ad}^0(\overline{\rho}))$.*

Proof. Liftability is a consequence of Proposition 6.15 and the smoothness of the minimally ramified lifting ring for representations of $T_{L,\tau}$ (Proposition 5.6 and Corollary 5.8). By Corollary 5.8, for $\tau \in \Sigma_e$ the dimension of the tangent space of $D^{\mathrm{m.r.},\nu}_{G_\tau}$ is $h^0(T_{L,\tau}, \mathrm{ad}\,\overline{W}_\tau) - 1 = h^0(T_{L,\tau}, \mathrm{ad}^0\,\overline{W}_\tau)$, and for $\tau \in \Sigma_c$ the dimension is $h^0(T_{L,\tau\oplus\tau^*}, \mathrm{ad}\,\overline{W}_{\tau\oplus\tau^*}) - 1 = h^0(T_{L,\tau\oplus\tau^*}, \mathrm{ad}^0\,\overline{W}_{\tau\oplus\tau^*})$. For $\tau \in \Sigma'_n$, by Proposition 5.6 the dimension of the tangent space of $D^{\mathrm{m.r.}}_{G_\tau}$ is

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$h^0(T_{L,\tau}, \operatorname{ad} \overline{W}_\tau)$. Using Proposition 6.15, we see that the dimension of the tangent space of the minimally ramified deformation condition is

$$\sum_{\tau \in \Sigma_e} h^0(T_{L,\tau}, \operatorname{ad}^0 \overline{W}_\tau) + \sum_{\tau \in \Sigma_c} h^0(T_{L,\tau\oplus\tau^*}, \operatorname{ad}^0 \overline{W}_{\tau\oplus\tau^*}) + \sum_{\tau \in \Sigma_n'} h^0(T_{L,\tau}, \operatorname{ad} \overline{W}_\tau).$$

It remains to identify this quantity with $h^0(\Gamma_L, \operatorname{ad}^0(\overline{\rho}))$. Using Lemma 6.13

$$\begin{aligned} H^0(\Gamma_L, \operatorname{End}(V)) &= \operatorname{Hom}_{k[\Gamma_L]}(V, V) \\ &= \bigoplus_{\tau \in \Sigma_e \cup \Sigma_n} \operatorname{Hom}_{T_{L,\tau}}(\overline{W}_\tau, \overline{W}_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \operatorname{Hom}_{T_{L,\tau\oplus\tau^*}}(\overline{W}_{\tau\oplus\tau^*}, \overline{W}_{\tau\oplus\tau^*}) \\ &= \bigoplus_{\tau \in \Sigma_e \cup \Sigma_n} H^0(T_{L,\tau}, \operatorname{End}(\overline{W}_\tau)) \oplus \bigoplus_{\tau \in \Sigma_c} H^0(T_{L,\tau\oplus\tau^*}, \operatorname{End}(\overline{W}_{\tau\oplus\tau^*})). \end{aligned}$$

We are interested in $H^0(\Gamma_L, \operatorname{ad}^0(\overline{\rho}))$: the elements $\psi \in H^0(\Gamma_L, \operatorname{End}(V))$ compatible with the pairing on V in the sense that for $v, v' \in V$

$$\langle \psi v, \psi v' \rangle = \langle v, v' \rangle.$$

The pairing on $V_\tau = \tau \otimes \overline{W}_\tau$ is induced by the pairings on $\overline{W}_\tau$ and τ when $\tau \in \Sigma_e$, and is induced by the pairings on $\overline{W}_{\tau\oplus\tau^*}$ and $\tau \oplus \tau^*$ when $\tau \in \Sigma_c$. When $\tau \in \Sigma_n'$, the pairing on $V_\tau \oplus V_{\tau^*}$ comes from the $\Gamma_{L,\tau}$-isomorphism $V_\tau \simeq V_{\tau^*}^\vee$ which in turn comes from the $T_{L,\tau}$-isomorphism $\overline{W}_\tau \simeq \overline{W}_{\tau^*}^\vee$. So ψ is compatible with the pairing if and only if the following hold.

- When $\tau \in \Sigma_e$, the associated $\psi_\tau \in H^0(T_{L,\tau}, \operatorname{End}(\overline{W}_\tau))$ is compatible with the pairing on $\overline{W}_\tau$.
- When $\tau \in \Sigma_c$, the associated $\psi_\tau \in H^0(T_{L,\tau\oplus\tau^*}, \operatorname{End}(\overline{W}_{\tau\oplus\tau^*}))$ is compatible with the pairing on $\overline{W}_{\tau\oplus\tau^*}$.
- When $\tau \in \Sigma_n'$, the associated ψ_τ and ψ_{τ^*} are identified by duality and the isomorphism $\overline{W}_\tau \simeq \overline{W}_{\tau^*}^\vee$.

In the first two cases, $\operatorname{ad}^0 \overline{W}_\tau$ and $\operatorname{ad}^0 \overline{W}_{\tau\oplus\tau^*}$ are the symplectic or orthogonal Lie algebra, consisting exactly of endomorphisms compatible with the pairing on $\overline{W}_\tau$. In the third, we just choose one of ψ_τ and ψ_{τ^*} without restriction, which determines the other. Thus we see

$$H^0(\Gamma_L, \operatorname{ad}^0(\overline{\rho})) = \bigoplus_{\tau \in \Sigma_e} H^0(T_{L,\tau}, \operatorname{ad}^0 \overline{W}_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} H^0(T_{L,\tau\oplus\tau^*}, \operatorname{ad}^0 \overline{W}_{\tau\oplus\tau^*}) \oplus \bigoplus_{\tau \in \Sigma_n'} H^0(T_{L,\tau}, \operatorname{ad} \overline{W}_\tau).$$

□

The Corollary is a more precise version of Theorem 1.1.

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