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THE RANGE OF GROUP ALGEBRA HOMOMORPHISMS

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ABSTRACT. A characterisation of the range of a homomorphism between two commutative group algebras is presented which implies, among other things, that this range is closed. The work relies mainly on the characterisation of such homomorphisms achieved by P. J. Cohen.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are commutative semisimple Banach algebras with carrier spaces $\Phi_{\mathfrak{A}}$ and $\Phi_{\mathfrak{B}}$ respectively and $\nu\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is an algebra homomorphism. Then $\nu^*(\Phi_{\mathfrak{B}}) \subseteq \Phi_{\mathfrak{A}} \cup \{0\}$, and if $b \in \nu(\mathfrak{A})$, then the Gel'fand transform $\hat{b}$ of b satisfies

(i) If $\varphi \in \Phi_{\mathfrak{B}}$ has $\nu^*(\varphi) = 0$, then $\hat{b}(\varphi) = 0$, and

(ii) If $\varphi_1, \varphi_2 \in \Phi_{\mathfrak{B}}$ have $\nu^*(\varphi_1) = \nu^*(\varphi_2) \in \Phi_{\mathfrak{A}}$, then $\hat{b}(\varphi_1) = \hat{b}(\varphi_2)$.

When each of $\mathfrak{A}$ and $\mathfrak{B}$ is the group algebra of a locally compact Abelian group, say $\mathfrak{A} = L^1(G_1)$ and $\mathfrak{B} = L^1(G_2)$, then we can identify $\Phi_{\mathfrak{A}}$ and $\Phi_{\mathfrak{B}}$ with Γ_1 and Γ_2, the dual groups of G_1 and G_2 respectively. This identifies the Gelf'and and Fourier transforms on each of $L^1(G_1)$ and $L^1(G_2)$. The main result of this paper is that in this situation we have a converse to the above.

THEOREM A. *Suppose G_1 and G_2 are locally compact Abelian groups and ν is an algebra homomorphism $L^1(G_1) \longrightarrow L^1(G_2)$. Then*

$$\nu\big(L^1(G_1)\big) = \{f \in L^1(G_2) : \nu^*(\gamma_1) = 0 \Longrightarrow \hat{f}(\gamma_1) = 0 \\ \text{and } \nu^*(\gamma_1) = \nu^*(\gamma_2) \Longrightarrow \hat{f}(\gamma_1) = \hat{f}(\gamma_2)\ (\gamma_1, \gamma_2 \in \Gamma_2)\}.$$

Since this expresses $\nu\big(L^1(G)\big)$ as an intersection of the kernels of a set of continuous linear functionals on $L^1(G_2)$, we immediately have the following.

COROLLARY B. *The range of a homomorphism between commutative group algebras is closed.*

The first three sections of this paper are devoted to developing ideas leading to a proof of Theorem A. The starting point for this discussion is Cohen's characterization of homomorphisms between commutative group algebras, to be stated in Section 1. The fourth section of this paper concerns the development of results of the above type for classes of Banach algebras other than the commutative group algebras.

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1. **Notation and preliminary results.** Most of the notation and basic results used can be found in the book of Rudin [12], or alternatively the books of Hewitt and Ross [7] and Reiter [11]. We will mainly follow Rudin and refer, where possible, to results therein. In particular, $G, G_1, \ldots$ will always denote locally compact Abelian groups, and their dual groups will be $\Gamma, \Gamma_1, \ldots$ respectively. We will use + for the group product in all locally compact Abelian groups.

If H is a subgroup of G, then we will denote the quotient mapping $G \longrightarrow G/H$ by Q_H and the annihilator of H in Γ by $H^\perp$. If $x \in G$, τ_x will denote the translation function on G given by $\tau_x(y) = x + y$. A nonempty subset E of G is called a *coset in* G when $E - E + E = E$, for then E is a translate of the subgroup $E - E$ of G. The terms *subcoset*, *index* and coset *generated by* a set $X \subseteq G$ then assume their obvious meanings, and the last of these will be denoted by $\mathrm{Aff}(X)$.

If $\mathfrak{X}(\Lambda)$ is a translation-invariant set of functions defined on any locally compact Abelian group Λ, then we define $\mathfrak{X}(E)$ to be the corresponding set of functions on a closed coset E of a locally compact Abelian group Γ; that is, $\mathfrak{X}(E)$ consists of those $f: E \longrightarrow \mathbb{C}$ for which $f \circ \tau_{-\gamma} \in \mathfrak{X}(E - E)$ for some, and hence all, $\gamma \in E$. If $\mathfrak{X}(E - E)$ has additional topological and/or algebraic structure that is translation-invariant in nature, this can be carried over to $\mathfrak{X}(E)$, so that $\mathfrak{X}(E)$ is isomorphic to $\mathfrak{X}(E - E)$. In particular we have $A(E)$, the *Fourier algebra* on E, whose carrier space is E, and whose multiplier algebra is $B(E)$, the *Fourier-Stieltjes algebra* on E. The *coset ring* of E, denoted by $\mathcal{R}(E)$, can likewise be obtained by viewing $\mathcal{R}(E - E)$ as a set of characteristic functions. Clearly $\mathcal{R}(E)$ is the boolean ring generated by the (relatively) open subcosets of E, and owing to the Idempotent Measure Theorem of P. J. Cohen, [2, Theorem 1] or [12, Theorem 3.1.3], we have that the idempotents in $B(E)$ consist of the characteristic functions of elements of $\mathcal{R}(E)$.

A more significant consequence of the Idempotent Measure Theorem is the characterisation of group algebra homomorphisms, again due to Cohen. A brief statement of this is that if $\nu: L^1(G_1) \longrightarrow L^1(G_2)$ is a homomorphism between group algebras, then the part of the graph of $\nu^*: L^\infty(G_2) \longrightarrow L^\infty(G_1)$ that lies within $\Gamma_2 \times \Gamma_1$ is actually an element of $\mathcal{R}\big((\Gamma_2 \times \Gamma_1)_{\mathrm{d}}\big)$. (Here Γ_{d} denotes the group Γ with its discrete topology imposed.)

To be able to use the original, more tractible statement of this theorem, we recall some of the relevant definitions. A map between cosets $\psi: E_1 \longrightarrow E_2$ is called *affine* if $\psi(x_1 + x_2 - x_3) = \psi(x_1) + \psi(x_2) - \psi(x_3)$ for any $x_1, x_2, x_3 \in E_1$. A map ψ from $X \subseteq \Gamma_2$ into Γ_1 is called *piecewise affine* if there exists disjoint $S_1, \ldots, S_n \in \mathcal{R}(\Gamma_2)$ such that $X = \bigcup_1^n S_k$ and each $\psi|_{S_k}$ has a continuous affine extension—that is, each is the restriction of a continuous affine map whose domain is a coset containing S_k. Recall from [1] that if X and Y are locally compact spaces, then $f: X \longrightarrow Y$ is called *proper* if $f^{-1}(C) \subseteq X$ is compact whenever $C \subseteq Y$ is compact. With these definitions, we have the following theorem, which also contains the converse to the above "brief statement". It originally appeared in Cohen's paper [3, Theorem 1], and can also be found in Chapter 4 of Rudin's book [12].

THEOREM 1.1. *If $\nu: L^1(G_1) \longrightarrow M(G_2)$ is a nonzero algebra homomorphism, then there is a set $Y \in \mathcal{R}(\Gamma_2)$ and a piecewise affine map $\alpha: Y \longrightarrow \Gamma_1$ such that for each*

$f \in L^1(G_1)$, $\widehat{\nu(f)} = \hat{f} \circ \alpha$ on Y and $\widehat{\nu(f)}$ is zero off Y. Conversely, any such piecewise affine map determines a homomorphism $\nu\colon L^1(G_1) \longrightarrow M(G_2)$, and $\nu\big(L^1(G_1)\big) \subseteq L^1(G_2)$ if and only if α is proper.

As in [12], we use $\hat{\nu}$ for the corresponding homomorphism $A(\Gamma_1) \longrightarrow B(\Gamma_2)$, and abbreviate the relation stated in Theorem 1.1 to $\hat{\nu}(f) = f \circ \alpha$. The maps ν^* and α of Theorems A and 1.1 are related, in that for $\gamma \in \Gamma_2$,

$$\begin{aligned} \gamma \notin Y &\Longrightarrow \hat{f}\big(\nu^*(\gamma)\big) = \widehat{\nu(f)}(\gamma) = 0 \quad \big(f \in L^1(G_2)\big) \\ &\Longrightarrow \nu^*(\gamma) = 0 \quad \text{and} \\ \gamma \in Y &\Longrightarrow \hat{f}\big(\nu^*(\gamma)\big) = \widehat{\nu(f)}(\gamma) = \hat{f}\big(\alpha(\gamma)\big) \quad \big(f \in L^1(G_2)\big) \\ &\Longrightarrow \alpha(\gamma) = \nu^*(\gamma). \end{aligned}$$

Hence $Y = \Big\{\gamma \in \Gamma_2 : \hat{f}(\gamma) \neq 0, \Big(f \in \nu\big(L^1(G_1)\big)\Big)\Big\}$ and $\alpha = \nu^*|_Y$. Due to this Banach algebraic characterization, we will further develop the case for more general commutative semisimple Banach algebras before returning to the group algebra case.

If $\mathfrak{A}$ is a commutative semisimple Banach algebra, define the *hull* of a set $X \subseteq \mathfrak{A}$ to be $Z_{\mathfrak{A}}(X) = \{\varphi \in \Phi_{\mathfrak{A}} : \varphi(x) = 0, (x \in X)\} = X^{\perp} \cap \Phi_{\mathfrak{A}}$, and the *kernel* of a set $E \subseteq \Phi_{\mathfrak{A}}$ to be $\mathcal{I}_{\mathfrak{A}}(E) = \{a \in A : \varphi(a) = 0, (\varphi \in E)\} = \bigcap_{\varphi \in E} \ker \varphi$. As with most notation of this type, the subscript will usually be omitted. It is clear that $Z(X)$ is always a closed subset of $\Phi_{\mathfrak{A}}$ and $\mathcal{I}(E)$ is always a closed ideal of $\mathfrak{A}$. Then $\mathfrak{A}$ is called *regular* if $E = Z\big(\mathcal{I}(E)\big)$ for any closed set $E \subseteq \Phi_{\mathfrak{A}}$. It it well known that that $L^1(G)$ is regular.

PROPOSITION 1.2. *Let $\nu\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism between commutative semisimple Banach algebras. Then $Y = \{\varphi \in \Phi_{\mathfrak{B}} : \varphi \circ \nu \neq 0\} = \Phi_{\mathfrak{B}} \setminus Z\big(\nu(\mathfrak{A})\big)$ is open in $\Phi_{\mathfrak{B}}$, and closed if $\mathfrak{A}$ is unital. Also, $\alpha = \nu^*|_Y$ is a continuous, proper, closed map $Y \longrightarrow \Phi_{\mathfrak{A}}$ with $Z(\ker \nu) = Z\Big(\mathcal{I}\big(\alpha(Y)\big)\Big)$, the closure of $\alpha(Y)$ in the hull-kernel topology on $\Phi_{\mathfrak{A}}$, so that if $\mathfrak{A}$ is regular, then $Z(\ker \nu) = \alpha(Y)$.*

PROOF. Clearly Y is open and $\alpha\colon Y \longrightarrow \Phi_{\mathfrak{A}}$ is continuous. If $\mathfrak{A}$ has unit e, then $\chi_Y = \widehat{\nu(e)} \in \hat{\mathfrak{B}} \subseteq C_0(\Phi_{\mathfrak{B}})$, and so Y is clopen. To show α to be proper, suppose $C \subseteq \Phi_{\mathfrak{A}}$ is compact. For $\varphi \in C$, take $a_\varphi \in \mathfrak{A}$ with $\varphi(a_\varphi) > 1$ and set $K_\varphi = \{\varphi_1 \in \Phi_{\mathfrak{A}} : \hat{a}_\varphi(\varphi_1) \geq 1\}$, a compact neighbourhood of φ. Similarly $\alpha^{-1}(K_\varphi) = \{\varphi_2 \in \Phi_{\mathfrak{B}} : \widehat{\nu(a)}(\varphi_2) \geq 1\}$ is also compact. Take a finite set $\varphi_1, \ldots, \varphi_n$ such that $K_{\varphi_1}, \ldots, K_{\varphi_n}$ cover C, and then $\alpha^{-1}(C) \subseteq \alpha^{-1}(K_{\varphi_1}) \cup \cdots \cup \alpha^{-1}(K_{\varphi_n})$, which is compact, so $\alpha^{-1}(C)$ is compact.

Then from [1, Section 1.10.10, Proposition 15 and Section 1.10.1, Proposition 7], or by a straightforward calculation, we have that α is closed. The remaining statements are clear from definitions. ∎

REMARK. In the case that $\nu\colon L^1(G_1) \longrightarrow M(G_2)$ and $\nu\big(L^1(G_1)\big) \not\subseteq L^1(G_2)$, there is no conflict between Theorem 1.1 and Proposition 1.2, since the first concerns $Y \subseteq \Gamma_2$ and $\alpha\colon Y \longrightarrow \Gamma_1$, which will not be proper, whereas the second concerns $Y \subseteq \Phi_{M(G_2)}$ and $\alpha\colon Y \longrightarrow \Gamma_1$, which will be proper.

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Suppose $\mathfrak{B}$ is a commutative semisimple Banach algebra, with U an open subset of $\Phi_{\mathfrak{B}}$ and ψ a map from U into a set X. Define

$$\kappa_{\mathfrak{B}}(\psi) = \{b \in \mathfrak{B} : \hat{b} = 0 \text{ off } U \text{ and } \hat{b}(\varphi_1) = \hat{b}(\varphi_2) \text{ whenever } \psi(\varphi_1) = \psi(\varphi_2)\}.$$

This will be written $\kappa(\psi)$ when no confusion is likely, and the notation $\hat{\kappa}(\psi)$ will denote $\{\hat{b} : b \in \kappa(\alpha)\}$. We can view $\kappa(\psi)$ as the closed subalgebra of those $b \in I(\Phi_{\mathfrak{B}} \setminus U)$ for which $\hat{b} \circ \psi^{-1}$ is a well-defined function on $\psi(U)$.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are commutative semisimple Banach algebras, $\nu: \mathfrak{A} \longrightarrow \mathfrak{B}$ is an algebra homomorphism and Y and α are obtained from ν as in Proposition 1.2. We have seen from the opening discussion of this paper that $\nu(\mathfrak{A}) \subseteq \kappa(\alpha)$. Thus we can consider $\kappa(\alpha)$ as a convenient upper bound on $\overline{\nu(\mathfrak{A})}$. Theorem A asserts that if $\mathfrak{A}$ and $\mathfrak{B}$ are both group algebras, then $\nu(\mathfrak{A}) = \kappa(\alpha)$.

As a model for the proof of Theorem A, it is instructive to consider the case where $\mathfrak{A} = C_0(X)$, for some locally compact Hausdorff topological space X, and $\mathfrak{B}$ is semisimple. Suppose $b \in \kappa(\alpha)$, then since α is proper, $\hat{b} \circ \alpha^{-1} \in C_0\big(\alpha(Y)\big)$. Moreover, $\alpha(Y)$ is closed in X, so it is a simple matter to apply the Tietze Extension Theorem on the one-point compactification of X to give an extension $g \in C_0(X)$ of $\hat{b} \circ \alpha^{-1}$. Then $\hat{b} = g \circ \alpha = \widehat{\nu(g)}$, and since $\mathfrak{B}$ is semisimple, $b = \nu(g)$. Hence $\kappa(\alpha) = \nu(\mathfrak{A})$.

The use of "extension" results analogous to the Tietze Extension Theorem will be used to prove Theorem A. In fact, it is clear that the following theorem is equivalent to Theorem A.

THEOREM 1.3. *If $Y \in R(\Gamma_2)$ and $\alpha: Y \longrightarrow \Gamma_1$ is a proper piecewise affine map, then for any $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1}: \alpha(Y) \longrightarrow C$ has an extension in $A(\Gamma_1)$.*

We will prove this theorem in three stages. Firstly when Y is an open coset and α is affine, then for certain cases in which α has an affine extension, and finally for the general case as stated above. Each case will be proven by obtaining properties of $\alpha(Y)$, then characterizing the behaviour of $\hat{f} \circ \alpha^{-1}$, and finally finding an extension. Throughout, the notation ν, Y and α will be as in Theorem 1.1. Here is the case where α is affine.

LEMMA 1.4. *If Y is an open coset in Γ_2 and $\alpha: Y \longrightarrow \Gamma_1$ is affine, proper and continuous, then for any $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$.*

PROOF. Since Y is closed and α is closed and affine, $\alpha(Y)$ is a closed coset. Take $\gamma_0 \in Y$, then $\Lambda = \alpha^{-1}\{\alpha(\gamma_0)\} - \gamma_0$ is a compact subgroup of $Y - \gamma_0$ and $Y_1 = Q_\Lambda(Y)$ is an open coset in Γ_2/Λ. Clearly $\hat{f} \in A(\Gamma_2)$ is constant on cosets of Λ, so by [12, Theorem 2.7.1] $f = 0$ a.e. off the open subgroup $H = \Lambda^\perp$ of G_2. Put $f_1 = f|_H \in L^1(H)$, then for all $\gamma + \Lambda \in \Gamma_2/\Lambda = H^\perp$, we have by [12, Equation 2.7.1(2)] that $\hat{f}_1(\gamma + \Lambda) = \hat{f}(\gamma)$. Thus $\hat{f}_1 \circ Q_\Lambda = \hat{f}$ and $\hat{f}_1$ is zero off Y_1.

Now, $\alpha \circ Q_\Lambda^{-1}: Y_1 \longrightarrow \alpha(Y)$ is a well defined affine bijection. Furthermore $\alpha \circ Q_\Lambda^{-1}$ is continuous and proper, and hence is an affine homeomorphism onto $\alpha(Y)$. Thus $g \longmapsto g \circ \alpha \circ Q_\Lambda^{-1}$ is an isomorphism $A\big(\alpha(Y)\big) \longrightarrow A(Y_1)$, so $\hat{f} \circ \alpha^{-1} = \hat{f}_1 \circ \big(\alpha \circ Q_\Lambda^{-1}\big)^{-1} \in A\big(\alpha(Y)\big)$. But $A\big(\alpha(Y)\big) = A(\Gamma_1)|_{\alpha(Y)}$, by [12, Theorem 2.7.4], so $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$. ∎

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It should be noted that the action of $\nu\colon L^1(G_1) \longrightarrow L^1(G_2)$ is essentially given by the homomorphism $T_{H_1}\colon L^1(G_1) \longrightarrow L^1(G_1/H_1)$, where $H_1 = \big[\alpha(Y) - \alpha(Y)\big]^\perp$ and T_{H_1} is the transformation that averages over cosets of H_1 with respect to the Haar measure on H_1—see [11, Sections 3.4–3.6]. The extension result in Lemma 1.4 relies on T_{H_1} being an epimorphism with $\hat{T}_{H_1}(\hat{f}) = \hat{f}|_{H_1^\perp}$.

COROLLARY 1.4.1. *If Γ_2 is connected, and $\nu\colon L^1(G_1) \longrightarrow L^1(G_2)$ is a nonzero homomorphism, then for some open subgroup H of G_2,*

$$\nu\big(L^1(G_1)\big) = \kappa(\alpha) = \{f \in L^1(G_2) : f = 0 \text{ a.e. off } H\} \cong L^1(H).$$

PROOF. Since $\mathcal{R}(\Gamma_2) = \{\emptyset, \Gamma_2\}$ we have that $Y = \Gamma_2$ and α is affine, so by the lemma, $\nu\big(L^1(G_1)\big) = \kappa(\alpha)$. Then with Λ and H as above, the conditions $f \in \kappa(\alpha)$ and $f = 0$ a.e. off H are each equivalent to $\hat{f}$ being constant on cosets of Λ. ∎

In the case where G_2 is also connected, this reduces to $\nu\big(L^1(G_1)\big) = L^1(G_2)$, and since the Euclidean groups $\mathbb{R}^n$ are the only connected locally compact Abelian groups with connected dual, we have the following.

COROLLARY 1.4.2. *If $G_2 = \mathbb{R}^n$ for some $n > 0$, then ν is onto.* ∎

2. **The coset ring and piecewise affine maps.** As a starting point for considering general piecewise affine maps, we have from the discussion in [12, Section 4.3.4] that any set in the coset ring of a locally compact Abelian group Γ is a finite disjoint union of sets in

$$\mathcal{R}_0(\Gamma) = \Big\{E_0 \setminus \Big(\bigcup_{1\le k\le m} E_k\Big) : E_0 \subseteq \Gamma \text{ is an open coset and } E_1, \ldots, E_m \text{ are open subcosets of infinite index in } E_0\Big\}.$$

Clearly, by [12, Lemma 4.3.3], $\emptyset \notin \mathcal{R}_0(\Gamma)$. If $S = E_0 \setminus \big(\bigcup_1^m E_k\big) \in \mathcal{R}_0(\Gamma)$, then for all $\gamma \in E_0 - E_0$, $S \cap (S+\gamma) = E_0 \setminus \big(\bigcup_1^m E_k \cup \bigcup_1^m (E_k+\gamma)\big) \in \mathcal{R}_0(\Gamma)$. So if E is any coset containing S, then $E \cap (E+\gamma) \neq \emptyset$, so $\gamma \in E - E$. Thus $E_0 - E_0 \subseteq E - E$, and so $E_0 \subseteq E$. Hence $E_0 = \mathrm{Aff}(S)$, the coset generated by S.

By applying this result plus the decomposition of [12, Section 4.3.4] to the original definition of "piecewise affine" we obtain the following characterisation.

LEMMA 2.1. *If $X \in \mathcal{R}(\Gamma_2)$ then $\psi\colon X \longrightarrow \Gamma_1$ is piecewise affine if and only if there are disjoint $S_1, \ldots, S_n \in \mathcal{R}_0(\Gamma_2)$ such that $X = S_1 \cup \cdots \cup S_n$ and for each k, $\psi|_{S_k}$ has a continuous affine extension $\psi_k\colon \mathrm{Aff}(S_k) \longrightarrow \Gamma_1$.* ∎

The following pair of lemmas will be used to obtain information about the affine maps $\psi_1, \ldots, \psi_n$ from ψ. Each lemma allows us to "smudge" a set $S \in \mathcal{R}_0(\Gamma)$ to cover a slightly larger set. In the first case the slightly larger set is $\mathrm{Aff}(S)$, and in the second it is $S + \Lambda$, for Λ a compact subgroup of Γ.

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LEMMA 2.2. *Let* $S \in R_0(\Gamma)$ *and put* $E_0 = \mathrm{Aff}(S)$, *then there is a finite subset* F *of* $E_0 - E_0$ *such that* $E_0 = S + F$.

PROOF. We proceed by induction on m to show that if $S = E_0 \setminus \left(\bigcup_1^m E_k\right) \in R_0(\Gamma)$ then there is a finite set $F \subseteq E_0 - E_0$ with $S + F = E_0$. If $m = 0$, then $S = E_0$ so that $F = \{e\}$ suffices. Now let $m > 0$ and put $S' = E_0 \setminus \left(\bigcup_1^{m-1} E_k\right) \in R_0(\Gamma)$. By hypothesis, there is a finite set $F' \subseteq E_0 - E_0$ with $S' + F' = E_0$. Since E_m is of infinite index in E_0, there exists $\gamma \in (E_0 - E_0) \setminus (E_m - E_m + F' - F')$, so that $(E_m + F') \cap (E_m + F' + \gamma) = \emptyset$. Put $F = F' \cup (F' + \gamma)$.

Since $S' \subseteq S \cup E_m$, we have $E_0 = S' + F' \subseteq (S + F') \cup (E_m + F') \subseteq E_0$, so that $(S + F') \cup (E_m + F') = E_0$, and similarly $(S + F' + \gamma) \cup (E_m + F' + \gamma) = E_0$. But $(E_m + F') \cap (E_m + F' + \gamma) = \emptyset$, due to the choice of γ, so that $S + F = (S + F') \cup (S + F' + \gamma) = E_0$. ∎

LEMMA 2.3. *If* $S \in R(\Gamma)$ *and* Λ *is a compact subgroup of* Γ, *then there is a finite subset* F *of* Λ *such that* $S + \Lambda = S + F$.

PROOF. Let $E_1, \ldots, E_n$ be open cosets such that S is in the Boolean ring generated by $\{E_1, \ldots, E_n\}$. Let $\Xi = \bigcap_1^n \left(E_k - E_k\right)$, an open subgroup of Γ with $S + \Xi = S$. Since Λ is compact and $\Lambda \cap \Xi$ is open in Λ, $\Lambda \cap \Xi$ is of finite index in Λ. Let $F \subseteq \Lambda$ be a finite set with $\left(\Lambda \cap \Xi\right) + F = \Lambda$, then since $S \subseteq S + \left(\Lambda \cap \Xi\right) \subseteq S + \Xi = S$, we have $S + F = S + \left(\Lambda \cap \Xi\right) + F = S + \Lambda$. ∎

PROPOSITION 2.4. *If* $S \in R_0(\Gamma_2)$ *and* $\psi\colon \mathrm{Aff}(S) \longrightarrow \Gamma_1$ *is affine such that* $\psi|_S$ *is proper, then* ψ *is proper.*

PROOF. Assume first that $\mathrm{Aff}(S)$ is a subgroup and ψ is a homomorphism, then let $F \subseteq \mathrm{Aff}(S)$ be such that $S + F = \mathrm{Aff}(S)$, as in Lemma 2.2. Note that $(\psi|_S)^{-1}(\cdot) = S \cap \psi^{-1}(\cdot)$, so that for any compact $C \subseteq \Gamma_1$,

$$\psi^{-1}(C) = \bigcup_{\gamma \in F} \psi^{-1}(C) \cap (\gamma + S) = \bigcup_{\gamma \in F} \left(\psi^{-1}\left(C - \psi(\gamma)\right) \cap S\right) + \gamma,$$

which is compact. Hence ψ is proper. The general case follows by translation. ∎

COROLLARY 2.4.1. *Suppose* $S \in R_0(\Gamma_2)$ *and* $\psi\colon S \longrightarrow \Gamma_1$ *is proper with an affine extension* ψ'. *Then* $\psi'\left(\mathrm{Aff}(S)\right)$ *is a closed coset in* Γ_1 *and* $\psi(S) \in R\left(\psi'\left(\mathrm{Aff}(S)\right)\right)$.

PROOF. Without loss, we can assume that ψ' has domain $E_0 = \mathrm{Aff}(S)$, so that by Lemma 2.4, ψ' is proper. Hence $E = \psi'(E_0)$ is a closed coset in Γ_1. Now, as in Lemma 1.4 there is a compact subgroup Λ of $E_0 - E_0$ such that $\psi' \circ Q_\Lambda^{-1}\colon E_0/\Lambda \longrightarrow E$ is an affine homeomorphism. Then by Lemma 2.3, there is a finite set $F \subseteq \Lambda$ with $S + \Lambda = S + F$, giving $S + \Lambda \in R(E_0)$. Hence $Q_\Lambda(S) \in R(E_0/\Lambda)$ and $\psi(S) = \psi'(S) \in R(E)$. ∎

Note that if E is a closed coset in Γ, then any $X \in R(E)$ is a closed set in $R(\Gamma_d)$. Define $R_d(\Gamma) = R(\Gamma_d)$ and $R_c(\Gamma) = \{x \subseteq \Gamma : X \text{ is closed and } X \in R_d(\Gamma)\}$. By combining Lemma 2.1 with the preceding corollary, we obtain the following.

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COROLLARY 2.4.2. *If* $X \in \mathcal{R}(\Gamma_2)$ *and* $\psi\colon X \longrightarrow \Gamma_1$ *is proper and piecewise affine then* $\psi(X) \in \mathcal{R}_c(\Gamma_1)$. ■

A similar result holds for piecewise affine maps that are not necessarily proper. It can be shown that the range of such a map $X \longrightarrow \Gamma_1$ is an element of $\mathcal{R}_d(\Gamma_1)$.

3. **Proof of the main theorem.** In this section we apply the analysis of the coset ring and piecewise affine maps presented in Section 2 to complete the proof of Theorem 1.3. We begin with the second stage of Theorem 1.3.

LEMMA 3.1. *Suppose* $Y \in \mathcal{R}_0(\Gamma)$ *and* $\alpha\colon Y \longrightarrow \Gamma_1$ *has an affine extension* $\alpha_1\colon \mathrm{Aff}(Y) \longrightarrow \Gamma_1$. *Then for any* $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1}$ *has an extension in* $A(\Gamma_1)$.

PROOF. Put $E = \mathrm{Aff}(Y)$, then by Lemma 2.4, $\alpha_1\colon E \longrightarrow \Gamma_1$ is proper, so as in Lemma 1.4, there is a compact subgroup Λ of $E - E$ such that $\alpha_1 \circ Q_\Lambda^{-1}\colon E/\Lambda \longrightarrow \alpha(E)$ is an affine homeomorphism. Define $\tilde{f}\colon \Gamma_2 \longrightarrow \mathbb{C}$ to be the unique function that agrees with $\hat{f}$ on Y, is constant on cosets of Λ, and is zero off $Y + \Lambda$. That is,

$$\tilde{f}(\gamma) = \begin{cases} f(\gamma - \lambda) & \text{when } \lambda \in \Lambda \text{ is such that } \gamma - \lambda \in Y; \\ 0 & \text{if } \gamma \notin Y + \Lambda. \end{cases}$$

To show $\tilde{f} \in A(\Gamma_2)$, we require sets $S_1, \ldots, S_n \in \mathcal{R}(\Gamma_2)$ such that $\bigcup_1^n S_j = Y + \Lambda$ and each $\tilde{f} \cdot \chi_{S_k} \in A(\Gamma_2)$, for then $\tilde{f} = \tilde{f} \cdot \chi_{S_1 \cup \cdots \cup S_n} = \sum_{k=1}^n \tilde{f} \cdot \chi_{S_k} \cdot \prod_{j=1}^{k-1} \chi_{\Gamma \setminus S_j} \in A(\Gamma)$.

By Lemma 2.3, there is a finite set $F \subseteq \Lambda$ such that $Y + \Lambda = \bigcup_{\lambda \in F}(Y + \lambda)$. Then for each $\lambda \in F$, $\tilde{f} \cdot \chi_{Y+\lambda} = f \circ \tau_{-\lambda} \in A(\Gamma_2)$, so the sets $\{Y + \lambda\}_{\lambda \in F} \subseteq \mathcal{R}(\Gamma_2)$ are as required, giving $\tilde{f} \in A(\Gamma_2)$. Furthermore, the defined properties of $\tilde{f}$ mean that $\tilde{f} \in \hat{\kappa}(\alpha_1)$. Hence by Lemma 1.4, $\tilde{f} \circ \alpha_1^{-1}$ has an extension in $A(\Gamma_1)$, and since $\tilde{f} \circ \alpha_1^{-1}$ is itself an extension of $\hat{f} \circ \alpha^{-1}$, we are done. ■

So we have proven Theorem A for the case where the decomposition of α given in Lemma 2.1 yields just one piece. In the general case, we will apply Lemma 3.1 to each piece of a "multi-piece" piecewise affine map, and then combine them. Lemma 3.3 below is crucial in that it allows us to carry out this last step.

If $\mathfrak{A}$ is a Banach algebra, recall that a *bounded approximate right identity* for $\mathfrak{A}$ is a bounded net $\{e_n\}_{n\in\Delta} \subseteq \mathfrak{A}$ such that for each $a \in \mathfrak{A}$, $ae_n \longrightarrow a$.

LEMMA 3.2. *If* $\mathcal{I}$ *and* $\mathcal{J}$ *are closed ideals of a Banach algebra* $\mathfrak{A}$ *and* $\mathcal{I}$ *has a bounded approximate right identity* $\{e_n\}_{n\in\Delta}$, *then* $\mathcal{I} + \mathcal{J}$ *is a closed ideal of* $\mathfrak{A}$.

PROOF. Clearly $\mathcal{I} + \mathcal{J}$ is an ideal of $\mathfrak{A}$. Let π be the natural isomorphism $\mathcal{J}/(\mathcal{I} \cap \mathcal{J}) \longrightarrow (\mathcal{I} + \mathcal{J})/\mathcal{I}$. Clearly π is continuous. Let $y \in \mathcal{J}$ be such that $\|y + \mathcal{I}\| < 1$, so that there exists $x \in \mathcal{I}$ with $\|y - x\| < 1$. Then

$$\begin{aligned} \|\pi^{-1}(y + \mathcal{I})\| &= \inf\{\|y + z\| : z \in \mathcal{I} \cap \mathcal{J}\} \\ &\le \inf_{n\in\Delta} \|y - ye_n\| \\ &\le \inf_{n\in\Delta}(\|y - x\| + \|x - xe_n\| + \|e_n\|\,\|x - y\|) \\ &< 1 + 0 + \sup_{n\in\Delta} \|e_n\| \end{aligned}$$

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and so π^{-1} is continuous. Hence $(I + J)/I$ is complete, and it follows that $I + J$ is a closed ideal of $\mathfrak{A}$. ∎

LEMMA 3.3. *If* $X, Y \in R_c(\Gamma)$ *then* $I(X) + I(Y) = I(X \cap Y)$.

PROOF. By [10, Theorem 13], $I(X)$ has a bounded approximate identity, so that by Lemma 3.2, $I(X)+I(Y)$ is a closed ideal of $A(\Gamma)$. Furthermore, we have $Z\big(I(X)+I(Y)\big) = X\cap Y \in R_c(\Gamma)$. By [5, Theorem 3.9], $X\cap Y$ is a set of spectral synthesis—that is, $I(X\cap Y)$ is the only ideal whose hull is $X \cap Y$—giving $I(X) + I(Y) = I(X \cap Y)$. ∎

COROLLARY 3.3.1. *If* $X, Y \in R_c(\Gamma)$ *and* $g_1, g_2 \in A(\Gamma)$ *are such that* $g_1|_{X\cap Y} = g_2|_{X\cap Y}$, *then there exists* $g \in A(\Gamma)$ *with* $g|_X = g_1|_X$ *and* $g|_Y = g_2|_Y$.

PROOF. Since $g_1 - g_2 \in I(X \cap Y)$, there exist $f_1 \in I(X)$, $f_2 \in I(Y)$ with $f_1 - f_2 = g_1 - g_2$. Then $g = g_1 - f_1 = g_2 - f_2$ is as required. ∎

This corollary enables us to find a common extension to the pieces of $\hat{f} \circ \alpha^{-1}$, and thus complete the proof of Theorem 1.3, and consequently that of Theorem A.

PROOF OF THEOREM 1.3. By Lemma 2.1 there are disjoint $S_1, \ldots, S_n \in R_0(\Gamma_2)$ such that $\bigcup_1^n S_k = Y$ and each $\alpha|_{S_k}$ is proper with an affine extension $\alpha_k\colon \mathrm{Aff}(S_k) \longrightarrow \Gamma_1$. For each $1 \leq k \leq n$, $\hat{f} \cdot \chi_{S_k} \in \hat{\kappa}(\alpha|_{S_k})$, so by Lemma 3.1, there exists $g_k \in A(\Gamma_1)$ such that $g_k|_{\alpha(S_k)} = \hat{f} \circ (\alpha|_{S_k})^{-1} = \hat{f} \circ \alpha^{-1}|_{\alpha(S_k)}$.

By repeatedly applying Corollary 3.3.1, we obtain $g \in A(\Gamma)$ such that for each $1 \leq k \leq n$, $g|_{\alpha(S_k)} = g_k|_{\alpha(S_k)}$. Then $g|_{\alpha(Y)} = f \circ \alpha^{-1}$, as required. ∎

We can now apply Theorem A to obtain a further property of homomorphisms between commutative group algebras.

THEOREM 3.4. *If* $\nu\colon L^1(G_1) \longrightarrow L^1(G_2)$ *is an algebra homomorphism and* J *is a closed ideal in* $L^1(G_1)$, *then* $\nu(J)$ *is a closed subalgebra of* $L^1(G_2)$.

PROOF. Let $I = \ker \nu = I\big(\alpha(Y)\big)$. By Corollary 2.4.2 and [10, Theorem 13] I has a bounded approximate identity. So by Lemma 3.2 $I + J$ is a closed ideal of $L^1(G_1)$. By Corollary B and the Open Mapping Theorem, $\nu\big(L^1(G_1) \setminus (I + J)\big)$ is open in $\kappa(\alpha)$. But $\nu\big(L^1(G_1) \setminus (I + J)\big) = \kappa(\alpha) \setminus \nu(J)$, so that $\nu(J)$ is closed in $\kappa(\alpha)$, and hence in $L^1(G_2)$. Finally, $\nu(J)$ is an ideal of $\kappa(\alpha)$, and hence a subalgebra of $L^1(G_2)$. ∎

4. **Final remarks.** Given that Theorem A and Corollary B present Banach-algebraic properties of commutative group algebras, it is natural to consider whether these results admit generalisations to other classes of Banach algebras. The possibility that either the domain or codomain algebras could be allowed to be any commutative semisimple Banach algebra is too general. In the first case, the group algebra $L^1(G)$ on any infinite locally compact Abelian group G has a proper dense subalgebra $\mathfrak{A}$ with a Banach algebra norm such that $\mathfrak{A} \hookrightarrow L^1(G)$ is continuous: if G is non-discrete, let $\mathfrak{A}$ be the Segal algebra $L^1(G) \cap L^2(G)$ of [11, Section 6.2]; and if G is discrete, let $\mathfrak{A}$ be the Beurling algebra $\ell^1(G, \omega)$ of [11, Section 6.3], where $\omega\colon G \longrightarrow [1, \infty]$ is an unbounded submultiplicative

weight on G. In the second case, such a group algebra can be continuously injected into $C_0(\Gamma)$ via the Fourier transform, and this monomorphism has proper dense range.

The possibility of generalising to algebras on non-abelian locally compact groups seems more promising. Here there are two obvious cases to consider, the first being the group algebras $L^1(G)$ on non-abelian groups, and the second being the Fourier algebras $A(G)$ on non-abelian groups. However, in each case we only have partial results analogous to Cohen's theorem on homomorphisms between group algebras.

In the case of homomorphisms between noncommutative group algebras, the existing classification results only deal with homomorphisms with norm bounded by some small constant—see for instance [6] or [9]. As in the case dealt with in Lemma 1.4 the homomorphisms of norm 1 are essentially given by the map $T_H: L^1(G) \longrightarrow L^1(G/H)$ of [11, Sections 3.4–3.6], and are easily shown to have closed range.

The homomorphisms $\nu: A(G_1) \longrightarrow A(G_2)$ between the Fourier algebras of two locally compact groups have been characterised in [8] in the case where G_1 is a finite extension of an abelian group. Here the characterisation is precisely that obtained by Cohen for the abelian case, given an appropriate definition of "piecewise affine". For such G_1, the ideals with bounded approximate identity are those with hull in $\mathcal{R}_c(G_1)$, and so the proof in Sections 2 and 3 of the present paper can be carried through with only minor changes. For more general G_1, it seems possible that the result will still hold. Certainly the example given in [8] of a homomorphism where α is not piecewise affine does have closed range of the type described in Theorem A—it is an isomorphism. So it seems that an alternative method of proof must be sought. A second potential obstacle is that due to the results in [4], in that if we are to have bounded approximate identities in any but the most trivial ideals of a Fourier algebra, we require the group to be amenable.

Finally, we should note that in situations not too far removed from Theorem A, we do not get the result we might expect. Recall from [12, Theorem 4.6.2] that for G_1 and G_2 locally compact abelian groups, any homomorphism $\nu: L^1(G_1) \longrightarrow L^1(G_2)$ has a natural extension $\tilde{\nu}: M(G_1) \longrightarrow M(G_2)$ given by $\tilde{\nu}(\mu)\hat{} = \hat{\mu} \circ \alpha$ on Y and $\tilde{\nu}(\mu)\hat{} = 0$ off Y. Put

$$\tilde{\kappa}(\alpha) = \{\mu \in M(G_2) : \hat{\mu}|_{\Gamma_2 \setminus Y} = 0 \text{ and } \alpha(\gamma_1) = \alpha(\gamma_2) \Longrightarrow \hat{\mu}(\gamma_1) = \hat{\mu}(\gamma_2)\},$$

a closed subalgebra of $M(G_2)$ containing $\tilde{\nu}\big(M(G_1)\big)$. Then we do not always have $\tilde{\nu}\big(M(G_1)\big) = \tilde{\kappa}(\alpha)$. For instance, let $\alpha: \mathbb{Z} \longrightarrow (\mathbb{Z}+1/2) \cup (\sqrt{2}\mathbb{Z}) \subseteq \mathbb{R}$ be a piecewise affine bijection and let $\nu: L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{T})$ and $\tilde{\nu}: M(\mathbb{R}) \longrightarrow M(\mathbb{T})$ be the homomorphisms determined by α. By Theorem A, ν is an epimorphism. However, $\chi_{\alpha^{-1}(\mathbb{Z}+1/2)} \in B(\mathbb{Z}) = \tilde{\kappa}(\alpha)\hat{}$ cannot be expressed as $F \circ \alpha$ for any $F \in B(\mathbb{R})$, due to the uniform continuity of Fourier-Stieltjes transforms, and so $\tilde{\nu}\big(M(\mathbb{R})\big) \neq \tilde{\kappa}(\alpha)$.

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