# CLIFFORD SEMIGROUPS OF LEFT QUOTIENTS

*by* VICTORIA GOULD†

(Received 26 June, 1985)

**1. Introduction.** Several definitions of a semigroup of quotients have been proposed and studied by a number of authors. For a survey, the reader may consult Weinert's paper **[8]**. The motivation for many of these concepts comes from ring theory and the various notions of rings of quotients. We are concerned in this paper with an analogue of the classical ring of quotients, introduced by Fountain and Petrich in **[3]**.

We recall that a ring $Q$ with identity is a classical ring of left quotients of its subring $R$ if every non-zero-divisor in $R$ has an inverse in $Q$ and every element of $Q$ may be written as $a^{-1}b$, where $a$, $b$ are elements of $R$. Here $a^{-1}$ is the inverse of $a$ in the group of units of $Q$, that is, $aa^{-1}=a^{-1}a=1$. In the definition of a semigroup of left quotients, given in Section 2, we concentrate on inverses within arbitrary subgroups of a semigroup and not just inverses within the group of units. Thus semigroups of quotients need not have an identity. Following the terminology for rings, we say that if $Q$ is a semigroup of (left) quotients of its subsemigroup $S$, then $S$ is a (left) order in $Q$.

It is a natural question to ask for characterisations of orders in particular classes of semigroups. The main theorem of **[3]** characterises orders in completely 0-simple semigroups; in this paper we consider the corresponding problem for Clifford semigroups.

It is clear from the definition that the notion of a semigroup of left quotients extends that of a group of left quotients, where $G$ is a group of left quotients of a subsemigroup $S$ if every element of $G$ can be written as $a^{-1}b$, where $a, b \in S$. We recall from **[1]** that a semigroup $S$ has a group of left quotients if and only if $S$ is right reversible and cancellative.

We remind the reader that a *Clifford semigroup* is an inverse semigroup with central idempotents. However, Theorem IV.2.1 of **[5]** gives an alternative description of Clifford semigroups as semilattices of groups, which enables us in Section 3 to describe left orders in Clifford semigroups in terms of semilattices and left orders in groups.

Our result bears a superficial resemblance to Theorem 3.4 of **[6]**, which states that a cancellative semigroup $S$ which is a semilattice $Y$ of left reversible semigroups $S_\alpha$ is embedded in a semilattice $Y$ of groups $G_\alpha$, where, for each $\alpha \in Y$, $G_\alpha$ is the group of right quotients of $S_\alpha$. However, the essential difference is that Osondu considers only those orders $S$ which are themselves cancellative. If $S$ is a semilattice $Y$ of left reversible semigroups $S_\alpha$, $\alpha \in Y$, then it is easy to see that $S$ is left reversible. So if in addition $S$ is cancellative, then $S$ has a group $G$ of right quotients. The purpose of **[6]** is to study the relationship between the group $G$ and the groups of right quotients $G_\alpha$ of $S_\alpha$, $\alpha \in Y$.

It is well known that, up to isomorphism, rings of left quotients of a given subring are unique and, correspondingly, the group of left quotients of a right reversible, cancellative

† The author acknowledges the support of the Science and Engineering Research Council in the form of a Research Studentship.

semigroup is unique. Unfortunately, this result is not true of semigroups of left quotients in general. In Section 4 we give an example of a semigroup which is a left order in two non-isomorphic Clifford semigroups.

To overcome the problem of uniqueness, the concepts of a stratified semigroup of (left) quotients and of a stratified (left) order were introduced in [4]. Theorem 3.3 of [4] gives that stratified semigroups of left quotients are unique up to isomorphism.

If $S$ is a left order in a Clifford semigroup $Q$, then $S$ is not necessarily stratified in $Q$, but we show in Theorem 4.3 that $S$ is a stratified left order in some Clifford semigroup $Q'$, which is therefore unique (up to isomorphism).

From Theorem 3.1, if $S$ is a left order in a Clifford semigroup, then $S$ is a semilattice of right reversible, cancellative semigroups. In particular, $S$ is a semilattice of cancellative semigroups and so is separative. In Section 5 we give an alternative characterisation of left orders in Clifford semigroups as separative semigroups satisfying a strong reversibility condition. We finish in Section 6 with a consideration of some special cases.

This paper forms part of the author's doctoral dissertation, written under the supervision of Dr J. B. Fountain, whose careful guidance was greatly appreciated.

**2. Preliminaries.** The generalisations $\mathscr{R}^*$, $\mathscr{L}^*$ and $\mathscr{H}^*$ of Green's relations $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{H}$ play an important role in what follows.

We recall that the relation $\mathscr{R}^*$ is defined on a semigroup $S$ by the rule that $a\,\mathscr{R}^*\,b$ if and only if $a\,\mathscr{R}\,b$ in some oversemigroup of $S$. The relation $\mathscr{L}^*$ is defined dually. Lemma 2.1 gives us an elementary characterisation of $\mathscr{R}^*$ and $\mathscr{L}^*$.

LEMMA 2.1 [2]. *The following conditions are equivalent for a semigroup $S$*:
(i) $a\,\mathscr{R}^*\,b$ $(a\,\mathscr{L}^*\,b)$
(ii) *for all* $x, y \in S^1$,

$$xa = ya \Leftrightarrow xb = yb \qquad (ax = ay \Leftrightarrow bx = by).$$

It is easy to see from this lemma that $\mathscr{R}^*$ is a left congruence and $\mathscr{L}^*$ is a right congruence. Thus the intersection of $\mathscr{R}^*$ and $\mathscr{L}^*$ is an equivalence relation, denoted by $\mathscr{H}^*$.

We say that an element $a$ of a semigroup $S$ is *square-cancellable* if $a\,\mathscr{H}^*\,a^2$. Using Lemma 2.1 we see that $a$ is square-cancellable if and only if, for all $x, y \in S^1$, $xa^2 = ya^2$ implies that $xa = ya$, and $a^2x = a^2y$ implies that $ax = ay$. This provides the justification for our terminology.

Let $a$ be an element of a semigroup $S$. If $S$ is a subsemigroup of a semigroup $Q$, then $a$ is in a subgroup of $Q$ if and only if $a\,\mathscr{H}\,a^2$ in $Q$. If $a\,\mathscr{H}\,a^2$ in $Q$ then clearly $a\,\mathscr{H}^*\,a^2$ in $S$; thus the condition that $a$ is square-cancellable is a necessary condition for $a$ to be in a subgroup of an oversemigroup.

In our theory of semigroups of quotients we consider square-cancellable elements as playing a role analogous to that of non-zero-divisors in the theory of rings of quotients. If $r$ is an element of a ring $R$ then the condition that $r$ is a non-zero-divisor is a necessary condition for $r$ to be in the group of units of a ring $Q$ containing $R$.

We now give the definition of a semigroup of (left, right) quotients. Let $S$ be a subsemigroup of a semigroup $Q$. Then $Q$ is a *semigroup of left quotients* of $S$ if

(i) every element of $Q$ can be written as $a^{-1}b$, where $a, b \in S$ and $a^{-1}$ is the inverse of $a$ in a subgroup of $Q$,

(ii) every square-cancellable element of $S$ is in a subgroup of $Q$.

*Semigroups of right quotients* are defined dually. If $Q$ is a semigroup of left (right) quotients of $S$ then we say that $S$ is a *left* (*right*) *order* in $Q$. A semigroup $Q$ is a *semigroup of quotients* of its subsemigroup $S$ and $S$ is an *order* in $Q$ if $S$ is both a left order and a right order in $Q$.

The main aim of this paper is to characterise left orders in Clifford semigroups. We rely continually on Theorem IV.2.1 of [5] which states that a semigroup is a Clifford semigroup if and only if it is a semilattice of groups. Using this result it is easy to see that if $S$ is a Clifford semigroup then, in $S$, $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D}$. Further, since the $\mathcal{H}$-classes of a semigroup are maximal subgroups, it is clear that if $S$ is a semilattice $Y$ of groups $G_\alpha$, $\alpha \in Y$, then the groups $G_\alpha$ are the $\mathcal{H}$-classes of $Q$. Hence a Clifford semigroup has a unique decomposition as a semilattice of groups.

All the left orders we are concerned with in this paper are semilattices of cancellative semigroups. As in the case of semilattices of groups, semilattices of cancellative semigroups may be described in a more appealing manner. To be specific they are separative semigroups.

We recall that a semigroup $S$ is *separative* if, for any $x, y \in S$,

$$x^2 = xy \quad \text{and} \quad y^2 = yx \quad \text{imply} \quad x = y,$$

and

$$x^2 = yx \quad \text{and} \quad y^2 = xy \quad \text{imply} \quad x = y.$$

LEMMA 2.1 [7]. *In a separative semigroup $S$,*

$$xa = ya \text{ if and only if } ax = ay \text{ for all elements } a, x, y \text{ of } S.$$

We define the relation $\mathcal{L}^\dagger$ on a semigroup $T$ by

$$a \mathcal{L}^\dagger b \text{ if, for all } x, y \in T, \qquad ax = ay \text{ if and only if } bx = by.$$

The relation $\mathcal{R}^\dagger$ is defined dually and we denote by $\mathcal{H}^\dagger$ the intersection of the relations $\mathcal{L}^\dagger$ and $\mathcal{R}^\dagger$. Clearly for any semigroup we have $\mathcal{L}^* \subseteq \mathcal{L}^\dagger$, $\mathcal{R}^* \subseteq \mathcal{R}^\dagger$ and $\mathcal{H}^* \subseteq \mathcal{H}^\dagger$. If $T$ is a monoid then $\mathcal{L}^* = \mathcal{L}^\dagger$, $\mathcal{R}^* = \mathcal{R}^\dagger$ and $\mathcal{H}^* = \mathcal{H}^\dagger$.

As an immediate consequence of Lemma 2.1 we have the following corollary.

COROLLARY 2.2. *Let $S$ be a separative semigroup. Then $\mathcal{L}^\dagger = \mathcal{R}^\dagger = \mathcal{H}^\dagger$ on $S$.*

THEOREM 2.3 [7]. *A semigroup $S$ is separative if and only if $S$ is a semilattice of cancellative semigroups. If so, $\mathcal{H}^\dagger$ is the greatest band congruence on $S$ all of whose classes are cancellative.*

COROLLARY 2.4. *For a separative semigroup $S$, $\mathcal{L}^* = \mathcal{R}^* = \mathcal{H}^* = \mathcal{L}^\dagger = \mathcal{R}^\dagger = \mathcal{H}^\dagger$.*

*Proof.* We show that if $a, b, x \in S$, $a \mathcal{L}^\dagger b$ and $ax = a$, then $bx = b$.

Suppose that $a$, $b$, $x$ are elements of $S$ satisfying the above conditions. From

Theorem 2.3, $S$ is a semilattice $Y$ of cancellative semigroups $S_\alpha$, $\alpha \in Y$. If $b \in S_\beta$, $x \in S_\gamma$, then from $ax = a$ we have $axb = ab$ and so $bxb = b^2$ and $\gamma\beta = \beta$. Thus $b$, $bx \in S_\beta$ and since $S_\beta$ is cancellative, $bx = b$.

We make heavy use of the notion of reversibility, which we define for subsets of a semigroup and not simply subsemigroups. If $T$ is a subset of a semigroup $S$ then $T$ is *left* (*right*) *reversible* if, given any elements $a$, $b$ of $T$, there exist elements $u$, $v$ in $T$ with $au = bv$ ($ua = vb$). If $T$ is both left and right reversible, then we say that $T$ is *reversible.*

**3. The main result.** This section is devoted to the proof of the following theorem.

THEOREM 3.1. *A semigroup $S$ is a left order in a semilattice $Y$ of groups $G_\alpha$, $\alpha \in Y$, if and only if $S$ is a semilattice $Y$ of right reversible, cancellative semigroups $S_\alpha$, $\alpha \in Y$.*

*Proof.* Suppose first that $S$ is a left order in $Q$, where $Q$ is a semilattice $Y$ of groups $G_\alpha$, $\alpha \in Y$. By the above comments, $\mathcal{H} = \mathcal{L} = \mathcal{R}$ in $Q$ and the groups $G_\alpha$ are the $\mathcal{H}$-classes of $Q$.

For $\alpha \in Y$, we define $S_\alpha$ to be $G_\alpha \cap S$. Let $\alpha \in Y$. To see that $S_\alpha$ is non-empty, choose $g \in G_\alpha$; by the definition of a semigroup of left quotients, $g = a^{-1}b$ for some $a$, $b$ in $S$. Let $\beta, \gamma \in Y$ be such that $a \in S_\beta$, $b \in S_\gamma$. Then $a^{-1} \in G_\beta$ and so $\alpha = \beta\gamma$, which gives $ab \in S_\alpha$ and so $S_\alpha \neq \emptyset$. Thus, for any $\alpha \in Y$, $S_\alpha$ is the non-empty intersection of two subsemigroups of $Q$, giving that $S_\alpha$ is a subsemigroup.

If $a^{-1}b \in G_\alpha$, then $a^{-1}b = a^{-1}be_\alpha$, where $e_\alpha$ is the identity of $G_\alpha$. Since the idempotents of $Q$ are central and $S_\alpha$ is non-empty, $a^{-1}b = a^{-1}c^{-1}cb$ for some $c \in S_\alpha$. Now if $a \in S_\beta$, $b \in S_\gamma$ we have $\beta\gamma = \alpha$ and so $a^{-1}c^{-1}$, $ca$, $cb$ are elements of $G_\alpha$. Then

$$a^{-1}c^{-1}ca = a^{-1}e_\alpha a = e_\alpha a^{-1}a = e_\alpha e_\beta$$

and, similarly,

$$caa^{-1}c^{-1} = ce_\beta c^{-1} = cc^{-1}e_\beta = e_\alpha e_\beta.$$

But $e_\alpha e_\beta$ is an idempotent in $G_\alpha$; so $e_\alpha e_\beta = e_\alpha$ and $a^{-1}c^{-1} = (ca)^{-1}$. Thus any $q \in G_\alpha$ may be written as $x^{-1}y$ for some $x, y \in S_\alpha$; that is, $S_\alpha$ is a left order in $G_\alpha$. Hence $S_\alpha$ is a right reversible cancellative subsemigroup of $S$. It is clear that $S$ is a semilattice $Y$ of the semigroups $S_\alpha$, $\alpha \in Y$.

Conversely, assume that $S$ is a semilattice $Y$ of right reversible, cancellative semigroups $S_\alpha$, $\alpha \in Y$. Since $S$ is separative, Corollary 2.4 gives that $\mathcal{H}^* = \mathcal{H}^\dagger$ on $S$. By Theorem 2.3, $\mathcal{H}^*$ is the greatest band congruence on $S$, all of whose classes are cancellative.

We define a relation $Y^S$ on $S$ by

$$aY^Sb \text{ if and only if } a, b \in S_\alpha \text{ for some } \alpha \in Y.$$

Clearly, $Y^S$ is a congruence on $S$, $S/Y^S$ is the semilattice $Y$ and the $Y^S$-classes are the semigroups $S_\alpha$, $\alpha \in Y$. So certainly $Y^S$ is a band congruence with cancellative congruence classes. By the above comments, $Y^S \subseteq \mathcal{H}^*$.

We note that every element of $S$ is square-cancellable, for if $a \in S_\alpha$, $x \in S_\beta$, $y \in S_\gamma$ and

$xa^2 = ya^2$, then $\alpha\beta = \alpha\gamma$ and so $ax, xa, ya \in S_{\alpha\beta}$. But $xa^2x = ya^2x$ and $S_{\alpha\beta}$ is cancellative, which gives that $xa = ya$. Since $S$ is separative, $\mathcal{H}^* = \mathcal{R}^* = \mathcal{R}^\dagger$ and so $a\,\mathcal{H}^*\,a^2$, that is, $a$ is square-cancellable.

We now proceed to construct a semigroup $Q$ of left quotients of $S$. For each $\alpha \in Y$, $S_\alpha$ has a group of left quotients $G_\alpha$ and we may assume that $G_\alpha \cap G_\beta = \emptyset$ for all $\alpha, \beta \in Y$ with $\alpha \neq \beta$. If $\alpha \in Y$ and $a, b, c, d \in S_\alpha$ are such that $a^{-1}b = c^{-1}d$ in $G_\alpha$, then $b = ac^{-1}d$ and since $S_\alpha$ is a left order in $G_\alpha$, $ac^{-1} = x^{-1}y$ for some $x, y$ in $S_\alpha$. This gives that $xa = yc$ and $xb = yd$. Conversely, if $a, b, c, d, x, y \in S_\alpha$ and $xa = yc$, $xb = yd$, then $ac^{-1} = x^{-1}y$ and $b = x^{-1}yd$. So $b = ac^{-1}d$, giving $a^{-1}b = c^{-1}d$.

If $b \in S_\alpha$ and $c \in S_\beta$, then $bc, cb \in S_{\alpha\beta}$ and since $S_{\alpha\beta}$ is right reversible there exist $x', y' \in S_{\alpha\beta}$ with $x'cb = y'bc$. Putting $x = x'c$, $y = y'b$, one sees that $xb = yc$ and $x, y \in S_{\alpha\beta}$. Further, for any $a \in S_\alpha$, $d \in S_\beta$, $xa, yd \in S_{\alpha\beta}$ and so $(xa)^{-1}yd$ exists in $G_{\alpha\beta}$.

Let $Q = \bigcup_{\alpha \in Y} G_\alpha$. Define a product . on $Q$ by

$$a^{-1}b \,.\, c^{-1}d = (xa)^{-1}yd,$$

where if $a, b \in S_\alpha$, $c, d \in S_\beta$, then $x, y \in S_{\alpha\beta}$ are chosen such that $xb = yc$. We emphasize that the product $(xa)^{-1}yd$ is taken as the product in $G_{\alpha\beta}$.

To see that the product . is well-defined suppose that we have elements $a, b, c, d$ of $S_\alpha$, $m, n, p, q$ of $S_\beta$ such that

$$a^{-1}b = c^{-1}d \text{ in } G_\alpha, \qquad m^{-1}n = p^{-1}q \text{ in } G_\beta.$$

Then there are elements $x, y$ in $S_\alpha$ and $w, x$ in $S_\beta$ with

$$xa = yc, \tag{3.1}$$

$$xb = yd, \tag{3.2}$$

$$wm = zp, \tag{3.3}$$

$$wn = zq. \tag{3.4}$$

By definition,

$$a^{-1}b \,.\, m^{-1}n = (ha)^{-1}kn \in G_{\alpha\beta},$$
$$c^{-1}d \,.\, p^{-1}q = (uc)^{-1}vq \in G_{\alpha\beta},$$

where $h, k, u, v \in S_{\alpha\beta}$ and

$$hb = km, \tag{3.5}$$

$$ud = vp. \tag{3.6}$$

Since $S_{\alpha\beta}$ is right reversible, there are elements $s, t$ in $S_{\alpha\beta}$ with

$$sha = tuc. \tag{3.7}$$

Now $sx, sh \in S_{\alpha\beta}$; so, again by right reversibility, there are elements $l, r'$ of $S_{\alpha\beta}$ with $lsx = r'sh$. Putting $ls = l'$, we have $l' \in S_{\alpha\beta}$ and $l'x = r'sh$.

By (3.1) and (3.7),

$$l'yc = l'xa = r'sha = r'tuc.$$

This gives $(l'y)(cl') = (r'tu)(cl')$ and $l'y,\ cl',\ r'tu \in S_{\alpha\beta}$ so that $l'y = r'tu$. But then from (3.2) we have

$$r'shb = l'xb = l'yd = r'tud,$$

which gives $shb = tud$. From (3.5) and (3.6),

$$skm = shb = tud = tvp. \tag{3.8}$$

By a similar argument to the above, there are elements $l''$, $r''$ of $S_{\alpha\beta}$ with $l''w = r''sk$ and so, using (3.3) and (3.8), we find that

$$l''zp = l''wm = r''skm = r''tvp$$

and it follows that $l''z = r''tv$. Now, by (3.4),

$$r''skn = l''wn = l''zq = r''tvq,$$

giving that

$$skn = tvq. \tag{3.9}$$

Equations (3.7) and (3.9) give that $a^{-1}b \,.\, m^{-1}n = c^{-1}d \,.\, p^{-1}q$ and so the product . on $Q$ is well-defined.

Next we show that $Q$ is a semigroup, that is the multiplication . on $Q$ is associative. Let $a^{-1}b \in G_\alpha$, $c^{-1}d \in G_\beta$, $h^{-1}k \in G_\gamma$ and put

$$X = (a^{-1}b \,.\, c^{-1}d) \,.\, h^{-1}k$$

and

$$Y = a^{-1}b \,.\, (c^{-1}d \,.\, h^{-1}k).$$

By definition,

$$X = ((ra)^{-1}sd) \,.\, h^{-1}k,$$

where $r, s \in S_{\alpha\beta}$ and

$$rb = sc. \tag{3.10}$$

Then

$$X = (tra)^{-1}uk,$$

where $t, u \in S_{\alpha\beta\gamma}$ and

$$tsd = uh. \tag{3.11}$$

Considering $Y$ we have that

$$Y = a^{-1}b \,.\, ((vc)^{-1}wk),$$

where $v, w \in S_{\beta\gamma}$ and

$$vd = wh. \tag{3.12}$$

https://doi.org/10.1017/S0017089500006492 Published online by Cambridge University Press

Then

$$Y = (xa)^{-1}ywk,$$

where $x, y \in S_{\alpha\beta\gamma}$ and

$$xb = yvc. \tag{3.13}$$

It is clear that $X$, $Y$ are both members of $G_{\alpha\beta\gamma}$. Since $tra$, $xa \in S_{\alpha\beta\gamma}$ and $S_{\alpha\beta\gamma}$ is right reversible, there are elements $m$, $n$ in $S_{\alpha\beta\gamma}$ with $mtra = nxa$. Now $aY^s b$ and so $a \mathscr{H}^* b$ and $mtrb = nxb$. Then, by (3.10) and (3.13),

$$mtsc = mtrb = nxb = nyrc.$$

But $c \mathscr{H}^* d$ and so $mtsd = nyvd$, which, by (3.11) and (3.12), gives

$$muh = mtsd = nyvd = nywh.$$

Since $h \mathscr{H}^* k$, we have that $muk = nywk$, which together with $mtra = nxa$ gives that $X = Y$.

To see that $S$ is a subsemigroup of $Q$ we need only show that the multiplication on $Q$ extends that of $S$. Let $a \in S_\alpha$, $b \in S_\beta$. Then $a = a^{-1}a^2 \in G_\alpha$, $b = b^{-1}b^2 \in G_\beta$. In $Q$, $a^{-1}a^2 \,.\, b^{-1}b^2 = (xa)^{-1}yb^2$, where $x, y \in S_{\alpha\beta}$ and $xa^2 = yb$. But then $yb^2 = xa^2b$ and so, in $G_{\alpha\beta}$,

$$a^{-1}a^2 \,.\, b^{-1}b^2 = (xa)^{-1}yb^2 = (xa)^{-1}(xa)(ab) = e_{\alpha\beta}ab = ab,$$

where $e_{\alpha\beta}$ is the identity of $G_{\alpha\beta}$. Moreover, it is easy to see that if $a^{-1}b$, $c^{-1}d \in G_\alpha$, then $a^{-1}b \,.\, c^{-1}d$ is equal to the product of $a^{-1}b$, $c^{-1}d$ in $G_\alpha$. Thus we may omit the symbol . and write the product of two elements of $Q$ unambiguously as juxtaposition.

By construction, $Q$ is the union of groups $G_\alpha$, where, for each $\alpha \in Y$, $G_\alpha$ is the group of left quotients of $S_\alpha$. From the definition of multiplication in $Q$, if $a^{-1}b \in G_\alpha$, $c^{-1}d \in G_\beta$, then $a^{-1}bc^{-1}d \in G_{\alpha\beta}$. Hence $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ and so $Q$ is a semilattice $Y$ of groups $G_\alpha$, $\alpha \in Y$. Finally it is clear that $S$ is a left order in $Q$.

**4. Stratified left orders.** Theorem 3.1 shows that if $S$ is a semilattice of right reversible, cancellative semigroups, then for any decomposition of $S$ as a semilattice $Y$ of right reversible, cancellative semigroups $S_\alpha$, $\alpha \in Y$, we can construct a Clifford semigroup $Q$ of left quotients of $S$, where $Q$ is a semilattice $Y$ of groups $G_\alpha$, $\alpha \in Y$. However, such a decomposition of $S$ is not necessarily unique. For example, the set $\mathbb{N}$ of positive integers under multiplication is a semilattice $Y$ of reversible, cancellative semigroups, where $Y$ is the single element semilattice. But if $X$ is the two element semilattice $\{\alpha, \beta\}$, where $\alpha\beta = \beta$, and if

$$N_\alpha = \{n \in \mathbb{N}: n \text{ is odd}\}, \qquad N_\beta = \{n \in \mathbb{N}: n \text{ is even}\},$$

then $\mathbb{N} = N_\alpha \cup N_\beta$, $N_\alpha$, $N_\beta$ are right reversible and cancellative and $N_\alpha N_\beta \subseteq N_\beta$. Hence $\mathbb{N}$ is a semilattice $X$ of right reversible, cancellative semigroups $N_\alpha$, $N_\beta$. Thus a semilattice of right reversible, cancellative semigroups may have non-isomorphic semigroups of left quotients.

In order to overcome this problem of uniqueness we introduce the notion of a stratified left order. Let $S$ be a subsemigroup of a semigroup $Q$. Then $S$ is a *stratified left order* in $Q$ and $Q$ is a *stratified semigroup* of *left quotients* of $S$ if

(i) every element of $Q$ can be written as $a^{-1}b$, where $a, b \in S$ and $a \mathcal{R} b$ in $Q$,
(ii) every square-cancellable element of $S$ is in a subgroup of $Q$,
(iii) for any elements $a$, $b$ of $S$,

$$a \mathcal{R} b \text{ in } Q \text{ if and only if } a \mathcal{R}^* b \text{ in } S,$$

$$a \mathcal{L} b \text{ in } Q \text{ if and only if } a \mathcal{L}^* b \text{ in } S.$$

Clearly, if $S$ is a stratified left order in $Q$ then $S$ is a left order in $Q$. We have the obvious definitions of stratified (right) order and stratified semigroup of (right) quotients.

Our interest in stratified semigroups of left quotients is due to the following result.

THEOREM 4.1 [4]. *Let $S$ be a stratified left order in semigroups $Q$ and $Q'$. Then $Q$ and $Q'$ are isomorphic under an isomorphism whose restriction to $S$ is the identity map.*

In the proof of Theorem 3.1 we showed that if $S$ is a left order in a Clifford semigroup $Q$ and $q \in Q$, then $q$ could be written as $a^{-1}b$ for some $a$, $b$ in $S$ with $a \mathcal{R} b$ (indeed $a \mathcal{H} b$) in $Q$. Since $\mathcal{L} = \mathcal{R} = \mathcal{H}$ in $Q$ and $\mathcal{L}^* = \mathcal{R}^* = \mathcal{H}^*$ in $S$, the semigroup $S$ will be a stratified left order in $Q$ if and only if $\mathcal{H}_Q \cap (S \times S) = \mathcal{H}^*_S$. If this is the case, then by Theorem 4.1, $Q$ must be the unique stratified semigroup of left quotients of $S$.

The next lemma enables us to show that if a semigroup $S$ is a left order in a Clifford semigroup $Q$, then $S$ is also a stratified left order in some Clifford semigroup $Q'$.

LEMMA 4.2. *Let $S$ be a semilattice $Y$ of right reversible, cancellative semigroups $S_\alpha$, $\alpha \in Y$. Then $\mathcal{H}^*$ is a semilattice congruence on $S$ and the $\mathcal{H}^*$-classes of $S$ are right reversible and cancellative.*

*Proof.* From Theorem 2.3 and Corollary 2.4 we have that $\mathcal{H}^*$ is the greatest band congruence on $S$ all of whose classes are cancellative.

As in the proof of Theorem 3.1 we define the relation $Y^S$ on $S$ by

$$aY^Sb \text{ if and only if } a, b \in S_\alpha \text{ for some } \alpha \in Y.$$

Then $Y^S$ is a semilattice congruence contained in $\mathcal{H}^*$.

To see that $\mathcal{H}^*$ is a semilattice congruence, it is only necessary to show that for any $a$, $b$ in $S$, $ab \mathcal{H}^* ba$. But $Y^S$ is a semilattice congruence and so if $a, b \in S$, then $abY^Sba$. Hence $ab \mathcal{H}^* ba$ and $S/\mathcal{H}^*$ is a semilattice.

It remains to prove that the $\mathcal{H}^*$-classes of $S$ are right reversible. Let $a, b \in S$ and suppose that $a \mathcal{H}^* b$. Then $ab \mathcal{H}^* ba \mathcal{H}^* a^2 \mathcal{H}^* a$, since $\mathcal{H}^*$ is a semilattice congruence. But $abY^Sba$ and so if $ab, ba \in S_\alpha$, $\alpha \in Y$, then $S_\alpha \subseteq H^*_a$. Now $S_\alpha$ is right reversible, giving $cab = dba$ for some $c, d \in S_\alpha$. Then $ca, db \in H^*_a$ and so $H^*_a$ is right reversible.

We can now deduce the following theorem.

THEOREM 4.3. *A semigroup $S$ is a stratified left order in a Clifford semigroup if and only if $S$ is a semilattice of right reversible, cancellative semigroups.*

*Proof.* If $S$ is a stratified left order in a Clifford semigroup $Q$, then as $Q$ is a semilattice of groups, Theorem 3.1 gives that $S$ is a semilattice of right reversible, cancellative semigroups.

Conversely, if $S$ is a semilattice of right reversible, cancellative semigroups, then from Lemma 4.2, $S$ is a semilattice $X$ of right reversible, cancellative semigroups $S_\alpha$, $\alpha \in X$, where the semigroups $S_\alpha$ are the $\mathscr{H}^*$-classes of $S$. It follows from Theorem 3.1 that $S$ is a left order in a Clifford semigroup $Q$, where the group $\mathscr{H}$-classes of $Q$ are the groups of left quotients of the semigroups $S_\alpha$, that is, the $\mathscr{H}^*$-classes of $S$. It is then immediate from the construction of $Q$ that $\mathscr{H}_Q \cap (S \times S) = \mathscr{H}_S^*$. Hence, by earlier comments, $S$ is a stratified left order in $Q$.

**5. Alternative characterisations.** Since semilattices of groups and semilattices of cancellative semigroups both have more appealing characterisations as inverse semigroups with central idempotents and separative semigroups, respectively, it is interesting to see whether semilattices of right reversible, cancellative semigroups might also be described in a more pleasing way. Obviously, such semigroups are separative and it is easy to see that they are right reversible. For if $S$ is a semilattice $Y$ of right reversible semigroups $S_\alpha$, $\alpha \in Y$ and $a \in S_\alpha$, $b \in S_\beta$, then $ba, ab \in S_{\alpha\beta}$ and right reversibility of $S_{\alpha\beta}$ gives that $ca = db$ for some $c, d \in S_{\alpha\beta}$. However, given a cancellative semigroup $T$ which is not right reversible, by adjoining a zero to $T$ we obtain a semigroup that is separative and right reversible, but which has no semilattice decomposition into cancellative right reversible semigroups.

To avoid awkward examples of this kind we define a stronger version of reversibility. We say that a semigroup $S$ is *right $\mathscr{H}^*$-reversible* if for any $a$, $b$ in $S$ there exist elements $x$, $y$ in $S$ with $xa = yb$ and $x \,\mathscr{H}^*\, y \,\mathscr{H}^*\, ab$. The dual notion is *left $\mathscr{H}^*$-reversibility* and we say that a semigroup is *$\mathscr{H}^*$-reversible* if it is both right and left $\mathscr{H}^*$-reversible.

THEOREM 5.1. *The following conditions are equivalent for the semigroup $S$:*
(I) *$S$ is a left order in a Clifford semigroup;*
(II) *$S$ is a semilattice of right reversible, cancellative semigroups;*
(III) *$S$ is separative and the $\mathscr{H}^*$-classes of $S$ are right reversible;*
(IV) *$S$ is separative and right $\mathscr{H}^*$-reversible.*

*Proof.* (I)$\Leftrightarrow$(II). This is Theorem 3.1.

(II)$\Rightarrow$(III). Since $S$ is a semilattice of cancellative semigroups, $S$ is separative. Lemma 4.2 gives that the $\mathscr{H}^*$-classes of $S$ are right reversible.

(III)$\Rightarrow$(IV). As $S$ is separative, $S$ is a semilattice of cancellative semigroups. As in Lemma 4.2, $\mathscr{H}^*$ is a congruence on $S$ and $S/\mathscr{H}^*$ is a semilattice.

Let $a, b \in S$. Then $ba, ab \in H_{ab}^*$ as $\mathscr{H}^*$ is a semilattice congruence. But $H_{ab}^*$ is right reversible and so there are elements $c$, $d$ in $H_{ab}^*$ with $cbd = dab$. But then $cb \in H_{ab}^* H_b^* \subseteq H_{ab}^*$, $da \in H_{ab}^* H_a^* \subseteq H_{ab}^*$, so putting $x = cb$, $y = da$, we have $xa = yb$ and $x \,\mathscr{H}^*\, y \,\mathscr{H}^*\, ab$. Thus $S$ is right $\mathscr{H}^*$-reversible. We know that $\mathscr{H}^*$ is a semilattice congruence on $S$ and that the $\mathscr{H}^*$-classes of $S$ are cancellative. To see that they are right reversible, let $a, b \in S$ and suppose that $a \,\mathscr{H}^*\, b$. By right $\mathscr{H}^*$-reversibility, there are elements $x$, $y$ in $S$ with

https://doi.org/10.1017/S0017089500006492 Published online by Cambridge University Press

$x \mathscr{H}^* y \mathscr{H}^* ab$ and $xa = yb$. But $S/\mathscr{H}^*$ is a semilattice and so $ab \mathscr{H}^* a^2 \mathscr{H}^* a$. Hence $x \mathscr{H}^* y \mathscr{H}^* a$ and $H_a^*$ is right reversible.

**6. Some special cases.** If a semigroup is commutative it is of course reversible, indeed any of its subsemigroups is reversible. The following result is essentially Theorem II.6.6 of [**7**].

COROLLARY 6.1. *The following conditions are equivalent for a semigroup $S$:*
(i) *$S$ is commutative and separative;*
(ii) *$S$ is a semilattice of commutative, cancellative semigroups;*
(iii) *$S$ is a left order in a commutative regular semigroup;*
(iv) *$S$ is a subsemigroup of a commutative regular semigroup.*

*Proof.* (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv). These are clear.

(ii)$\Rightarrow$(iii). Let $S$ be a semilattice $Y$ of commutative, cancellative semigroups $S_\alpha$, $\alpha \in Y$. From Theorem 3.1, $S$ is a left order in a semigroup $Q$, where $Q$ is a semilattice $Y$ of groups $G_\alpha$, $\alpha \in Y$ and for each $\alpha \in Y$, $G_\alpha$ is the group of left quotients of $S_\alpha$. Since $S_\alpha$ is commutative, $G_\alpha$ is abelian, for each $\alpha \in Y$. It now follows as in Theorem II.6.6 of [**7**] that $Q$ is commutative.

(iv)$\Rightarrow$(i). Let $S$ be a subsemigroup of $Q$, where $Q$ is commutative and regular. Then $Q$ is a semilattice $Y$ of abelian groups $G_\alpha$, $\alpha \in Y$, and so certainly $Q$ is commutative and separative. Since $S$ is a subsemigroup of $Q$, we have that (i) holds.

We recall that a semigroup $S$ is *abundant* if every $\mathscr{R}^*$-class and every $\mathscr{L}^*$-class of $S$ contains an idempotent. If in addition the idempotents of $S$ commute, then $S$ is said to be *adequate*. Owing to the importance of the relations $\mathscr{R}^*$ and $\mathscr{L}^*$ in the theory of semigroups of quotients, it is interesting to consider abundant orders in Clifford semigroups. Clearly such orders must in fact be adequate.

PROPOSITION 6.2. *The following conditions are equivalent for an adequate semigroup $S$:*
(I) *$S$ is a left order in a Clifford semigroup;*
(II) *$\mathscr{R}^* = \mathscr{L}^*$ on $S$ and each $\mathscr{H}^*$-class is right reversible;*
(III) *each $\mathscr{H}^*$-class of $S$ contains an idempotent and is right reversible;*
(IV) *each $\mathscr{H}^*$-class of $S$ contains an idempotent and $S$ is right $\mathscr{H}^*$-reversible.*

*Proof.* (I)$\Rightarrow$(II). Since $S$ is separative, $\mathscr{R}^* = \mathscr{L}^* = \mathscr{H}^*$ on $S$. From Theorem 5.1, the $\mathscr{H}^*$-classes of $S$ are right reversible.

(II)$\Rightarrow$(III). This is immediate from Proposition 2.9 of [**2**].

(III)$\Rightarrow$(IV). Proposition 2.9 of [**2**] gives that $S$ is a strong semilattice of cancellative monoids. Thus $S$ is separative and since the $\mathscr{H}^*$-classes of $S$ are right reversible, Theorem 5.1 gives that $S$ is right $\mathscr{H}^*$-reversible.

(IV)$\Rightarrow$(I). Again from Proposition 2.9 of [**2**], $S$ is separative and so the result follows from Theorem 5.1.

Finally we consider orders in Clifford semigroups. It is easy to see that if a reversible, cancellative semigroup $S$ is a left order in a group $G$, then $S$ is an order in $G$. For if $a^{-1}b \in G$, where $a, b \in S$, then by the left reversibility of $S$, there are elements $c$, $d$ in $S$ with $ac = bd$. Then $cd^{-1} = a^{-1}b$ and so $S$ is a right order in $G$. The situation is similar for stratified orders in Clifford semigroups.

PROPOSITION 6.3. *The following conditions are equivalent for the semigroup* $S$:

(i) *$S$ is an order in a Clifford semigroup*;

(ii) *$S$ is separative and $\mathcal{H}^*$-reversible*;

(iii) *$S$ is a semilattice of reversible, cancellative semigroups*;

(iv) *$S$ is a left order in a Clifford semigroup $Q_1$ and a right order in a Clifford semigroup $Q_2$*;

(v) *$S$ is a stratified order in a Clifford semigroup.*

*Proof.* The implications (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv) are clear from Theorem 5.1 and the implication (v)$\Rightarrow$(i) is obvious.

(ii)$\Rightarrow$(iii). Since $S$ is separative, $\mathcal{H}^*$ is a semilattice congruence on $S$ and the $\mathcal{H}^*$-classes of $S$ are reversible by Theorem 5.1.

(iv)$\Rightarrow$(v). By Theorem 5.1, the $\mathcal{H}^*$-classes of $S$ are reversible. From Theorems 3.1 and 4.3, $S$ has a stratified semigroup of left quotients $Q$, where $Q$ is a Clifford semigroup and the $\mathcal{H}$-classes of $Q$ are the groups of left quotients of the $\mathcal{H}^*$-classes of $S$. By the comment preceding this proposition, the group $\mathcal{H}$-classes of $Q$ are groups of quotients of the $\mathcal{H}^*$-classes of $S$ and so $S$ is an order in $Q$.

COROLLARY 6.4. *If $S$ is a stratified left order in a Clifford semigroup $Q_1$ and a stratified right order in a Clifford semigroup $Q_2$, then $Q_1$ is isomorphic to $Q_2$ and $Q_1$, $Q_2$ are semigroups of quotients of $S$.*

## REFERENCES

**1.** A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups,* Mathematical Surveys 7, Vol. 1 (American Mathematical Society, 1961).

**2.** J. B. Fountain, Adequate semigroups, *Proc. Edinburgh Math. Soc.* (2) **22** (1979), 113–125.

**3.** J. B. Fountain and M. Petrich, Completely 0-simple semigroups of quotients, *J. Algebra,* to appear.

**4.** V. A. R. Gould, Absolutely flat completely 0-simple semigroups of left quotients, submitted.

**5.** J. M. Howie, *An introduction to semigroup theory* (Academic Press, 1976).

**6.** K. E. Osondu, Semilattices of left reversible semigroups, *Semigroup Forum* **17** (1979), 139–152.

**7.** M. Petrich, *Introduction to semigroups* (Charles E. Merrill, 1973).

**8.** H. J. Weinert, *S*-sets and semigroups of quotients, *Semigroup Forum* **19** (1980), 1–78.

UNIVERSITY OF YORK
present address
SCHOOL OF MATHEMATICS
UNIVERSITY OF BRISTOL
UNIVERSITY WALK
BRISTOL BS8 1TW