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ZEROS OF DIFFERENTIAL POLYNOMIALS IN REAL MEROMORPHIC FUNCTIONS

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Abstract We investigate whether differential polynomials in real transcendental meromorphic functions have non-real zeros. For example, we show that if g is a real transcendental meromorphic function, $c \in \mathbb{R} \setminus \{0\}$ and $n \geqslant 3$ is an integer, then $g'g^n - c$ has infinitely many non-real zeros. If g has only finitely many poles, then this holds for $n \geqslant 2$. Related results for rational functions g are also considered.

Keywords: meromorphic function; differential polynomial; real zeros

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1. Introduction and results

Our starting point is the following result due to Sheil-Small [**15**], which solved a long-standing conjecture.

Theorem A. *Let f be a real polynomial of degree d. Then $f' + f^2$ has at least $d - 1$ distinct non-real zeros which are not zeros of f.*

In the special case where f has only real roots this theorem is due to Prüfer (see [**14**, Chapter V, p. 182] and [**15**] for further discussion of the result).

Theorem B below is an analogue of Theorem A for transcendental meromorphic functions. Here 'meromorphic' will mean 'meromorphic in the complex plane' unless explicitly stated otherwise. A meromorphic function is called real if it maps the real axis $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$.

Theorem B. *Let f be a real transcendental meromorphic function with finitely many poles. Then $f' + f^2$ has infinitely many non-real zeros which are not zeros of f.*

Theorem B is a special case of [**4**, Theorem 1.3]. The result that $f' + f^2$ has infinitely many non-real zeros if f is a real entire transcendental function follows from the main theorem of [**3**].

A corollary of Theorem A is the following result.

Corollary 1.1. *Let f be a real polynomial of degree d and $m \geqslant 2$ an integer. Then $f' + f^m$ has at least $d(m-1) - 1$ distinct non-real zeros which are not zeros of f. In particular, $f' + f^m$ has at most $d+1$ real zeros.*

In fact, putting $g(z) = f(w)^{m-1}$, where $w = z/(m-1)$, we have

$$g'(z) + g(z)^2 = f(w)^{m-2}(f'(w) + f(w)^m)$$

so that Corollary 1.1 follows from Theorem A applied to g, since g has degree $d(m-1)$.

The same argument yields the following corollary to Theorem B.

Corollary 1.2. *Let f be a real transcendental meromorphic function with finitely many poles and $m \geqslant 2$ an integer. Then $f' + f^m$ has infinitely many non-real zeros which are not zeros of f.*

We note that Theorem B does not hold for meromorphic functions with infinitely many poles, a simple example being $f(z) = -\tan z$. Similarly, Theorem A fails for rational functions.

In this paper we consider to what extent Corollary 1.1 holds for rational functions, and Corollary 1.2 for meromorphic functions f with infinitely many poles. The following theorem summarizes results on zeros of the differential polynomial $f' + f^m$ in the plane.

Theorem C. *Let f and g be non-constant and meromorphic and let $m \geqslant 3$ and $n \geqslant 1$ be integers. Then*

(i) *$f' + f^m$ has at least one zero which is not a zero of f, and infinitely many if f is transcendental;*

(ii) *if c is a non-zero complex number, $g^n g' - c$ has at least one zero, and infinitely many if g is transcendental.*

Hayman [**6**, Corollary to Theorem 9] proved that if f is a transcendental meromorphic function and $m \geqslant 5$, then $f' + f^m$ has infinitely many zeros, and his proof shows that $f' + f^m$ has infinitely many zeros which are not zeros of f. Mues [**11**] showed that Hayman's result remains valid for $m = 4$, and the first and second author [**2**] proved that this also holds for $m = 3$. In fact the transformation

$$w = cz, \qquad g(z) = \frac{1}{f(w)}, \qquad f'(w) + f(w)^m = c^{-1} g(z)^{-m}(c - g(z)^{m-2} g'(z)) \tag{1.1}$$

shows that part (i) of Theorem C may be deduced from part (ii), and part (ii) was proved for transcendental g and $n \geqslant 1$ in [**2**]. The assertion of part (ii) for non-constant rational functions g follows easily from a consideration of the behaviour of g and $g^n g' - c$ at infinity.

Theorem 1.3. *Let f be a real rational function of degree d and $m \geqslant 5$ an integer. Then $f' + f^m$ has at least $(m-4)d$ distinct non-real zeros which are not zeros of f. In particular, $f' + f^m$ has at most $4d$ real zeros.*

If m is odd, then $f' + f^m$ has at least $(m-3)d$ distinct non-real zeros which are not zeros of f so that $f' + f^m$ has at most $3d$ real zeros.

Theorem 1.4. *Let f be a real transcendental meromorphic function and $m \geqslant 5$ an integer. Then $f' + f^m$ has infinitely many non-real zeros which are not zeros of f.*

Theorem 1.4 improves Hayman's result in the case that f is real. We will show by examples that the restriction $m \geqslant 5$ in this theorem is best possible.

Using the transformation (1.1) we obtain the following results from Theorems 1.3 and 1.4.

Corollary 1.5. *Let g be a real rational function of degree d and $n \geqslant 3$ an integer. Then for every real $c \neq 0$ the equation $g^n g' = c$ has at least $d(n-2)$ distinct non-real solutions.*

Corollary 1.6. *Let g be a real transcendental meromorphic function and $n \geqslant 3$ an integer. Then for every real $c \neq 0$ the equation $g^n g' = c$ has infinitely many non-real solutions.*

The condition $n \geqslant 3$ is best possible in order to conclude that there are non-real solutions. However, for polynomials and transcendental meromorphic functions with finitely many poles we have the following theorem.

Theorem 1.7. *Let g be a real polynomial of degree d and $n \geqslant 2$ an integer. Then for every real $c \neq 0$ the number of distinct non-real solutions of the equation $g^n g' = c$ is at least $d(n-1)-1$ if n is even and at least $d(n-1)$ if n is odd. In particular, this equation has at most $2d$ real solutions.*

Theorem 1.8. *Let g be a real transcendental meromorphic function with finitely many poles and $n \geqslant 2$ an integer. Then for every real $c \neq 0$ the equation $g^n g' = c$ has infinitely many non-real solutions.*

Examples show that if $n = 1$, then the equation $g^n g' = c$ need not have non-real solutions.

For our last result, we return to the value distribution of $f' + f^m$. Hayman [**6**] proved not only that if f is a transcendental meromorphic function and $m \geqslant 5$ then $f' + f^m$ has infinitely many zeros, but that under these conditions $f' + f^m + c$ has infinitely many zeros for any $c \in \mathbb{C}$. Examples show that when m is 5 or 6 and $c \in \mathbb{R} \setminus \{0\}$ then $f' + f^m + c$ may fail to have non-real zeros, with f real, transcendental and meromorphic.

Theorem 1.9. *Let f be a real meromorphic function and set $G = f' + f^m + c$, where $c \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{N}$.*

Suppose that f is transcendental. If $m \geqslant 7$ then G has infinitely many non-real zeros which are not zeros of f. The same conclusion holds for $m \geqslant 4$ if $\bar{N}(r, f) = o(T(r, f))$ and thus in particular if f has finitely many poles.

Suppose finally that f is a non-constant rational function with $f' \not\equiv -c$. If $m \geqslant 6$ then G has at least one non-real zero which is not a zero of f, and the same conclusion holds for $m \geqslant 3$ if f is a polynomial.

2. A result from complex dynamics

One of our main tools is a result from holomorphic dynamics. Recall that the Fatou set of a nonlinear meromorphic function f is the set where the iterates $f^{\circ n}$ of f are defined and form a normal family. We say that $\zeta \in \mathbb{C}$ is a multiple fixed point of f (of multiplicity μ) if $f(z) - z$ has a multiple zero (of multiplicity μ) at ζ. This has to be slightly modified if $\zeta = \infty$. We say that ∞ is a multiple fixed point of f (of multiplicity μ) if $1/f(1/z)$ has a multiple fixed point (of multiplicity μ) at 0.

Lemma 2.1. *Let f be a nonlinear rational or transcendental meromorphic function and let ζ be a multiple fixed point of f of multiplicity μ. Then there are $\nu := \mu - 1$ components $U_1, U_2, \dots, U_\nu$ of the Fatou set of f satisfying $f(U_j) \subset U_j$, $\zeta \in \partial U_j$ and $f^{\circ n}(z) \to \zeta$ for all $z \in U_j$ as $n \to \infty$. Each U_j contains at least one singularity of f^{-1}, the inverse function of f.*

In addition, the U_j can be labelled such that if $z \in U_j$, then

$$\arg(f^{\circ n}(z) - \zeta) \to \theta_j := \frac{-\arg f^{(\nu+1)}(\zeta) - \pi + 2\pi j}{\nu}$$

as $n \to \infty$ if $\zeta \in \mathbb{C}$, while if $\zeta = \infty$ then, for some $\theta_0 \in \mathbb{R}$,

$$\arg(f^{\circ n}(z)) \to \theta_j := \frac{\theta_0 + 2\pi j}{\nu}$$

as $n \to \infty$.

The domains U_j appearing in Lemma 2.1 are called *Leau domains*. Note that the singularities of f^{-1} are precisely the critical and asymptotic values of f. If f is rational, then the only singularities of f^{-1} are the critical values. In fact, the domains U_j then also contain critical points, but we do not need this result.

Lemma 2.1 is due to Fatou, and can now be found in every textbook on complex dynamics. Excellent introductions to complex dynamics are [**10**, **16**]; see [**10**, § 10] or [**16**, § 3.5] for a proof and discussion of the results stated in Lemma 2.1. We note that [**10**, **16**] and most other textbooks on complex dynamics treat only the case where f is rational, but the proofs extend to the case of transcendental meromorphic functions (see also [**1**, § 4.3]).

To demonstrate how dynamics works we begin with a simple direct proof of Corollary 1.1 found by the second author in 1989 after reading Sheil-Small's paper. This dynamical proof is also reproduced in [**8**].

Proof of Corollary 1.1. Let

$$F(z) = z - \frac{1}{f(z/(m-1))^{m-1}}. \tag{2.1}$$

Then $F'(z) = 0$ gives $f(w) \neq 0$ and $f'(w) + f(w)^m = 0$ where $w = z/(m-1)$. We have

$$F(z) = z + cz^{-d(m-1)} + O(z^{-d(m-1)-1}), \quad z \to \infty,$$

where $c \in \mathbb{R} \setminus \{0\}$. This implies that ∞ is a multiple fixed point of f of multiplicity $\mu = d(m-1)+2$. Let $U_1, U_2, \dots, U_{\mu-1}$ be the Leau domains at ∞ and let $\theta_1, \theta_2, \dots, \theta_{\mu-1}$ be as in Lemma 2.1. Each U_j contains a critical value and, as F is real, this critical value and the corresponding critical point can be real only if θ_j is a multiple of π. This is the case for at most two of the $\mu - 1$ values of j, and thus F has at least $\mu - 3 = d(m-1) - 1$ non-real critical points.

As the total number of zeros of $f' + f^m$ is dm we obtain the result. □

The proof of Theorem 1.3 will use the same idea.

Proof of Theorem 1.3. Let $f(z) = P(z)/Q(z)$, where P has degree p and Q has degree q, so that $d = \max\{p, q\}$. We consider again the function F defined by (2.1). If $q < p = d$, then

$$F(z) = z + cz^{-(d-q)(m-1)} + O(z^{-(d-q)(m-1)-1}), \quad z \to \infty,$$

and the argument used in the proof of Corollary 1.1 shows that F has a multiple fixed point at ∞ and that the Leau domains associated to this fixed point contain at least $(d-q)(m-1) - 1$ non-real zeros of F'.

Let $\zeta_1, \dots, \zeta_k$ be the finite poles of f, with multiplicities $s_1, \dots, s_k$. Then $\zeta_j(m-1)$ is a fixed point of multiplicity $s_j(m-1)$ of F. Lemma 2.1 now yields that of the $s_j(m-1)-1$ Leau domains associated to ζ_j at most two can contain a real critical value, so at least $s_j(m-1) - 3$ of these domains give rise to non-real zeros of F'.

Overall this leads to $\sum_{j=1}^{k} s_j(m-1) - 3 = q(m-1) - 3k$ non-real zeros of F'. If $q = d$ we thus have $d(m-1)-3k \geqslant d(m-1)-3d = d(m-4)$ non-real zeros of F'. If $q < d$, then we obtain $(d-q)(m-1)-1+q(m-1)-3k = d(m-1)-1-3k \geqslant d(m-1)-1-3(d-1) = d(m-4)+2$ non-real zeros of F'.

If m is odd, then the number of Leau domains at each multiple fixed point is odd. Lemma 2.1 shows that each of them, with at most one exception, contains a non-real critical value. The above argument then shows that F' has at least $d(m-3)$ non-real zeros. □

3. Proof of Theorem 1.4

Our second main tool is the following result of Pang [**13**], which has already found many applications (see, for example, Zalcman's survey [**18**]).

Lemma 3.1. *Let f be a meromorphic function with unbounded spherical derivative, and $\kappa \in (-1, 1)$. Then there exist sequences $z_j \in \mathbb{C}$ and $a_j > 0$ such that*

$$a_j^{-\kappa} f(z_j + a_j z) \to h(z), \quad j \to \infty, \tag{3.1}$$

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uniformly on compact subsets of $\mathbb{C}$*, where* h *is a non-constant meromorphic function with bounded spherical derivative. Furthermore, one can choose*

$$z_j \to \infty \quad \text{and} \quad a_j \to 0, \quad j \to \infty. \tag{3.2}$$

The following result is an immediate consequence of the definition of the order of a meromorphic function, using the Ahlfors–Shimizu form of the Nevanlinna characteristic.

Lemma 3.2. *Let* f *be a meromorphic function with bounded spherical derivative. Then* f *is of order at most* 2.

Proof of Theorem 1.4. The proof is by contradiction, assuming that $f' + f^m$ has only finitely many non-real zeros which are not zeros of f.

The first step is to reduce the result to the case of functions with bounded spherical derivative. Suppose f has unbounded spherical derivative. We apply Lemma 3.1 with $\kappa = 1/(1-m)$. Then $\kappa m = \kappa - 1$ and we obtain

$$a_j^{-\kappa m} f(z_j + a_j z)^m \to h(z)^m.$$

In $\mathbb{C} \setminus h^{-1}(\{\infty\})$ we have

$$a_j^{-\kappa m} f'(z_j + a_j z) = a_j^{-\kappa+1} f'(z_j + a_j z) \to h'(z)$$

and

$$h'(z) + h(z)^m = \lim_{j\to\infty} a_j^{-\kappa m}(f'(z_j + a_j z) + f(z_j + a_j z)^m).$$

Suppose first that $|\operatorname{Im} z_j/a_j| \to \infty$. If $h' + h^m$ has a zero at ζ then $f' + f^m$ has a zero at $\eta_j = z_j + a_j\zeta_j$, where $\zeta_j \to \zeta$. But then $|\eta_j| \geqslant |z_j| - o(1) \to \infty$, using (3.2), and $|\operatorname{Im} \eta_j/a_j| \to \infty$, so that η_j is non-real. Thus η_j is a zero of f and ζ is a zero of h. Hence all zeros of $h' + h^m$ are zeros of h, which contradicts Theorem C.

Thus $|\operatorname{Im} z_j/a_j| \not\to \infty$ and we may assume that $\operatorname{Im} z_j/a_j \to s \in \mathbb{R}$. Putting $x_j = \operatorname{Re} z_j = z_j - \mathrm{i}\operatorname{Im} z_j$ we find that

$$a_j^{-\kappa} f(x_j + a_j z) = a_j^{-\kappa} f(z_j + a_j(z - \mathrm{i}\operatorname{Im} z_j/a_j)) \to h(z - \mathrm{i}s).$$

We may thus assume that $z_j \in \mathbb{R}$ so that h is real, since otherwise we can replace z_j by x_j and $h(z)$ by $h(z - \mathrm{i}s)$. We obtain a non-constant real meromorphic function h with bounded spherical derivative. Suppose that ζ is a non-real zero of $h' + h^m$. Then again $f' + f^m$ has a zero at $\eta_j = z_j + a_j\zeta_j$, where $\zeta_j \to \zeta$, so that ζ_j is non-real for large j. But this gives $\operatorname{Im} \eta_j = a_j(\operatorname{Im} \zeta_j) \neq 0$, as well as $|\eta_j| \geqslant |z_j| - o(1) \to \infty$, using (3.2). Hence η_j is a zero of f, and ζ is a zero of h. It follows that *all* non-real zeros of $h' + h^m$ are zeros of h. Theorem 1.3 implies that h is transcendental.

If f has bounded spherical derivative, then we put $h = f$. Again h is transcendental, but $h' + h^m$ may have finitely many non-real zeros which are not zeros of h.

As in the proof of Corollary 1.1 and Theorem 1.3 we consider the auxiliary function

$$F(z) = z - \frac{1}{h(z/(m-1))^{m-1}},$$

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and note that F' has finitely many non-real zeros and that poles of h give rise to multiple fixed points of F of multiplicity at least $m-1$. Lemma 2.1 implies that at least two of the (at least three) Leau domains associated to such a fixed point contain a non-real singularity of F^{-1}.

Since F has finite order and finitely many non-real critical values, Theorem 1 from [**2**] implies that all non-real asymptotic values correspond to logarithmic singularities of the inverse function F^{-1}. But according to the Denjoy–Carleman–Ahlfors Theorem [**12**, § 258], a function of order at most 2 can have at most four logarithmic singularities. Thus F has only finitely many non-real critical and asymptotic values, so we conclude that h has finitely many poles. We obtain a contradiction with Corollary 1.2. □

4. Proof of Theorems 1.7 and 1.8

Proof of Theorem 1.7. Without loss of generality we can assume that $c = 1$. We consider the function

$$G(z) = z - \frac{1}{n+1} g(z)^{n+1}$$

and note that the critical points of G are exactly the solutions of $g(z)^n g'(z) = 1$.

If ζ is a zero of g of multiplicity s, then ζ is a multiple fixed point of G of multiplicity $s(n+1)$. Lemma 2.1 and an argument as in the proof of Corollary 1.1 yield that the $s(n+1)-1$ associated Leau domains contain at least $s(n+1)-3$ non-real critical values. If n is odd and thus the number $s(n+1)-1$ of Leau domains is odd, then we even obtain $s(n+1)-2$ non-real critical values in these domains. Let $\zeta_1, \dots, \zeta_k$ be the zeros of g, with multiplicities $s_1, \dots, s_k$. If n is odd then we find that G' has at least $(n+1)\sum_{j=1}^{k} s_j - 2k = d(n+1) - 2k$ non-real zeros. Since $k \leqslant d$ this implies that G' has at least $d(n-1)$ non-real zeros.

Now we consider the case that n is even. If ζ is a simple zero of g, then the previous argument based on Lemma 2.1 will only yield that $n-2$ of the n associated Leau domains contain a non-real critical value, which does not give any information in the most interesting case where $n = 2$. We note, however, that if $g'(\zeta) < 0$, then $G^{(n+1)}(\zeta) = -n! g'(\zeta)^{n+1} > 0$. With U_j as in Lemma 2.1 we find that if $z \in U_j$, then

$$\arg(G^{\circ \ell}(z) - \zeta) \to \theta_j := \frac{-\arg G^{(n+1)}(\zeta) - \pi + 2\pi j}{n} = \frac{(2j-1)\pi}{n}$$

as $\ell \to \infty$. As n is even, $(2j-1)/n$ is never an integer and thus all n Leau domains at ζ contain a non-real critical value in this case.

Let K be the number of real simple zeros ζ of g for which $g'(\zeta) > 0$. Between two such zeros there must be a real simple zero ζ satisfying $g'(\zeta) < 0$ or a multiple zero of odd multiplicity. Let L be the number of real simple zeros ζ of g for which $g'(\zeta) < 0$ and let M be the number of real multiple zeros of odd multiplicity. Then

$$L + M + 1 \geqslant K. \tag{4.1}$$

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We denote by N the total number of multiple zeros of g and by P the number of non-real simple zeros. Thus $K+L+P$ is the total number of simple zeros of g. We find that

$$d \geqslant K+L+P+3M+2(N-M) = K+L+P+M+2N. \tag{4.2}$$

Denote by $\xi_1,\dots,\xi_N$ the multiple zeros of g, with multiplicities $t_1,\dots,t_N$. The above considerations based on Lemma 2.1 show that there are at least $(n+1)t_j-3$ non-real critical values of G contained in the Leau domains associated to ξ_j. Also, for a non-real simple zero of g each of the n associated Leau domains contains a non-real critical value of G. Overall the number ν of non-real critical points of G thus satisfies

$$\begin{aligned}\nu &\geqslant (n+1)\sum_{j=1}^{N} t_j - 3N + K(n-2) + Ln + Pn \\ &= (n+1)(d-K-L-P) - 3N + K(n-2) + Ln + Pn \\ &= (n-1)d + 2d - 3N - 3K - L - P.\end{aligned}$$

Using (4.2) and (4.1) we obtain

$$\begin{aligned}\nu &\geqslant (n-1)d + 2(K+L+P+M+2N) - 3N - 3K - L - P \\ &= (n-1)d - K + L + 2M + N + P \\ &\geqslant (n-1)d - (L+M+1) + L + 2M + N + P \\ &= (n-1)d - 1 + M + N + P.\end{aligned}$$

The conclusion follows since $M,N,P \geqslant 0$. □

To prove Theorem 1.8 we cannot use the reduction to functions of finite order based on Lemma 3.1, because there exists a non-constant real entire function, namely $h(z)=z$, with the property that all solutions of the equation $h'(z)h(z)^2=1$ are real. We need instead a direct proof that g is of finite order.

Proof of Theorem 1.8. In view of Corollary 1.6 it is enough to consider the case $n=2$. We can assume without loss of generality that $c=1$.

Suppose that the equation $g'(z)g(z)^2=1$ has finitely many non-real solutions. First we show that g has order at most 1. To accomplish this, we need the characteristic function in the upper half-plane as developed by Tsuji [**17**] and Levin and Ostrovskii [**9**] (see also [**5**]), and as used in [**4**]. For ψ meromorphic and non-constant in the closed upper half-plane $\operatorname{Im} z \geqslant 0$ and for $t \geqslant 1$ let $\mathfrak{n}(t,\psi)$ be the number of poles of ψ, counting multiplicity, in $\{z : |z-\mathrm{i}t/2| \leqslant t/2,\ |z| \geqslant 1\}$, and set

$$\mathfrak{N}(r,\psi) = \int_1^r \frac{\mathfrak{n}(t,\psi)}{t^2}\,\mathrm{d}t, \quad r \geqslant 1.$$

The Tsuji characteristic is

$$\mathfrak{T}(r,\psi) = \mathfrak{m}(r,\psi) + \mathfrak{N}(r,\psi),$$

where

$$\mathfrak{m}(r,\psi)=\frac{1}{2\pi}\int_{\sin^{-1}(1/r)}^{\pi-\sin^{-1}(1/r)}\frac{\log^+|\psi(r\sin\theta\mathrm{e}^{\mathrm{i}\theta})|}{r\sin^2\theta}\,\mathrm{d}\theta.$$

We refer the reader to [**4**, **5**, **9**, **17**] for the fundamental properties of the Tsuji characteristic, but note in particular that the lemma on the logarithmic derivative [**9**, p. 332] (see also [**5**, Theorem 3.2, p. 141]) gives

$$\mathfrak{m}(r,\psi'/\psi)=O(\log r+\log^+\mathfrak{T}(r,\psi)) \tag{4.3}$$

as $r\to\infty$ outside a set of finite measure.

The following lemma is a direct analogue for g of a result of Hayman from [**6**].

Lemma 4.1. *We have*

$$\mathfrak{T}(r,g)=O(\log r),\quad r\to\infty.$$

Proof. We follow Hayman's proof as in [**6**], but using the Tsuji characteristic and in particular (4.3). Let

$$\phi(z)=\tfrac{1}{3}g(z)^3.$$

Then ϕ has finitely many poles, and $\phi'-1$ has finitely many zeros in the open upper half-plane H, and so

$$\mathfrak{N}(r,\phi)+\mathfrak{N}\left(r,\frac{1}{\phi'-1}\right)=O(1). \tag{4.4}$$

Milloux's inequality [**7**, Theorem 3.2, p. 57] translates directly in terms of the Tsuji characteristic to give, using (4.3) and (4.4), outside a set of finite measure,

$$\mathfrak{T}(r,\phi)<\mathfrak{N}\left(r,\frac{1}{\phi}\right)-\mathfrak{N}_0\left(r,\frac{1}{\phi''}\right)+O(\log r+\log^+\mathfrak{T}(r,\phi)) \tag{4.5}$$

in which $\mathfrak{N}_0(r,1/\phi'')$ counts only zeros of ϕ'' which are not multiple zeros of $\phi'-1$. But all zeros of ϕ have multiplicity at least 3 and so are zeros of ϕ'' but not zeros of $\phi'-1$, and consequently each such zero contributes 2 to

$$\mathfrak{n}\left(r,\frac{1}{\phi}\right)-\mathfrak{n}_0\left(r,\frac{1}{\phi''}\right)$$

but at least 3 to $\mathfrak{n}(r,1/\phi)$. Hence (4.5) becomes

$$\begin{aligned}\mathfrak{T}(r,\phi)&<\tfrac{2}{3}\mathfrak{N}(r,1/\phi)+O(\log r+\log^+\mathfrak{T}(r,\phi))\\&<\tfrac{2}{3}\mathfrak{T}(r,\phi)+O(\log r+\log^+\mathfrak{T}(r,\phi)).\end{aligned}$$

Thus $3\mathfrak{T}(r,g)\leqslant\mathfrak{T}(r,\phi)+O(\log r)=O(\log r)$ initially outside a set of finite measure, and hence without exceptional set since $\mathfrak{T}(r,g)$ differs from a non-decreasing function by a bounded term [**17**] (see also [**4**, p. 980]). $\square$

Lemma 4.2. *The function g has order at most* 1.

Proof. This proof is almost identical to [**4**, Lemma 3.2] and to corresponding arguments in [**9**]. Lemma 4.1 and an inequality of Levin and Ostrovskii [**9**, p. 332] (see also [**4**, Lemma 2.2]) give

$$\int_R^\infty \frac{m_{0\pi}(r,g)}{r^3}\,\mathrm{d}r \leqslant \int_R^\infty \frac{\mathfrak{m}(r,g)}{r^2}\,\mathrm{d}r = O\biggl(\frac{\log R}{R}\biggr), \quad R \to \infty, \tag{4.6}$$

in which

$$m_{0\pi}(r,g) = \frac{1}{2\pi}\int_0^\pi \log^+ |g(r\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta.$$

But g is real on the real axis and has finitely many poles and so

$$T(r,g) = m(r,g) + O(\log r) = 2m_{0\pi}(r,g) + O(\log r)$$

and (4.6) now gives, as $R \to \infty$,

$$\frac{T(R,g)}{R^2} \leqslant 2\int_R^\infty \frac{T(r,g)}{r^3}\,\mathrm{d}r \leqslant 4\int_R^\infty \frac{\mathfrak{m}(r,g)}{r^2}\,\mathrm{d}r + O\biggl(\frac{\log R}{R^2}\biggr) = O\biggl(\frac{\log R}{R}\biggr),$$

from which the lemma follows. □

The rest of the proof is similar to the proof of Theorem 1.7. However, the arguments simplify considerably since we have restricted our attention to the case $n = 2$ and since we do not have to count the number of non-real critical points as precisely as in the proof of Theorem 1.7.

We put

$$G(z) = z - \tfrac{1}{3}g(z)^3.$$

Then the zeros of G' are exactly the solutions of $g(z)^2g'(z) = 1$, so all critical points of G, with finitely many exceptions, are real. By Lemma 4.2 and the Denjoy–Carleman–Ahlfors theorem, G has at most two finite asymptotic values. Thus the number of non-real singularities of G^{-1} is finite.

The fixed points of G are all multiple and they coincide with zeros of g. If g has finitely many zeros, then $g(z) = p(z)\exp(az)$, where p is a rational function and $a \in \mathbb{R} \setminus \{0\}$. For such g, it is easy to see that the equation $g'(z)g(z)^2 = 1$ has infinitely many non-real solutions.

Hence we may assume that g has infinitely many zeros. A non-real zero of g is a non-real multiple fixed point of G, and Lemma 2.1 implies that the Leau domains associated to it contain a non-real singularity of G^{-1}. A multiple zero of g is a multiple fixed point of G of multiplicity at least 6, and Lemma 2.1 yields that at least three of the associated Leau domains contain a non-real singularity of G^{-1}. As the number of non-real singularities of G^{-1} is finite, we see that only finitely many zeros of g are non-real or multiple, and thus all but finitely many of the zeros of g are real and simple. This implies that there are infinitely many real zeros ζ of g with $g'(\zeta) < 0$. As in the proof of Theorem 1.7 we see that there are at least two non-real singularities of G^{-1} contained in the Leau domains associated to such a point ζ. Since the number of non-real singularities of G^{-1} is finite, we deduce that there are only finitely many real zeros ζ of g satisfying $g'(\zeta) < 0$, a contradiction. □

5. Proof of Theorem 1.9

Let $m \geqslant 3$, let f, G and c be as in the hypotheses, and define g and H by

$$g = \frac{f' + c}{f^m}, \qquad H = \frac{g'}{g+1}.$$

Then

$$\frac{f' + c}{G} = \frac{g}{g+1}, \qquad f'' - \left(\frac{G'}{G}\right)(f' + c) = \frac{g'G}{(g+1)^2} = \frac{f^m g'}{g+1} = f^m H, \tag{5.1}$$

where the second equation in (5.1) is obtained by differentiating the first and multiplying by G. The function g is non-constant, poles of g are zeros of f, and $g(z) = -1$ implies $G(z) = 0$. We may assume that $H \not\equiv 0$, since $H \equiv 0$ implies that g is constant. Let $S(r, f)$ denote any quantity which is $o(T(r, f))$ as $r \to \infty$ outside a set of finite measure. The lemma on the logarithmic derivative then implies that $m(r, H) = S(r, f)$.

Lemma 5.1. *We have, as $r \to \infty$,*

$$(m-1)m(r, f) = m(r, f^{m-1}) \leqslant m\left(r, \frac{1}{H}\right) - \alpha \log r + S(r, f), \tag{5.2}$$

in which $\alpha = 0$ if f is transcendental; $\alpha = 2$ if $f(\infty) = \infty$; $\alpha = 2$ if $f(\infty) \in \mathbb{C} \setminus \{0\}$ and $g(\infty) \neq -1$; $\alpha = 1$ otherwise.

Proof. For transcendental f we apply the method of Clunie's lemma [**7**, Lemma 3.3, p. 68]. We have $|f^{m-1}H| \leqslant |H|$ if $|f| < 1$ and (5.1) yields

$$|f^{m-1}H| \leqslant \left|\frac{f''}{f}\right| + \left|\frac{G'}{G}\right|\left(\left|\frac{f'}{f}\right| + |c|\right)$$

if $|f| \geqslant 1$. Thus

$$|f^{m-1}H| \leqslant |H| + \left|\frac{f''}{f}\right| + \left|\frac{G'}{G}\right|\left(\left|\frac{f'}{f}\right| + |c|\right)$$

in any case and hence

$$m(r, f^{m-1}H) \leqslant m(r, H) + S(r, f) = S(r, f).$$

Now (5.2) follows on writing $f^{m-1} = (f^{m-1}H)/H$.

Suppose now that f is rational. If f has a pole of multiplicity $p \in \mathbb{N}$ at ∞, then g has a zero of multiplicity $p(m-1)+1$ and H a zero of multiplicity $p(m-1)+2$ at ∞, which gives (5.2) with $\alpha = 2$. If $f(\infty) \in \mathbb{C}$ then $m(r, f) = O(1)$ and H has at least a simple zero at ∞, while if $f(\infty) \in \mathbb{C} \setminus \{0\}$ and $g(\infty) \neq -1$, then the zero of H at ∞ is at least double. $\square$

From Jensen's formula and (5.2) we obtain at once, since $m(r, H) = S(r, f)$,

$$(m-1)T(r, f) \leqslant (m-1)N(r, f) - N\left(r, \frac{1}{H}\right) + N(r, H) - \alpha \log r + S(r, f).$$

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Now if f has a pole in $\mathbb{C}$ of multiplicity q, then g has a zero of multiplicity $mq-q-1 \geqslant 1$ and so H has a zero of multiplicity $mq-q-2=(m-1)q-2$. So the contribution to $(m-1)n(r,f)-n(r,1/H)$ from this pole is 2. Thus

$$(m-1)N(r,f)-N\left(r,\frac{1}{H}\right)=2\bar{N}(r,f)-N_0\left(r,\frac{g+1}{g'}\right),$$

where $N_0(r,(g+1)/g')$ counts zeros of $g'/(g+1)$ which are not poles of f. A pole of H is simple, and must be a zero or a pole of $g+1$, and poles of g are zeros of f. Thus

$$N(r,H)\leqslant \bar{N}\left(r,\frac{1}{f}\right)+\bar{N}_1\left(r,\frac{1}{g+1}\right),$$

where $\bar{N}_1(r,1/(g+1))$ counts zeros of $g+1$ which are not zeros of f. Combining the last three formulae yields

$$\begin{aligned}(m-1)T(r,f)\leqslant 2\bar{N}(r,f)-N_0\left(r,\frac{g+1}{g'}\right)+\bar{N}\left(r,\frac{1}{f}\right)\\ +\bar{N}_1\left(r,\frac{1}{g+1}\right)-\alpha\log r+S(r,f).\end{aligned}\qquad(5.3)$$

Write

$$\bar{N}_1\left(r,\frac{1}{g+1}\right)=\bar{N}_{1,\mathrm{R}}\left(r,\frac{1}{g+1}\right)+\bar{N}_{1,\mathrm{NR}}\left(r,\frac{1}{g+1}\right),$$

in which the subscripts 'R', 'NR' denote real and non-real zeros, respectively.

Lemma 5.2. *We have, as $r\to\infty$,*

$$\bar{N}_{1,\mathrm{R}}\left(r,\frac{1}{g+1}\right)\leqslant \bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{g+1}{g'}\right)+\beta\log r+S(r,f),\qquad(5.4)$$

in which $\beta=0$ if f is transcendental; $\beta=-1$ if f is rational and $g(\infty)=-1$; $\beta=1$ if f is rational and $g(\infty)\neq-1$.

Proof. Applying Rolle's theorem shows that between adjacent real zeros x_1, x_2 of $g+1$ there must be at least one point x_0 which is a pole of g or a zero of g' but not a zero of $g+1$. In the first case x_0 is a zero of f, while in the second case x_0 is either a pole of f or contributes to $N_0(r,(g+1)/g')$. If $g(\infty)=-1$ then the same argument may be applied on intervals of form $(-\infty,x_1),(x_2,\infty)$. □

Combining (5.3) and (5.4) yields

$$(m-1)T(r,f)\leqslant 3\bar{N}(r,f)+2\bar{N}\left(r,\frac{1}{f}\right)+\bar{N}_{1,\mathrm{NR}}\left(r,\frac{1}{g+1}\right)+(\beta-\alpha)\log r+S(r,f).\quad(5.5)$$

For transcendental f both assertions of the theorem follow at once from (5.5).

Suppose now that f is a rational function, and that all non-real zeros of $g+1$ are zeros of f. If $f(\infty)=\infty$ then $\beta-\alpha\leqslant-1$ and a contradiction arises from (5.5) if $m\geqslant 6$ or if $m\geqslant 3$ and f is a polynomial. Next, if $f(\infty)=0$ then $\beta-\alpha\leqslant 0$ and so (5.5) gives a contradiction for $m\geqslant 6$ since $\bar{N}(r,1/f)<(1-\varepsilon)T(r,f)$ as $r\to\infty$, for some $\varepsilon>0$. Finally, if $f(\infty)\in\mathbb{C}\setminus\{0\}$ then $\beta-\alpha\leqslant-1$ and again if $m\geqslant 6$ we obtain a contradiction from (5.5). □

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6. Examples

(1) $f(z) = \mathrm{e}^{-z} + 1$ gives $f' + f = 1$ so one cannot take $m = 1$ in Theorem 1.4.

(2) $f(z) = -\tan z$ gives $f' + f^2 = -1$, so one cannot take $m = 2$ in Theorem 1.4.

(3) $f(z) = 1/(2\sin z)$ gives

$$f'(z) + f(z)^3 = \frac{1 - 2\sin(2z)}{8\sin^3 z}$$

and this has only real zeros, so Theorem 1.4 does not hold with $m = 3$.

$g(z) = 1/f(z) = 2\sin z$ gives $g'(z)g(z) - 1 = 2\sin 2z - 1$ which has only real zeros. So one cannot put $n = 1$ in Theorem 1.8.

(4) For $f(z) = a - \tan a^4 z$ with $a = 12$ we obtain

$$w(t) = f' + f^4 = -a^4(1 + t^2) + (a + t)^4, \quad \text{where } t = -\tan a^4 z.$$

Indeed, $w(0) = 0$, $w'(0) > 0$ and $w(\pm\infty) = +\infty$. It follows that w has a negative root. Now $w(1) = -2 \times 12^4 + 13^4 = -12\,911 < 0$ which implies that w has two positive roots. Thus all four roots of w are real. The full preimage of the real line under $-\tan a^4 z$ is contained in the real line, so all roots of $f' + f^4$ are real. So Theorem 1.4 does not hold with $m = 4$.

Another example with this property is given by

$$f(z) = \frac{\sqrt{2} + \tan(4z)}{1 + \sqrt{2}\tan(4z)}$$

so that

$$f'(z) + f(z)^4 = -\frac{\tan^3(4z)(4\sqrt{2} + 7\tan(4z))}{(1 + \sqrt{2}\tan(4z))^4}.$$

Taking $g = 1/f$ we see that Corollary 1.6 does not hold with $n = 2$.

(5) For $f(z) = 1/z$ the function $f'(z) + f(z)^4 = (1 - z^2)/z^4$ has only real zeros, so the condition $m \geqslant 5$ in Theorem 1.3 cannot be weakened to $m \geqslant 4$ in order to conclude that $f' + f^m$ has non-real zeros. Equivalently, $g(z) = 1/f(z) = z$ shows that in Corollary 1.5 the condition $n \geqslant 2$ does not yield the existence of non-real solutions of $g'g^n = 1$.

An example of degree 2 with these properties is given by

$$f(z) = \frac{5z + 3}{2z^2 + 8z + 3}$$

and $g(z) = 1/f(z)$. Here 0 is a zero of $g'g^2 - 1$ of multiplicity 3, and the other three zeros of this function are also real. This can be seen by numerical computation, but can also be deduced from Lemma 2.1, since $G(z) = z - g(z)^3/3$ has four Leau domains associated to the two real zeros of g, each of them containing a critical point, and symmetric about the real axis since $g' \geqslant 0$ on $\mathbb{R}$. If one contained a non-real critical point, then it would also contain the complex conjugate of this point, and this leads to a contradiction since there are only four distinct critical points.

(6) The following examples show that for transcendental meromorphic f one cannot take $m = 5$ or $m = 6$ in Theorem 1.9. Let $f(z) = -\tan z$. Then

$$f'(z) + f(z)^6 + 1 = (\tan z + 1)(\tan z - 1)(\tan^2 z + 1)\tan^2 z$$

has only real zeros.

For the case $m = 5$ let

$$5a^4 - 10a^2 + 1 = 0, \qquad b = 5a - 10a^3, \qquad c = -a^5 - b. \tag{6.1}$$

Set $f(z) = a + \tan(bz) = a + t$. Then

$$G = f' + f^5 + c = t(t^2 + 1)(t^2 + 5at + 10a^2 - 1).$$

Equation (6.1) has a root

$$a = \sqrt{1 - 2/\sqrt{5}} = 0.3249\ldots$$

for which $25a^2 - 4(10a^2 - 1) > 0$, so that all zeros of G are real.

The example

$$f(z) = -16z^2 + 8z + 2, \qquad f'(z) + f(z)^2 - 12 = 256z^3(z - 1)$$

shows that Theorem 1.9 is sharp for polynomial f.

We suspect that Theorem 1.9 is not sharp for non-polynomial rational functions or for transcendental functions with finitely many poles.

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