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THE LIMITING BEHAVIOR OF SEQUENCES OF QUASICONFORMAL MAPPINGS

BY

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ABSTRACT. The limiting behavior of sequences of quasiconformal homeomorphisms of the n-sphere S^n is studied using a substitute to the Poincaré extension of Möbius transformations introduced by Tukia. Adapted versions of the limit set and the conical limit set known in the theory of Kleinian groups are utilized. Most of the results also hold for families of homeomorphisms of S^n with the convergence property introduced by Gehring and Martin.

1. **Introduction.** In [1] we have studied the limiting behavior of sequences of Möbius transformations of the n-sphere S^n. The behavior of the Poincaré extension at any one point in the unit ball B^{n+1} has been found to be crucial for the asymptotics of the sequence on $\bar{B}^{n+1}$. For quasiconformal mappings there is a substitute to the Poincaré extension introduced by Tukia [5]. It turns out that using his extension, practically all the results on Möbius transformations can be transferred to the quasiconformal case.

Let n be a positive integer and $K \geq 1$. By $Q(K, S^n)$ we denote the set of all K-quasiconformal homeomorphisms of S^n, which may be orientation reversing. For the definition of quasiconformality we refer to [8]. In the case $n = 1$, K-quasiconformality is defined using moduli of quadrilaterals as in [5, section 1F]. Our results will be given in Section 3. They are concerned with sequences in $Q(K, S^n)$. Adapted versions of the limit set and the conical limit set known in the theory of Kleinian groups will play a crucial role. Most of the results also hold if $Q(K, S^n)$ is replaced by any family of homeomorphisms of S^n with the so-called convergence property, see the remark at the end of Section 3. The proofs can be found in Section 4. Tukia's extension is explained below, while some useful properties of a projection which is essential for it are listed in Section 2.

Every Möbius transformation of S^n can be extended to a Möbius transformation of the closed hyperbolic space $\bar{B}^{n+1} = B^{n+1} \cup S^n$. For quasiconformal mappings, Tukia has introduced triple space T^n and a continuous projection $p: T^n \longrightarrow B^{n+1}$ as a substitute for hyperbolic space:

$$T^n = \{\, u = (u_1, u_2, u_3) \in (S^n)^3 : \; u_1, u_2, u_3 \text{ distinct} \,\},$$
$$p(u) = \text{orthogonal projection of } u_3 \text{ (in hyperbolic geometry)}$$
$$\text{onto the hyperbolic line joining } u_1 \text{ and } u_2.$$

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By componentwise action, every homeomorphism of S^n extends to one of T^n, which we will denote by the same symbol. The projection p can be extended to a map from $\bar{T}^n = T^n \cup S^n$ onto $\bar{B}^{n+1}$ by setting

$$p(x) = x \quad \text{for } x \in S^n.$$

It has nice properties in connection with quasiconformal maps (see Section 2).

The Tukia extension $\tilde{f}$ of a homeomorphism f of S^n is a continuous map from $\bar{B}^{n+1}$ into itself. To define it we need the following notation. For $X \subset B^{n+1}$ non-empty and bounded in the hyperbolic metric, let $Z(X)$ be the center of the smallest closed hyperbolic ball containing X. The extension is then defined by

$$\tilde{f}(x) = Z(pfp^{-1}(x)), \quad x \in B^{n+1}.$$

Note that by Property 1 in 2.1, $pfp^{-1}(x)$ is a compact subset of B^{n+1}, hence bounded in the hyperbolic metric. If f is K-quasiconformal, then by [5, Theorem 1], $\tilde{f}|B^{n+1}$ is a quasiisometry with respect to the hyperbolic metric, which we denote by d:

$$C^{-1}\, d(x,y) - M \le d(\tilde{f}(x), \tilde{f}(y)) \le C\, d(x,y) + M \qquad \forall x, y \in B^{n+1}$$

for some constants $C = C(n, K)$ and $M = M(n, K)$.

Everywhere in this paper, the projection p could be replaced by its symmetrized variant $\hat{p}$:

$$\begin{aligned}\hat{p}(u) = {} & \text{orthocenter (or barycenter) of the hyperbolic triangle} \\ & \text{with vertices } p(u_1, u_2, u_3), p(u_3, u_1, u_2), p(u_2, u_3, u_1).\end{aligned}$$

Since the Möbius group is triply transitive on S^n, it is easily seen that the three geodesics which go through one component of u and intersect the geodesic joining the other two components orthogonally meet at $\hat{p}(u)$. Also, with $ds = 2|dx|/(1-|x|^2)$ as the hyperbolic length element,

$$d(pu, \hat{p}u) = \frac{1}{2}\log 3 \qquad \forall u \in T^n.$$

Hence, $\hat{p}$ enjoys all the properties of p given in Proposition 2.1.

We shall state our results using just $\bar{T}^n$ and p: For a sequence (g_j) in $Q(K, S^n)$ we consider the mappings $pg_j\colon \bar{T}^n \longrightarrow \bar{B}^{n+1}$. The results could as well be formulated for the extensions $\tilde{g}_j\colon \bar{B}^{n+1} \longrightarrow \bar{B}^{n+1}$; one just has to make the obvious replacements $T^n \longrightarrow B^{n+1}$, $pg \longrightarrow \tilde{g}$, $u_0 \longrightarrow p(u_0)$. Note that for $x \in B^{n+1}$, $u \in p^{-1}(x)$, and $g \in Q(K, S^n)$,

$$d(pg(u), \tilde{g}(x)) \le m(n, K), \tag{1.1}$$

as follows from Proposition 2.1(3).

There is an essential qualitative difference in the behavior of a sequence (g_j) in the cases where $|pg_j(u_0)|$ converges to one and where it does not. Here, u_0 is an arbitrary point in T^n, for instance $u_0 = (-e_1, e_1, e_2)$. Property 3 in 2.1 shows that the condition $|pg_j(u_0)| \longrightarrow 1$ does not depend on the choice of the triple u_0 and that it is equivalent to $|pg_j^{-1}(u_0)| \longrightarrow 1$. The case where the condition is not fulfilled is essentially the uninteresting case, see Lemma 4.2. So we will be mainly concerned with the other case, the virtue of which can be seen, for instance, in Lemma 4.1.

There is another extension to which the results given here apply. Tukia and Väisälä [7] have shown that for every $g \in Q(K, S^n)$ there exists an extension $G: \bar{B}^{n+1} \longrightarrow \bar{B}^{n+1}$ such that G is K_1-quasiconformal and $G|B^{n+1}$ is L-bilipschitz in the hyperbolic metric, where K_1 and L depend only on n, K, and an arbitrary parameter $\varepsilon > 0$. (The disadvantage of this extension is that even with ε chosen, it is not explicit). Using compactness of the set of normalized quasiconformal mappings one shows that the extension satisfies a boundedness condition similar to (1.1). That is why the results hold for this extension as well.

2. **Properties of the projection.** We will need the following properties of the projection p. Recall that d denotes the hyperbolic metric of B^{n+1}.

PROPOSITION 2.1. *The projection $p: \bar{T}^n \longrightarrow \bar{B}^{n+1}$ has the following properties.*
1. If $C \subset B^{n+1}$ is compact, then $p^{-1}(C) \subset T^n$ is compact.
2. There is a universal constant b such that the following hold for $x \in S^n$, $u \in T^n$, $r > 0$:

$$\text{if } |u_i - x| \leq r \text{ for two } i, \text{ then } |pu - x| \leq br.$$
$$\text{if } |u_i - x| > r \text{ for two } i, \text{ then } |pu - x| > r/b.$$

3. There are constants $m = m(n, K)$ and $c = c(n, K)$ such that

$$c^{-1}\, d\,(pu, pv) - m \leq d\,(pf(u), pf(v)) \leq c\, d\,(pu, pv) + m$$

for all $f \in Q(K, S^n)$ and $u, v \in T^n$.
4. Let $L(x, y)$ denote the hyperbolic geodesic joining x and y. Then with the same constants as above,

$$c^{-1}\, d\,(pu, L(x, y)) - m \leq d\,(pf(u), L(fx, fy)) \leq c\, d\,(pu, L(x, y)) + m$$

for all $f \in Q(K, S^n)$, $u \in T^n$, and $x, y \in S^n$, $x \neq y$.
5. There is a continuous increasing function $\phi: [0, 2] \longrightarrow [0, 2]$ such that $\phi(0) = 0$ and

$$|pu - pv| \leq \phi\left(\max_{i=1,2,3} |u_i - v_i|\right) \quad \forall u, v \in \bar{T}^n.$$

For the proof of 1 through 3, see [5, sections 3A, 3C] and [6, section C]. The right hand inequality in 4 follows from the right inequality in 3 by choosing $z \in S^n$ in $v = (x, y, z)$ such that $d\,(pu, L(x, y)) = d\,(pu, pv)$. The left inequality then follows from the right one applied to f^{-1}. The proof of 5 is a simple compactness argument.

3. **Results.** To every (infinite) subset A of $Q(K, S^n)$ we associate a subset of S^n, which is the usual limit set if A is a group. Let $u_0 \in T^n$ be an arbitrary base point.

DEFINITION 3.1. The *limit set* Λ of the set $A \subseteq Q(K, S^n)$ is

$$\Lambda = \{ \, x \in S^n : \inf_{g \in A} |x - pg^{-1}(u_0)| = 0 \, \}$$

By Property 2.1(3), this definition does not depend on the choice of the point $u_0 \in T^n$. Note that $\Lambda = \emptyset$ if A is finite. The following theorem shows that Λ can be considered an analogue of the Julia set in iteration theory.

THEOREM 3.2. *Let A be a subset of $Q(K, S^n)$ and Λ its limit set. Then $\bar{T}^n \setminus \Lambda$ is the maximal open subset of $\bar{T}^n$ where $\{pg : g \in A\}$ is a normal family.*

Equivalently, $\bar{B}^{n+1} \setminus \Lambda$ is the maximal open subset of $\bar{B}^{n+1}$, where $\{ \tilde{g} : g \in A \}$ is normal. ($\tilde{g}$ is Tukia's extension explained in the introduction.) Of course, this is well-known in the case where A is a group and the argument of the transformations is restricted to S^n.

From now on we restrict to *sequences* in $Q(K, S^n)$. We generalize two notions that were introduced in connection with continued fractions, see [3], [4], and [1].

DEFINITION 3.3. Let n be a positive integer, $y \in S^n$, and (g_j) a sequence in $Q(K, S^n)$. We say that (g_j) *converges generally to* y if there exist sequences (u_j) and (v_j) in $\bar{T}^n$ (or S^n) such that

$$\lim pg_j(u_j) = \lim pg_j(v_j) = y \quad \text{and} \quad \liminf |pu_j - pv_j| > 0.$$

We call (g_j) a *restrained sequence* if there exist sequences (u_j) and (v_j) in S^n such that

$$\lim |g_j(u_j) - g_j(v_j)| = 0 \quad \text{and} \quad \liminf |u_j - v_j| > 0.$$

The following characterizes restrained sequences.

THEOREM 3.4. *For a sequence (g_j) in $Q(K, S^n)$ the following are equivalent.*
(a) (g_j) is a restrained sequence.
(b) $|pg_j(u_0)| \longrightarrow 1$
(c) $|pg_j(x) - pg_j(u_0)| \longrightarrow 0$ locally uniformly on $\bar{T}^n \setminus \Lambda$.
(d) For every $\varepsilon > 0$, $|pg_j(x_j) - pg_j(u_0)| \longrightarrow 0$ uniformly for all sequences (x_j) in $\bar{T}^n$ satisfying $|px_j - pg_j^{-1}(u_0)| \geq \varepsilon$ $(\forall j)$.
(e) For every pair of sequences (u_j), (v_j) in $\bar{T}^n$ such that $\liminf |pu_j - pv_j| > 0$, the following holds:

$$\lim_{j \to \infty} \min \{ \, |pg_j(u_j) - pg_j(u_0)|, \, |pg_j(v_j) - pg_j(u_0)| \, \} = 0.$$

If one of the conditions (a) through (e) holds and $pg_j(u_0)$ diverges, then there are at most two points in $\bar{T}^n$, where pg_j converges, and these points lie in Λ.

REMARK. In (e) one could as well require u_j and v_j to be in S^n. Statement (b) shows that (g_j) is a restrained sequence if and only if (g_j^{-1}) is.

A characterization of general convergence is given by the following.

THEOREM 3.5. *If Λ is the limit set of a sequence (g_j) in $Q(K, S^n)$ and $y \in S^n$, then the following statements are equivalent.*

(a) (g_j) converges generally to y.

(b) $\lim pg_j(u_0) = y$.

(c) $pg_j(x) \longrightarrow y$ uniformly in $\bar{T}^n \setminus \Lambda$.

(d) For every $\varepsilon > 0$, $pg_j(x_j) \longrightarrow y$ uniformly for all sequences (x_j) in $\bar{T}^n$ such that $|px_j - pg_j^{-1}(u_0)| \geq \varepsilon$ $(\forall j)$.

(e) For every pair of sequences (u_j), (v_j) in $\bar{T}^n$ such that $\liminf |pu_j - pv_j| > 0$, the following holds:

$$\lim_{j \to \infty} \min\{\, |pg_j(u_j) - y|\,,\ |pg_j(v_j) - y|\,\} = 0.$$

REMARKS. Statement (d) shows that there are always sequences (u_j), (v_j) in S^n satisfying the condition in the definition of general convergence. The equivalence of (a) and (b) shows that a general limit is unique if it exists. Theorems 3.4 and 3.5 together imply that if (g_j) is a restrained sequence, then $(pg_j(x))$, $x \in \bar{T}^n$, takes at most two different limits.

We give two corollaries which have already been proved for the case of continued fractions (which is essentially the case of $Q(1, S^2)$) [3, Theorems 4.3 and 4.8].

COROLLARY 3.6. *If (g_j) converges generally to a point y in S^n and there are sequences (u_j) and (v_j) in $\bar{T}^n$ such that neither $pg_j(u_j)$ nor $pg_j(v_j)$ accumulates at y, then $|pu_j - pv_j| \longrightarrow 0$.*

PROOF. Using (e) in the theorem, the counterassumption is easily seen to be contradictory. ■

COROLLARY 3.7. *If there exist three different points u, v, w in S^n such that*

$$\lim |g_j^{-1}(v) - g_j^{-1}(w)| = 0 \qquad \textit{and} \qquad \liminf |g_j^{-1}(u) - g_j^{-1}(v)| > 0,$$

then (g_j) converges generally to u.

PROOF. We show that $pg_j(u_0)$ converges to u. Assume the contrary and choose a subsequence such that

$$pg_i(u_0) \longrightarrow y \neq u, \quad pg_i^{-1}(u_0) \longrightarrow y' \quad (i \in I).$$

By the first part of the hypothesis and Lemma 4.2, y and y' are in S^n. Lemma 4.1 applied to $(g_i^{-1})_{i \in I}$ yields

$$g_i^{-1}(x) \longrightarrow y' \ (i \in I) \quad \forall x \in S^n \setminus \{y\}.$$

Together with the first part of the hypothesis again, this implies that both, $g_i^{-1}(u)$ and $g_i^{-1}(v)$ converge to y' for $i \in I$. This contradicts the second part of the hypothesis. ■

We have seen that for a restrained sequence, $|pg_j(x) - pg_j(u_0)|$ converges to zero locally uniformly in $\bar{T}^n \setminus \Lambda$. The points $x \in S^n$, where $|g_j(x) - pg_j(u_0)|$ converges to zero can actually be described geometrically. For $x \in S^n$ and $a \in (1, \infty)$, the *Stolz cone at x* is the set

$$C_a(x) = \{ y \in B^{n+1} : |y - x| < a(1 - |y|) \}.$$

DEFINITION 3.8. The *conical limit set* Λ_c of a sequence (g_j) is defined by

$$\Lambda_c = \{ x \in S^n : \exists a \in (1, \infty) \text{ and a subsequence of } (g_j) \text{ such that } pg_i^{-1}(u_0) \text{ converges to } x \text{ inside } C_a(x) \}.$$

By Proposition 2.1(4), this definition is equivalent to saying that $x \in S^n$ belongs to Λ_c if there is a subsequence of (g_j) and a hyperbolic geodesic L with endpoint x such that $pg_i^{-1}(u_0) \longrightarrow x$ and $d(pg_i^{-1}(u_0), L)$ is bounded.

THEOREM 3.9. *Let (g_j) be a restrained sequence in $Q(K, S^n)$ and $x \in S^n$. Then*

$$|g_j(x) - pg_j(u_0)| \longrightarrow 0 \iff x \notin \Lambda_c.$$

COROLLARY 3.10. *Assume (g_j) is a restrained sequence in $Q(K, S^n)$ and* card $(\Lambda) > 1$. *Then $|g_j(x) - pg_j(u_0)|$ diverges for all $x \in \Lambda_c$. In particular, every generally convergent sequence diverges on Λ_c.*

PROOF. Let $x \in \Lambda_c$. Choose a subsequence $(g_i)_{i \in I}$ such that $pg_i^{-1}(u_0) \longrightarrow z \in \Lambda \setminus \{x\}$. Lemma 4.1 implies

$$|g_i(x) - pg_i(u_0)| \longrightarrow 0 \quad (i \in I).$$

But by the theorem, $|g_j(x) - pg_j(u_0)|$ does not converge to zero. ∎

REMARK. There is a more general situation for which most of the results given here are true. In [2] Gehring and Martin have introduced the so-called convergence property: An infinite family of homeomorphisms of S^n has the *convergence property* if every infinite subfamily contains a sequence (f_j) of distinct elements such that either

(A) there is a homeomorphism f such that $f_j \longrightarrow f$ and $f_j^{-1} \longrightarrow f^{-1}$ uniformly or

(B) there exist $x_0, y_0 \in S^n$ such that $f_j(x) \longrightarrow y_0$ and $f_j^{-1}(x) \longrightarrow x_0$ locally uniformly on $S_n \setminus \{x_0\}$ and on $S_n \setminus \{y_0\}$, respectively.

By [2, Theorem 3.2], $Q(K, S^n)$ has the convergence property.

All the results given in the present section, except for Theorem 3.9 and its corollary, remain valid if $Q(K, S^n)$ is replaced by any family of homeomorphisms of S^n with the convergence property.

Of course, in Lemma 4.2, the limit g need not belong to the family. The proofs remain the same except that (4.2) in the proof of Lemma 4.1 is shown using (B) instead of Proposition 2.1(3). (The alternative (A) cannot occur because of $|pg_j(u_0)| \longrightarrow 1$.)

Since the convergence property is invariant under conjugation by an arbitrary homeomorphism and

$$\Lambda_c(hg_jh^{-1}) = h(\Lambda_c(g_j))$$

is *not* true for general homeomorphisms h, Theorem 3.9 does not generalize.

4. **Proofs of the theorems.** From now on we will abbreviate $Q(K, S^n)$ by Q. First we need two lemmas.

LEMMA 4.1. *If* (g_j) *is a sequence in* Q *such that* $\lim |pg_j(u_0)| = 1$, *then for every* $\varepsilon > 0$ *the following holds:*

$$\lim_{j\to\infty} |pg_j(x_j) - pg_j(u_0)| = 0$$

uniformly for all sequences (x_j) *in* $\bar{T}^n$ *such that*

$$|px_j - pg_j^{-1}(u_0)| \geq \varepsilon \quad \forall j. \tag{4.1}$$

In particular, $|pg_j(x) - pg_j(u_0)| \longrightarrow 0$ *locally uniformly in* $\bar{T}^n \setminus \Lambda$.

PROOF. First we show that for all $\varepsilon > 0$,

$$\min\{\, |g_j(x_j) - pg_j(u_0)|,\, |g_j(y_j) - pg_j(u_0)|\,\} \longrightarrow 0 \tag{4.2}$$

uniformly for all pairs of sequences (x_j), (y_j) in S^n such that $|x_j - y_j| \geq \varepsilon$.

We may assume $\varepsilon < 1$. Choose points $z_j \in S^n$ such that $|z_j - x_j| \geq \varepsilon$ and $|z_j - y_j| \geq \varepsilon$. Set $v_j = (x_j, y_j, z_j)$. It follows that the hyperbolic distance $d(pv_j, pu_0)$ is uniformly bounded. By Proposition 2.1(3), the same holds for $d(pg_j(v_j), pg_j(u_0))$, hence $|pg_j(v_j) - pg_j(u_0)|$ converges to zero uniformly for all pairs $((x_j), (y_j))$ in question. Because $pg_j(u_0)$ approaches the boundary of B^{n+1}, (4.2) follows by Proposition 2.1(2).

Let now (x_j) be a sequence in S^n satisfying (4.1). For every j there is a component of u_0, say y_j, such that

$$|y_j - pg_j(u_0)| \geq c_1 > 0 \quad \text{and} \quad |x_j - g_j^{-1}(y_j)| \geq c_2 > 0,$$

where c_2 depends on ε. Applying (4.2) to the pair (x_j), $(g_j^{-1}(y_j))$ yields

$$\min\{\, |g_j(x_j) - pg_j(u_0)|,\, |y_j - pg_j(u_0)|\,\} \longrightarrow 0$$

uniformly for all (x_j) satisfying (4.1). Thus, $|g_j(x_j) - pg_j(u_0)| \longrightarrow 0$ uniformly for all sequences (x_j) in S^n such that (4.1) holds.

The general case $x_j \in \bar{T}^n$ follows from what has been proved, since by (4.1) and Property 2.1(2), at least two components of x_j uniformly stay away from $pg_j^{-1}(u_0)$. ∎

LEMMA 4.2. *Let* (g_j) *be a sequence in* Q *such that* $|pg_j(u_0)|$ *does not converge to one. Then there exists a subsequence* $(g_i)_{i\in I}$ *and an element* g *of* Q *such that*

$$g_i \longrightarrow g, \quad g_i^{-1} \longrightarrow g^{-1} \quad \textit{uniformly on } S^n.$$

PROOF: There is a subsequence such that $pg_i(u_0)$ is bounded in the hyperbolic metric. By [6, Lemma C1] there exists $g \in Q$ such that for a subsequence, $g_i \longrightarrow g$ uniformly on S^n. Since $pg_j^{-1}(u_0)$ is bounded as well, we can choose a subsequence such that also g_i^{-1} converges to an element of Q, which is readily seen to be the inverse of g. ∎

PROOF OF THEOREM 3.4. (a) $\Rightarrow$ (b). Assume $|pg_j(u_0)|$ does not converge to one. Then by Lemma 4.2 there is a subsequence such that $g_i \longrightarrow g \in Q$ uniformly on S^n. Pick a

subsequence again such that also u_i and v_i from Definition 3.3 converge to points u and v in S^n, respectively. The condition $|g_i(u_i) - g_i(v_i)| \longrightarrow 0$ then implies $g(u) = g(v)$. Hence $u = v$, in contradiction to $\liminf |u_j - v_j| > 0$.
(b) $\Rightarrow$ (c). This follows from Lemma 4.1.
(c) $\Rightarrow$ (d). Lemma 4.2 and (c) imply (b), which in turn implies (d) by Lemma 4.1.
(d) $\Rightarrow$ (e). If $\liminf |pu_j - pv_j| > 0$ one can choose a sequence (x_j) such that $x_j \in \{u_j, v_j\}$ $\forall j$ and $\liminf |px_j - pg_j^{-1}(u_0)| > 0$. Then by (d), $|pg_j(x_j) - pg_j(u_0)|$ converges to zero and the assertion follows.
(e) $\Rightarrow$ (a). Choose four different points a, b, c, d on S^n. Setting

$$u_j = \begin{cases} a & \text{if } |g_j(a) - pg_j(u_0)| \leq |g_j(b) - pg_j(u_0)| \\ b & \text{otherwise} \end{cases}$$

and similarly for v_j, where a and b are replaced by c and d, it is clear that $\liminf |u_j - v_j| > 0$, while it follows from (e) that $\lim |g_j(u_j) - g_j(v_j)| = 0$.

Now assume that (b) holds and $pg_j(u_0)$ diverges. There exist two different points $y, w \in S^n$ and subsequences such that

$$pg_i(u_0) \longrightarrow y \quad (i \in I), \qquad pg_l(u_0) \longrightarrow w \quad (l \in L).$$

Extracting subsequences again one can arrange that also

$$pg_i^{-1}(u_0) \longrightarrow y' \quad (i \in I), \qquad pg_l^{-1}(u_0) \longrightarrow w' \quad (l \in L),$$

where $y', w' \in \Lambda$. Now Lemma 4.1 implies

$$pg_i(x) \longrightarrow y \ \ \forall x \in \bar{T}^n \setminus \{y'\} \quad \text{and} \quad pg_l(x) \longrightarrow w \ \ \forall x \in \bar{T}^n \setminus \{w'\}.$$

Thus, $pg_j(x)$ can at most converge for $x = y'$ or $x = w'$. ■

PROOF OF THEOREM 3.5. (a) $\Rightarrow$ (b). Since $pg_j(u_j)$ and $pg_j(v_j)$ both converge to y, statement (e) in Theorem 3.4 implies that the same is true of $pg_j(u_0)$.
(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) follow from Lemma 4.1.
(d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a) are analogous to the proof of the corresponding implications in Theorem 3.4. ■

PROOF OF THEOREM 3.2. a) Let (g_j) be any sequence from A. By choosing a subsequence we may assume that $pg_j^{-1}(u_0)$ converges to some $y \in \bar{B}^{n+1}$. Consider two cases.

(i) If $y \in B^{n+1}$, then $\{pg_j^{-1}(u_0)\}$ is bounded in the hyperbolic metric. By Property 2.1(3), the same holds for $\{pg_j(u_0)\}$. From [6, Lemma C1] it follows that for a subsequence, g_i converges (uniformly in the euclidean metric of S^n) to some $g \in Q$. Thus by Proposition 2.1(5), $pg_i \longrightarrow pg$ uniformly on $\bar{T}^n$.

(ii) If $y \in S^n$, then there is a subsequence such that $pg_i^{-1}(u_0) \longrightarrow y \in \Lambda$ and $pg_i(u_0) \longrightarrow w$ for some $w \in S^n$. Theorem 3.5 implies

$$pg_i(x) \longrightarrow w \quad \text{locally uniformly in } \bar{T}^n \setminus \{y\} \supseteq \bar{T}^n \setminus \Lambda.$$

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b) Let x be an arbitrary point in Λ. There exists a sequence (g_j) in A such that $pg_j^{-1}(u_0) \longrightarrow x$. Thus, every neighborhood of x in S^n contains at least two components of $g_j^{-1}(u_0)$ for j large enough. It follows that $\operatorname{diam}(g_j(V)) > 1$ for every neighborhood $V \subset S^n$ of x and all large j. So, A is not a normal family in any neighborhood of x. ■

PROOF OF THEOREM 3.9. Assume $x \in \Lambda_c$. Let $L(x, y)$ denote the hyperbolic geodesic with endpoints x and y in S^n. There is a subsequence such that $pg_i^{-1}(u_0) \longrightarrow x$ and $d(pg_i^{-1}(u_0), L(x, y))$ is bounded for every $y \in S^n \setminus \{x\}$. By Property 2.1(4), $d(pu_0, L(g_ix, g_iy))$ is also bounded. Thus, $|g_ix - g_iy| \geq \delta(y) > 0 \ \forall y \in S^n \setminus \{x\}$. Since at least two of the components of u_0 are different from x, Proposition 2.1(2) implies that $|g_i(x) - pg_i(u_0)| \geq \delta' > 0$ for all i.

Conversely, if $|g_j(x) - pg_j(u_0)|$ does not converge to zero, then for a subsequence $|g_i(x) - pg_i(u_0)| \geq \varepsilon > 0$ $(\forall i \in I)$. Using Proposition 2.1(2) and taking a suitable subsequence we get

$$|g_i(x) - g_i(y_k)| \geq \varepsilon' > 0 \quad \forall i \in I, \ \ k = 1, 2,$$

where y_1 and y_2 denote two different components of u_0. It follows that $d(pu_0, L(g_ix, g_iy_k))$ is bounded for $k = 1, 2$ and all i. By Property 2.1(4), the same holds for $d(pg_i^{-1}(u_0), L(x, y_k))$. Because $pg_i^{-1}(u_0)$ approaches the boundary at infinity, it must converge to x, and hence, $x \in \Lambda_c$. ■

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