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DILATIONS OF ONE PARAMETER SEMIGROUPS OF POSITIVE CONTRACTIONS ON L^p SPACES

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ABSTRACT. It is proved in this note, that a strongly continuous semigroup of (sub)positive contractions acting on an L^p-space, for $1 < p < \infty$ $p \neq 2$, can be dilated by a strongly continuous group of (sub)positive isometries in a manner analogous to the dilation M. A. Akçoglu and L. Sucheston constructed for a discrete semigroup of (sub)positive contractions. From this an improvement of a von Neumann type estimation, due to R. R. Coifman and G. Weiss, on the transfer map belonging to the semigroup is deduced.

1. **Introduction.** Let $(X, \mathfrak{A}, \mu)$ be a measure space and let, for $1 < p < \infty$, $L^p(X, \mu)$ be the Lebesgue space of equivalence classes of p-integrable functions on X.

For a positive contraction T acting on $L^p(X, \mu)$ M. A. Akçoglu and L. Sucheston constructed in [AS] a dilation:

There exists another measure space $(\Omega, \mathfrak{B}, \nu)$ and a positive invertible isometry S of $L^p(\Omega, \nu)$ such that

$$D \circ T^n = P \circ S^n \circ D, \quad n \in \mathbb{Z}^+,$$

where $D: L^p(X, \mu) \longrightarrow L^p(\Omega, \nu)$ is a suitable positive isometric embedding and $P: L^p(\Omega, \nu) \longrightarrow L^p(\Omega, \nu)$ a positive contractive projection. (Here and in the sequel we call an operator acting between L^p-spaces positive if it is positivity preserving, that is, it maps nonnegative functions to nonnegative ones. Similarly, the phrases, contractive respectively contraction, are used for operators, which do not increase norms.)

R. R. Coifman and G. Weiss in [CW], *cf.* their Theorem 4.16, derived from this a von Neumann type inequality:

Let m be a finitely supported function on $\mathbb{Z}^+$ and let $C > 0$ be such that

$$\Big(\sum_{k=-\infty}^{\infty}\Big|\sum_{l=-\infty}^{\infty} m(l)g(k-l)\Big|^p\Big)^{1/p} \leq C\Big(\sum_{k=-\infty}^{\infty}|g(k)|^p\Big)^{1/p}, \quad g \in l_p(\mathbb{Z}),$$

then

$$\Big\|\sum_{n=0}^{\infty} m(n)T^n(f)\Big\|_p \leq C\big\|f\big\|_p, \quad f \in L^p(X, \mu).$$

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An approximation process then serves for deriving a continuous analogue (see Corollary 4.17, [CW]): Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup of positive contractions on $L^p(X, \mu)$. If m is a function, integrable with respect to the Lebesgue measure λ, with compact support contained in $\mathbb{R}^+$, and if $C \geq 0$ is such that

$$\Big(\int_{\mathbb{R}} |m \star g(y)|^p \, d\lambda(y)\Big)^{1/p} \leq C\Big(\int_{\mathbb{R}} |g(y)|^p \, d\lambda(y)\Big)^{1/p}, \quad g \in L^p(\mathbb{R}\lambda)$$

then

$$\Big\| \int_0^\infty m(t) T_t f \, d\lambda(t) \Big\|_p \leq C\|f\|_p, \quad f \in L^p(X, \mu).$$

Here, for $m \in L^1(\mathbb{R}, \lambda)$ and $g \in L^p(\mathbb{R}, \lambda)$, the convolution product $m \star g$, as an element of $L^p(\mathbb{R}, \lambda)$, is defined by $m \star g(x) = \int_{\mathbb{R}} m(y) g(x - y) \, d\lambda(y)$, $x \in \mathbb{R}$.

For $p = 2$ dilation theorems and a von Neumann inequality hold true under solely the contraction hypothesis on T, we refer the reader to the book of B. Sz.-Nagy and C. Foiaş [SNF] for this. In this note we shall merely be interested in the cases $1 < p < \infty$, $p \neq 2$, and we will first establish a continuous analogue of the Akçoglu-Sucheston dilation theorem:

THEOREM 1. *Let $\{T_t : t \geq 0\}$, with $T_0 = \mathrm{id}$, be a strongly continuous semigroup of positive contractions acting on $L^p(X, \mu)$, where $1 < p < \infty$. Then there exists a measure space $(\Omega, \mathfrak{B}, \nu)$ and a strongly continuous group of isometries $\{S_s : s \in \mathbb{R}\}$, with $S_0 = \mathrm{id}$, acting on $L^p(\Omega, \nu)$ such that*

$$D \circ T_t = P \circ S_t \circ D, \quad t \geq 0,$$

where D is an isometric embedding of $L^p(X, \mu)$ in $L^p(\Omega, \nu)$ and $P\colon L^p(\Omega, \nu) \longrightarrow L^p(\Omega, \nu)$ a contractive projection; further D, P and the isometries $\{S_s : s \in \mathbb{R}\}$ can be chosen to be positive.

Clearly, for $m \in L^1(\mathbb{R}^+, \lambda)$, the integral

$$\int_0^\infty m(t) T_t \, d\lambda(t)$$

is convergent in $B\big(L^p(X, \mu)\big)$, the space of bounded operators on $L^p(X, \mu)$, with respect to the strong operator topology. It defines an algebra homomorphism

$$F\colon L^1(\mathbb{R}^+, \lambda) \longrightarrow B\big(L^p(X, \mu)\big),$$
$$m \longmapsto \int_0^\infty m(t) T_t \, d\lambda(t), \quad m \in L^1(\mathbb{R}^+, \lambda).$$

Interesting applications of Coifman and Weiss's von Neumann type estimates on this transfer map F, to the harmonic analysis of semigroups, are given by M. Cowling in [Co]. As a consequence of our Theorem 1 we shall slightly improve those estimates to obtain p-complete bounded ones (see Section 4 for definitions):

THEOREM 2. *Let $1 < p < \infty$ and $\{T_t : t \geq 0\}$ be as in Theorem 1. If for some $n \in \mathbb{N}$ and some matrix $(m_{i,j})_{i,j=1}^n \in M_n \otimes L^1(\mathbb{R}, \lambda)$, whose entries $m_{i,j} \in L^1(\mathbb{R}, \lambda)$ have support in $\mathbb{R}^+$, there exists a constant $C \geq 0$ such that for any n elements $g_1, \ldots, g_n \in L^p(\mathbb{R}, \lambda)$*

$$\Big(\sum_{i=1}^n \int_{\mathbb{R}} \Big|\sum_{j=1}^n m_{i,j} \star g_j(y)\Big|^p \, d\lambda(y)\Big)^{1/p} \leq C\Big(\sum_{j=1}^n \int_{\mathbb{R}} |g_j(y)|^p \, d\lambda(y)\Big)^{1/p}$$

then for all $f_1, \ldots, f_n \in L^p(X, \mu)$

$$\Big(\sum_{i=1}^n \Big\|\sum_{j=1}^n F(m_{i,j}) f_j\Big\|_p^p\Big)^{1/p} \leq C\Big(\sum_{j=1}^n \|f_j\|_p^p\Big)^{1/p}.$$

As R. Coifman, R. Rochberg and G. Weiss observed in [CRW] these methods cover the more general case of semigroups of subpositive contractions (for the definition of a subpositive contraction see page 54 of [CRW]). In fact, if T is only a subpositive contraction, then in the generalization of the Akçoglu-Sucheston dilation theorem the operator S will be an invertible isometry, P a contractive projection and D an isometric embedding, where P and D are still positive. If we use this as an ingredient for a generalization of our Theorem 1 to the case when the semigroup $\{T_t : t \geq 0\}$ consists of subpositive contractions then, except that the projection P and the operators S_s, $s \in \mathbb{R}$, may only be subpositive contractions, all other assertions of our Theorem 1 remain valid, when $1 < p < \infty$ and $p \neq 2$. For $p = 2$ we neither know whether the, as invertible isometries in this case unitary, operators of the dilating group $\{S_s : s \in \mathbb{R}\}$ can be chosen to be subpositive contractions, nor do we know whether the projection P has any (sub)positivity properties. This is due to the fact that for $p \neq 2$ a precise description of the range of a contractive projection on an L^p-space is available, compare *e.g.* Chapter 6, §17, Theorem 3 of [La], whereas this is not the case for $p = 2$.

The norm estimation in Theorem 2, and its proof is not changed at all under this hypothesis.

2. **An Ultraproduct Construction.**

In this section we apply a Banach space ultraproduct construction to a set of dilations, which exist by the dilation theorem of Akçoglu and Sucheston. We refer the reader to S. Heinrich's survey [He] and to Section 3 of [AS] for definitions and results concerning Banach space ultraproducts.

Throughout this section, as almost always in this note, the letters T S, P, D stand for positive operators between L^p-spaces, D will denote an isometric embedding, P a contractive projection, T will be a contraction and S an invertible isometry.

LEMMA 1. *Let $\mathbb{Q}$ denote the rational numbers. If $\{T_t : t \geq 0\}$ is a semigroup in $B\big(L^p(X, \mu)\big)$, which fulfills the requirements of Theorem 1, then there exists a measure space $(\Omega^\circ, \mathfrak{B}^\circ, \nu^\circ)$ and a group $\{S_s^\circ : s \in \mathbb{Q}\}$ of positive isometries of $L^p(\Omega^\circ, \nu^\circ)$ such that*

$$D^\circ \circ T_s = P^\circ \circ S_s^\circ \circ D^\circ, \quad s \in \mathbb{Q}^+,$$

for a suitable positive isometric embedding $D^\circ\colon L^p(X,\mu) \longrightarrow L^p(\Omega^\circ,\nu^\circ)$ and a positive contractive projection P° acting on $L^p(\Omega^\circ,\nu^\circ)$.

PROOF. For a finite set $B \subset \mathbb{Q}$ let $U_B := \{n \in \mathbb{N} : ns \in \mathbb{Z} \quad \forall s \in B\}$. Then the set of all sets $\{U_B : B \subset \mathbb{Q}, \quad B \text{ finite}\}$ is closed under finite intersections and thus constitutes the basis of some filter $\mathfrak{F}$, which is contained in some ultrafilter $\mathfrak{U}$ on $\mathbb{N}$.

According to Theorem 1.1 of [AS], for any $n \in \mathbb{N}$, there exists a dilation of $\{T_{1/n}^k : k \in \mathbb{Z}^+\}$:

$$D_{1/n} \circ T_{1/n}^k = P_{1/n} \circ S_{1/n}^k \circ D_{1/n}, \quad k \in \mathbb{Z}^+,$$

where $D_{1/n}, P_{1/n}, S_{1/n}$ are as above, $S_{1/n}$ acting on some space $L^p(\Omega_{1/n},\mu_{1/n})$.

If we define, for $s \in \mathbb{Q}$, $S_{n,s} \in B\big(L^p(\Omega_{1/n},\mu_{1/n})\big)$ by

$$S_{n,s} = \begin{cases} S_{1/n}^{ns} & \text{if } ns \in \mathbb{Z} \\ \text{id} & \text{if } ns \notin \mathbb{Z}, \end{cases}$$

and if $B = \{s_1,\dots,s_k\} \subset \mathbb{Q}^+$ is a finite subset, then for $s \in B$ and $n \in U_B$ the following diagram commutes:

$$\begin{array}{ccccc} L^p(X,\mu) & & \xrightarrow{\quad T_{1/n}^{ns} \quad} & & L^p(X,\mu) \\ \big\downarrow D_{1/n} & & & & \big\downarrow D_{1/n} \\ L^p(\Omega_{1/n},\mu_{1/n}) & \xrightarrow{S_{n,s}} & L^p(\Omega_{1/n},\mu_{1/n}) & \xrightarrow{P_{1/n}} & L^p(\Omega_{1/n},\mu_{1/n}). \end{array}$$

We may form ultraproducts of the spaces and operators involved. From [DCK] we know, that there exist measure spaces $(\Omega^\circ,\mathfrak{B}^\circ,\nu^\circ)$ and $(X^\circ,\mathfrak{A}^\circ,\mu^\circ)$ such that

$$\prod_{\mathfrak{U}} L^p(X,\mu) = L^p(X^\circ,\mu^\circ)$$
$$\prod_{\mathfrak{U}} L^p(\Omega_{1/n},\mu_{1/n}) = L^p(\Omega^\circ,\nu^\circ)$$

can be identified as Banach lattices in each of the two cases.

Let I denote the canonical inclusion

$$I\colon L^p(X,\mu) \longrightarrow \prod_{\mathfrak{U}} L^p(X,\mu) = L^p(X^\circ,\mu^\circ)$$

and let denote

$$D^\circ = \prod_{\mathfrak{U}} D_{1/n} \circ I,$$
$$P^\circ = \prod_{\mathfrak{U}} P_{1/n},$$
$$S_s^\circ = \prod_{\mathfrak{U}} S_{n,s}, \quad s \in \mathbb{Q}.$$

Then all the asserted properties of the operators and spaces involved are rather immediate.

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To give examples let us check that

$$S\colon \mathbb{Q} \longrightarrow B\Big(\prod_{\mathfrak{U}} L^p(\Omega_{1/n}, \mu_{1/n})\Big), \quad s \longmapsto S_s,$$

is a group homomorphism and that we obtained a dilation of the semigroup $\{T_s : s \in \mathbb{Q}^+\}$.

If $s, s' \in \mathbb{Q}$ are given and if $f \in \prod_{\mathfrak{U}} L^p(\Omega_{1/n}, \mu_{1/n})$ is represented by a sequence $(f_n)_{n\in\mathbb{N}}$, which is possible by Theorem 3.2 of [AS], then for $n \in U_{\{s,s'\}}$ there holds true:

$$\begin{aligned} S_{n,s+s'}(f_n) &= S_{1/n}^{n(s+s')}(f_n) \\ &= S_{1/n}^{ns}\big(S_{1/n}^{ns'}(f_n)\big) \\ &= S_{n,s}\big(S_{n,s'}(f_n)\big). \end{aligned}$$

Since $\mathfrak{F} \subset \mathfrak{U}$, the sequences

$$\big(S_{n,s+s'}(f_n)\big)_{n\in\mathbb{N}} \text{ and } \Big(S_{n,s}\big(S_{n,s'}(f_n)\big)\Big)_{n\in\mathbb{N}}$$

represent the same elements in $\prod_{\mathfrak{U}} L^p(\Omega_{1/n}, \mu_{1/n})$, and thus

$$S_s^\circ \circ S_{s'}^\circ(f) = S_{s+s'}^\circ(f).$$

Furthermore, for $s \in \mathbb{Q}^+$, the commutativity of the above diagram for $n \in U_{\{s\}}$ implies, as one can see using a reasoning analogical to the one just given,

$$D^\circ \circ T_s = P^\circ \circ S_s^\circ \circ D^\circ.$$

3. **Continuity.** We continue to use the notation set up in the introduction and in the previous section.

If $\mathbb{Q}$ is endowed with the topology it inherits as a subset from $\mathbb{R}$, then the ultraproduct, constructed in the last section $\prod_{\mathfrak{U}} L^p(\Omega_{1/n}, \mu_{1/n}) = L^p(\Omega^\circ, \nu^\circ)$, may be too large for the representation $S^\circ\colon \mathbb{Q} \longrightarrow B\big(L^p(\Omega^\circ, \nu^\circ)\big)$ to be continuous, when $B\big(L^p(\Omega^\circ, \nu^\circ)\big)$ is endowed with the strong operator topology.

We will use here that, for $1 < p < \infty$, the spaces involved are reflexive and we shall restrict the representation S° to the subspace $\big(L^p(\Omega^\circ, \nu^\circ)\big)_c$ of continuously translating elements. That is to the space of those elements $f \in L^p(\Omega^\circ, \nu^\circ)$, such that $s \longmapsto S_s^\circ(f)$ is continuous from $\mathbb{Q}$ to $L^p(\Omega^\circ, \nu^\circ)$ with its norm topology. Furthermore, we use the uniform convexity of L^p to obtain that the dilation structure is not destroyed by restricting.

The arguments we shall use were developed by I. Glicksberg and K. de Leeuw in much wider context in [GdL]. For the convenience of the reader we present a simple approach sufficient for our case.

Let G be a commutative topological group acting by a possibly not continuous but uniformly bounded representation π on a reflexive Banach space E. Thus we are given an algebraic group homomorphism π from G to the invertible elements of $B(E)$ with $C := \sup_{\pi\in G} \|\pi(x)\| < \infty$.

Let $\mathfrak{U}(e)$ be the system of neighbourhoods of the identity e in the topological group G and denote for $U \in \mathfrak{U}(e)$

$$U_\pi = \{\pi(x) : x \in U\}^{-wot}$$

the closure of $\pi(U)$ in the weak operator topology in $B(E)$ and let

$$\Gamma = \bigcap_{U \in \mathfrak{U}(e)} U_\pi.$$

Since G is abelian, $\Gamma \subset B(E)$ is a set of commuting operators and a compact topological space, when endowed with the weak operator topology. Furthermore if E is given its weak topology then the action of Γ on E, *i.e.* the map

$$\begin{aligned} (S, \xi) &\longmapsto S\xi \\ \Gamma \times E &\longrightarrow E \end{aligned}$$

is separately continuous.

We have the following description of the subspace $E_c = \{\xi \in E : x \longmapsto \pi(x)\xi$ is continuous from G to $E\}$ of continuously translating elements of E.

LEMMA 2. *E_c coincides with the set of Γ-fixed points in E.*

PROOF. If $\xi \in E_c$, then for $\epsilon > 0$ there exists an $U \in \mathfrak{U}(e)$ such that $\|\pi(x)\xi - \xi\| < \epsilon$ for all $x \in U$. But then $\|S\xi - \xi\| < \epsilon$ for all $S \in U_\pi$ and it follows that $S\xi = \xi$ for all $S \in \Gamma$.

On the other hand, if $\{x_\alpha\}_{\alpha\in I}$ is a net in G converging to the identity, then the accumulation points, there exits at least one since Γ is compact, of the net of operators $\{\pi(x_\alpha)\}_{\alpha\in I}$ are in Γ. If ξ is fixed by all elements of Γ we hence have

$$\xi = \lim_\alpha \pi(x_\alpha)\xi.$$

For $\xi \in E$ let $\operatorname{conv}\{\pi(U)\xi\}^-$ be the norm closure of the convex hull of $\pi(U)\xi$, which is a bounded set, weakly closed because of its convexity and hence weakly compact in the reflexive space E. Then

$$C_\xi := \bigcap_{U \in \mathfrak{U}(e)} \operatorname{conv}\{\pi(U)\xi\}^-$$

is a nonvoid, convex weakly compact Γ-invariant subset of E reducing to $C_\xi = \{\xi\}$ if $\xi \in E_c$. Furthermore

$$\begin{aligned} C_{\alpha\xi} &= \alpha C_\xi, \quad \alpha \in \mathbb{C}, \xi \in E, \\ C_{\xi+\eta} &\subset C_\xi + C_\eta, \quad \xi, \eta \in E. \end{aligned}$$

PROPOSITION 1. *Let E be a reflexive Banach space, G a commutative topological group and $\pi: G \longrightarrow B(E)$ a uniformly bounded representation of G as above. Then there exists a projection $Q \in B(E)$ with range E_c such that:*

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i) $\pi(x)Q = Q\pi(x), \quad x \in G$,
ii) $\|Q\| \leq \sup_{x\in G} \|\pi(x)\|$,
iii) $Q\xi \in C_\xi, \quad \xi \in E$.

PROOF. The considerations done before the statements of Lemma 2 and of the proposition show that for all $\xi \in E$ the Markov-Kakutani fixed point theorem see *e.g.* Chapter IV, Appendix, Theorem 1 of [Bour], may be applied to the action of Γ on C_ξ. Hence there exists at least one point in C_ξ fixed by all $S \in \Gamma$, which shows $C_\xi \cap E_c \neq \emptyset$.

We claim that $C_\xi \cap E_c$ contains exactly one element. If there are $\eta, \zeta \in C_\xi \cap E_c$ then for any $\epsilon > 0$ there exists $U_0 \in \mathfrak{U}(e)$ such that for all $x \in U_0$

$$\|\pi(x)\eta - \eta\| < \epsilon \text{ and } \|\pi(x)\zeta - \zeta\| < \epsilon,$$

and there are approximations $\tilde{\eta}, \tilde{\zeta} \in \operatorname{conv}\{\pi(U_0)\xi\}$ such that

$$\|\tilde{\eta} - \eta\| < \epsilon \text{ and } \|\tilde{\zeta} - \zeta\| < \epsilon.$$

We may write $\tilde{\eta} = \sum_{i=1}^n \lambda_i \pi(y_i)\xi$ and $\tilde{\zeta} = \sum_{j=1}^m \mu_j \pi(z_j)\xi$ where $y_1, \ldots, y_n, z_1, \ldots, z_m \in U_0$ and $\lambda_1, \ldots, \lambda_n > 0$, $\mu_1, \ldots, \mu_m > 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{j=1}^m \mu_j = 1$. Then

$$\begin{aligned}
\Big\|\eta - \sum_{j=1}^m \mu_j \pi(z_j)\tilde{\eta}\Big\| &\leq \Big\|\eta - \sum_{j=1}^m \mu_j \pi(z_j)\eta\Big\| + \Big\|\sum_{j=1}^m \mu_j \pi(z_j)(\eta - \tilde{\eta})\Big\| \\
&\leq \sum_{j=1}^m \mu_j \sup_j \|\eta - \pi(z_j)\eta\| + \sum_{j=1}^m \mu_j \sup_j \|\pi(z_j)\| \, \|\eta - \tilde{\eta}\| \\
&\leq \epsilon + C\epsilon.
\end{aligned}$$

Similarly

$$\Big\|\zeta - \sum_{i=1}^n \lambda_i \pi(y_i)\tilde{\zeta}\Big\| \leq (1 + C)\epsilon.$$

Since G is commutative

$$\sum_{j=1}^m \mu_j \pi(z_j)\tilde{\eta} = \sum_{j=1}^m \sum_{i=1}^n \mu_j \lambda_i \pi(z_j y_i)\xi = \sum_{i=1}^n \lambda_i \pi(y_i)\tilde{\zeta},$$

coincide and we infer

$$\|\eta - \zeta\| \leq 2(1 + C)\epsilon,$$

which proves the claim since $\epsilon > 0$ is arbitrary.

We may define a map $Q: E \longrightarrow E$ by requiring for $\xi \in E$

$$\{Q\xi\} = E_c \cap C_\xi.$$

By its definition it is clear that $Q\xi \in C_\xi$ for all $\xi \in E$ and the properties of the sets C_ξ mentioned just before the statement of the proposition ensure that Q is a linear projection onto E_c.

That Q is in the commutant of $\pi(G)$ is implied by

$$C_{\pi(x)\xi} = \pi(x)C_\xi, \quad x \in G, \xi \in E,$$

which again is a consequence of the commutativity of G.

Finally the norm estimation ii) is obvious from

$$\sup\{\|\eta\| : \eta \in C_\xi\} \leq \sup_{x\in G} \|\pi(x)\| \, \|\xi\|, \quad \xi \in E.$$

The following lemma is needed to prove that the dilation structure is not destroyed by projecting onto the continuously translating part.

LEMMA 3. *For $f \in L^p(X,\mu)$, $S^\circ \circ D^\circ(f)\colon s \longmapsto S^\circ_s \circ D^\circ(f)$ is continuous from $\mathbb{Q}$ to $L^p(\Omega^\circ, \nu^\circ)$ with its norm topology.*

PROOF. The space $\prod_{\amalg} L^p(\Omega_{1/n}, \mu_{1/n}) = L^p(\Omega^\circ, \nu^\circ)$ is uniformly convex. Thus to $\epsilon > 0$ there exists $\eta_p(\epsilon) > 0$ such that for any $f, h \in L^p(\Omega^\circ, \nu^\circ)\|f\|_p = 1$, $\|h\|_p = 1$ and $\|\frac{1}{2}(f+h)\|_p \geq 1 - \eta_p(\epsilon)$ imply $\|f-h\|_p \leq \epsilon$.

Since $\{S^\circ : s \in \mathbb{Q}\}$ is a group of isometries, it suffices to show, for $f \in L^p(X,\mu)$, the continuity from the right of the map $s \longmapsto S^\circ_s\big(D^\circ(f)\big)$ at $s = 0$.

Given $f \in L^p(X,\mu)$, with $\|f\|_p = 1$, and $\epsilon > 0$ there exists $\delta > 0$ such that $\|T_s f - f\|_p \leq 2\eta_p(\epsilon)$ when $0 \leq s \leq \delta$. Hence, for $s \in \mathbb{Q} \cap [0, \delta)$

$$\begin{aligned}
\left\|S^\circ_s\big(D^\circ(f)\big) + D^\circ(f)\right\|_p &\geq \|P^\circ \circ S^\circ_s \circ D^\circ(f) + P^\circ \circ D^\circ(f)\|_p \\
&= \|D^\circ \circ T_s(f) + D^\circ(f)\|_p = \|T_s(f) + f\|_p \\
&= \left\|2f - \big(f - T_s(f)\big)\right\|_p \geq \|2f\|_p - \|T_s(f) - f\|_p \\
&\geq 2 - 2\eta_p(\epsilon).
\end{aligned}$$

Since $\left\|S^\circ_s\big(D^\circ(f)\big)\right\|_p = \|D^\circ(f)\|_p = 1$, we infer $\left\|S^\circ_s\big(D^\circ(f)\big) - D^\circ(f)\right\|_p \leq \epsilon$.

PROOF OF THEOREM 1. Since $L^p(\Omega^\circ, \nu^\circ)$ is reflexive we may apply Proposition 1 to our representation S of the additive group $\mathbb{Q}$ with its usual topology as a subset of $\mathbb{R}$ to obtain a projection Q with range $Y := \big(L^p(\Omega^\circ, \nu^\circ)\big)_c$ in the commutant of $\{S_s : s \in \mathbb{Q}\}$. It follows from iii) that Q is positive and it is of norm at most one by ii). The above lemma asserts that its range includes $D^\circ\big(L^p(X,\mu)\big)$. Therefore,

$$Q \circ D^\circ \circ T_s = Q \circ P^\circ_{|Y} \circ S^\circ_s \circ Q \circ D^\circ, \quad s \in \mathbb{Q}^+.$$

The range of Q is a sublattice, as follows from Lemma 6, Chapter 6, §17, of Lacey's book [La], and it is closed, since Q is a contractive projection. As an abstract L^p-space it is, by the Bohnenblust-Nakano theorem, see *e.g.* Chapter 5, §15, Theorem 3 of [La], isometrically and order isomorphic to $L^p(\Omega, \nu)$, for some measure space $(\Omega, \mathfrak{B}, \nu)$, by a linear map Φ, say. Define

$$\begin{aligned}
D &= \Phi^{-1} \circ D^\circ, \\
P &= \Phi^{-1} \circ P^\circ{}_{|Y} \circ \Phi, \\
S_s &= \Phi^{-1} \circ Q \circ S^\circ_{s|Y} \circ \Phi, \quad s \in \mathbb{Q}.
\end{aligned}$$

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By continuity, the representation $S_{\cdot}$ can be extended to a continuous representation of $\mathbb{R}$, still acting on $L^p(\Omega,\nu)$. By abuse of notation this extension will still be denoted $S_{\cdot}$.

For all $f \in L^p(X,\mu)$ we obtain

$$D \circ T_t(f) = P \circ S_t \circ D(f), \quad t \in \mathbb{R},$$

since both sides are continuous functions of $t \in \mathbb{R}$ and the above equality is valid for the dense subgroup $\mathbb{Q}$.

In the last part of this section we indicate the changes necessary, for proving the analogue of Theorem 1, given a semigroup $\{T_t : t \geq 0\}$ of subpositive contractions. In this case it can be shown, by the same reasoning as above, that Q is a contractive projection. From iii) of our proposition it can be seen to be a subpositive contraction, even. Anyway, in the case $1 < p < \infty$ and $p \neq 2$, the structural description of the range of contractive contractions on L^p-spaces, *cf. e.g.* Chapter 6, §17, Theorem 3 [La], guarantees that, for a direct sum $U \colon L^p(\Omega^\circ,\nu^\circ) \longrightarrow L^p(\Omega^\circ,\nu^\circ)$ of unitary multiplication operators, UY is isometrically and order isomorphic to $L^p(\Omega,\nu)$, for some measure space $(\Omega,\mathfrak{B},\nu)$, by an isomorphism, which we again call Φ. We note that $U_{|Y}$ is invertible and acts as the identity on the range of D, since Q does, as follows from Lemma 3. All we have to do is a further conjugation of $P^\circ{}_{|Y}$ and of $\{Q \circ S^\circ_{s\,|Y} : s \in \mathbb{Q}\}$ with $U_{|Y}$ in the definition of P and of $\{S_s : s \in \mathbb{Q}\}$ before conjugating the respective results with Φ.

If $p = 2$, then the closed subspace Y is isomorphic to $l^2(I)$ for some set I. In this case no assertion on (sub)positivity properties of the involved operators is made and we simply transport the group $\{S_s : s \in \mathbb{Q}\}$ by means of this isomorphism.

The completion to a representation of $\mathbb{R}$ and the last conclusion on the strong continuity of this representation can be done exactly as before.

4. **Estimates on Transference.** For the convenience of the reader we recall the notion of p-complete boundedness from [Fe] and [Pi]. Let for a Banach space E denote $B(E)$ the Banach space of bounded linear operators on E.

For $n \in \mathbb{N}$ and an $n \times n$ matrix $(m_{i,j})_{i,j=1}^n \in M_n \otimes B(E)$ of operators $m_{i,j} \in B(E)$ denote

$$\|(m_{i,j})_{i,j=1}^n\|_{(n)} = \sup\Big\{ \Big(\sum_{i=1}^n \Big\|\sum_{j=1}^n m_{i,j}(g_j)\Big\|^p\Big)^{1/p} : \Big(\sum_{j=1}^n \|g_j\|^p\Big)^{1/p} \leq 1, g_1,\ldots,g_n \in E\Big\}.$$

If $S \subset B(E)$ is a subspace, then we call a linear operator $u \colon S \longrightarrow B(F)$, where F is an other Banach space, p-completely bounded if there exists a finite constant $C > 0$ such that for all $n \in \mathbb{N}$ and for all $(m_{i,j})_{i,j=1}^n \in M_n \otimes B(E)$

$$\Big\|\big(u(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)} \leq C\|(m_{i,j})_{i,j=1}^n\|_{(n)}.$$

We let denote $\|u\|_p$ the least such constant.

Since the results presented in this section, and their proofs, are almost verbatim the same in the more general situation when one is concerned with an amenable locally compact group and a left Haar measure on it, instead of the locally compact abelian (hence amenable) additive group $\mathbb{R}$ with the (translation invariant) Lebesgue measure, we chose this generality and we let denote G a locally compact amenable group endowed with a Haar measure λ on it.

For $m \in L^1(G,\lambda)$ and $g \in L^p(G,\lambda)$ the convolution product $m \star g(x) = \int_G m(y)g(y^{-1}x)\,d\lambda(y)$, $x \in G$, is defined. Further $\Lambda_p\colon m \longmapsto (g \longmapsto m \star g)$ is injective from $L^1(G,\lambda)$ into $B\big(L^p(G,\lambda)\big)$ and we may consider $\Lambda_p\big(L^1(G,\lambda)\big)$ as a normed subspace of $B\big(L^p(G,\lambda)\big)$. A continuous representation of G on $L^p(\Omega,\mu)$ is, by definition, a group homomorphism π mapping G into the invertible operators on $L^p(\Omega,\mu)$, which is continuous, when those are endowed with the strong operator topology. If, furthermore, π is uniformly bounded, *i.e.* $\sup_{x\in G}\|\pi(x)\| < \infty$, then its extension by integration, $\Lambda_\pi\colon L^1(G,\lambda) \longrightarrow B\big(L^p(\Omega,\mu)\big)$, is defined by

$$\Lambda_\pi(m)f = \int_G m(x)\pi(x)f\,d\lambda(x), \quad f \in L^p(\Omega,\mu),\ m \in L^1(G,\lambda),$$

and it is an algebra homomorphism.

THEOREM 3. *Let $\pi\colon G \longrightarrow B\big(L^p(\Omega,\mu)\big)$ be a continuous uniformly bounded representation of G. Then $\Lambda_\pi\colon \Lambda_p\big(L^1(G,\lambda)\big) \longrightarrow B\big(L^p(\Omega,\mu)\big)$ is a p-completely bounded algebra homomorphism with norm $\|\Lambda_\pi\|_p \leq \sup_{x\in G}\|\pi(x)\|^2$.*

PROOF. We apply the amenability of the group G in a manner closely related to a Følner-Leptin condition. This seems to be due to C. Herz [Hz], compare also [CW].

For $1 < p < \infty$ let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$, then for $\alpha \in L^p(G,\lambda)$ and $\beta \in L^q(G,\lambda)$ the convolution $\beta \star \alpha^\vee$, where $\alpha^\vee(x) := \alpha(x^{-1})$, $\quad x \in G$, coincides λ almost everywhere with a continuous function vanishing at infinity and $\|\beta \star \alpha^\vee\|_\infty \leq \|\alpha\|_p\,\|\beta\|_q$.

Since G is amenable, there exist nets $(\alpha_\tau)_{\tau\in\Delta} \subset L^p(G,\lambda)$ and $(\beta_\tau)_{\tau\in\Delta} \subset L^q(G,\lambda)$ with

$$\sup_{\tau\in\Delta}\|\alpha_\tau\|_p \leq 1, \sup_{\tau\in\Delta}\|\beta_\tau\|_q \leq 1$$

such that

$$\lim_{\tau\in\Delta}\beta_\tau \star \alpha_\tau^\vee = 1$$

uniformly on compact sets.

It follows that, whenever $m \in L^1(G,\lambda)$, $f \in L^p(\Omega,\mu)$ and $g \in L^q(\Omega,\mu)$ are given, then there holds true:

$$\begin{aligned}
&\int_\Omega \Lambda_\pi(m)f(\omega)g(\omega)\,d\mu(\omega)\\
&\qquad = \lim_{\tau\in\Delta}\int_\Omega\int_G \beta_\tau \star \alpha_\tau^\vee(x)m(x)\pi(x)f(\omega)g(\omega)\,d\lambda(x)\,d\mu(\omega)\\
&\qquad = \lim_{\tau\in\Delta}\int_\Omega \Lambda_\pi\big((\beta_\tau \star \alpha_\tau^\vee)\cdot m\big)f(\omega)g(\omega)\,d\mu(\omega),
\end{aligned}$$

where $\cdot$ denotes the pointwise product of functions. This is an abuse of the dominated convergence theorem, since Δ might be uncountable. But here we are concerned with one, later with finitely many, integrable functions on G. They all vanish λ almost everywhere outside a σ-compact subgroup, for which one can arrange Δ to be a sequence. Similar arguments justify the use of Fubinis theorem in the argumentations given below.

If we denote by $\pi^t\colon G \longrightarrow B\big(L^q(\Omega,\mu)\big)$ the representation adjoint to π, given by

$$\int_\Omega f(\omega)\big(\pi^t(x)g\big)(\omega)\,d\mu(\omega) = \int_\Omega \big(\pi(x^{-1})f\big)(\omega)g(\omega)\,d\mu(\omega), \quad x \in G,$$

then

$$\int_\Omega \Lambda_\pi(\beta \star \alpha^\vee \cdot m)f(\omega)g(\omega)\,d\mu(\omega) = \int_\Omega \int_G \int_G m(yx)F^\omega(x^{-1})G^\omega(y)\,d\lambda(x)\,d\lambda(y)\,d\mu(\omega),$$

where

$$F^\omega(x) = \alpha(x)\pi(x^{-1})f(\omega), \quad x \in G, \quad \omega \in \Omega$$
$$G^\omega(y) = \beta(y)\pi^t(y^{-1})g(\omega), \quad y \in G, \quad \omega \in \Omega.$$

Finally for $f_1,\ldots,f_n \in L^p(\Omega,\mu)$, $g_1,\ldots,g_n \in L^q(\Omega,\mu)$ and $(m_{i,j})_{i,j=1}^n \in M_n \otimes \Lambda_p\big(L^1(G,\lambda)\big)$ we may estimate, with $F^\omega_{\tau,j}$ and $G^\omega_{\tau,i}$ defined in analogy to the above functions F^ω and G^ω:

$$\begin{aligned}
&\Big|\sum_{i,j=1}^n \int_\Omega \Lambda_\pi(m_{i,j})(x)f_j(\omega)g_i(\omega)\,d\mu(\omega)\Big| \\
&\quad= \lim_{\tau\in\Delta}\Big|\sum_{i,j=1}^n \int_\Omega \Lambda_\pi\big((\beta_\tau \star \alpha_\tau^\vee)\cdot m_{i,j}\big)f_j(\omega)g_i(\omega)\,d\mu(\omega)\Big| \\
&\quad= \lim_{\tau\in\Delta}\Big|\int_\Omega \sum_{i,j=1}^n \int_G\int_G m_{i,j}(yx)F^\omega_{\tau,j}(x^{-1})G^\omega_{\tau,i}(y)\,d\lambda(x)\,d\lambda(y)\,d\mu(\omega)\Big| \\
&\quad\le \lim_{\tau\in\Delta}\int_\Omega \Big\{\Big\|\big(\Lambda_p(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)}\Big(\sum_{j=1}^n \int_G |F^\omega_{\tau,j}(x)|^p\,d\lambda(x)\Big)^{1/p} \\
&\qquad\quad \cdot \Big(\sum_{i=1}^n \int_G |G^\omega_{\tau,i}(y)|^q\,d\lambda(y)\Big)^{1/q}\Big\}\,d\mu(\omega) \\
&\quad\le \Big\|\big(\Lambda_p(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)} \lim_{\tau\in\Delta}\Big\{\Big(\sum_{j=1}^n \int_G |\alpha_\tau(x)|^p \int_\Omega |\pi(x^{-1})f_j(\omega)|^p\,d\mu(\omega)\,d\lambda(x)\Big)^{1/p} \\
&\qquad\quad \cdot \Big(\sum_{i=1}^n \int_G |\beta_\tau(y)|^q \int_\Omega |\pi^t(y^{-1})g_i(\omega)|^q\,d\mu(\omega)\,d\lambda(y)\Big)^{1/q}\Big\} \\
&\quad\le \Big\|\big(\Lambda_p(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)} \lim_{\tau\in\Delta}\Big(\sup_{x\in G}\|\pi(x^{-1})\|\,\|\alpha_\tau\|_p\Big(\sum_{j=1}^n \|f_j\|_p^p\Big)^{1/p} \\
&\qquad\quad \cdot \sup_{y\in G}\|\pi^t(y^{-1})\|\,\|\beta_\tau\|_q\Big(\sum_{i=1}^n \|g_i\|_q^q\Big)^{1/q}\Big) \\
&\quad\le \sup_{x\in G}\|\pi(x)\|^2\,\Big\|\big(\Lambda_p(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)}\Big(\sum_{j=1}^n \|f_j\|_p^p\Big)^{1/p}\Big(\sum_{i=1}^n \|g_i\|_q^q\Big)^{1/q}.
\end{aligned}$$

PROOF OF THEOREM 2. Let $\{T_t : t \ge 0\}$ be the semigroup under consideration. We have to prove that for a matrix $(m_{i,j})_{i,j=1}^n \in M_n \otimes L^1(\mathbb{R}^+,\lambda)$

$$\Big\|\big(F(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)} \le \Big\|\big(\Lambda_p(m_{i,j})\big)_{i,j=1}^n\Big\|_{(n)}.$$

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Let

$$D \circ T_t = P \circ S_t \circ D, \quad t \geq 0,$$

be a dilation according to Theorem 1. Since $\{T_t : t \geq 0\}$ and $\{S_s : s \in \mathbb{R}\}$ are strongly continuous (semi)groups:

$$\begin{aligned} D \circ F(m) &= D \circ \int_0^\infty m(t) T_t \, d\lambda(t) \\ &= \int_0^\infty m(t) D \circ T_t \, d\lambda(t) \\ &= \int_0^\infty m(t) D \circ S_t \circ P \, d\lambda(t) \\ &= D \circ \int_0^\infty m(t) S_t \, d\lambda(t) \circ P \\ &= D \circ \Lambda_S(m) \circ P, \quad m \in L^1(\mathbb{R}^+, \lambda). \end{aligned}$$

Hence, by Theorem 3, applied to the continuous representation S of $\mathbb{R}$

$$\begin{aligned} \left\| \bigl(F(m_{i,j})\bigr)_{i,j=1}^n \right\|_{(n)} &= \left\| \bigl(D \circ F(m_{i,j})\bigr)_{i,j=1}^n \right\|_{(n)} \\ &= \left\| \bigl(D \circ \Lambda_S(m_{i,j}) \circ P\bigr)_{i,j=1}^n \right\|_{(n)} \\ &\leq \left\| \bigl(\Lambda_S(m_{i,j})\bigr)_{i,j=1}^n \right\|_{(n)} \\ &\leq \left\| \bigl(\Lambda_p(m_{i,j})\bigr)_{i,j=1}^n \right\|_{(n)}, \end{aligned}$$

which completes the proof of Theorem 2.

REFERENCES

[AS] M. A. Akçoglu and L. Sucheston, *Dilations of positive contractions on L^p-spaces*, Canad. Math. Bull. **20**(1977), 285–292.

[Bour] N. Bourbaki, *Elements of Mathematics, Topological Vector Spaces, Chapters 1–5*, Springer, Berlin, Heidelberg, New York, 1987.

[CRW] R. R. Coifman, R. Rochberg and G. Weiss, *Applications of transference: The L^p version of von Neumanns inequality and Littlewood-Paley-Stein theory*. In: Linear spaces and Approximation, (eds. P. L. Butzer and B. Sz.-Nagy), Birkhäuser, Basel 1978, 53–67.

[CW] R. R. Coifman and G.Weiss, *Transference methods in analysis*, CBMS regional conference series in mathematics No. 31, Amer. Math. Soc., Providence, R.I., 1977.

[Co] M. Cowling, *Harmonic analysis on semigroups*, Ann. of Math. **117**(1983), 267–283.

[DCK] D. Dacunha Castelle and J.-L. Krivine, *Applications des ultraproduits à l'étude des éspaces et des algèbres de Banach*, Studia Math. XLI (1972), 315–334.

[Fe] G. Fendler, *Herz Schur multipliers and coefficients of uniformly bounded representations, Thesis*, University of Heidelberg, 1987.

[GdL] I. Glicksberg and K. de Leeuw, *The decomposition of certain group representations*, J. Anal. Math. **15**(1965), 135–192.

[He] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313**(1980), 72–104.

[Hz] C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23**(1973), 91–123.

[La] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer, Berlin, Heidelberg, New York, 1974.

[Pi] G. Pisier, *Completely bounded maps between sets of Banach spaces operators*, Indiana Univ. Math. J. **39**(1990), 251–277.

[SNF] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó, Budapest, 1970.

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