BULL. AUSTRAL. MATH. SOC. 86A05
VOL. 8 (1973), 423-433.

The response of lake levels to an unsteady wind stress

B.J. Noye

This paper presents a theoretical investigation into the forced oscillations produced in an elongated lake by wind stresses varying in time. Analysis of the appropriate hydrodynamical equations of motion, in the absence of friction, and the equation of continuity give an estimate of the response function of the longitudinal component of the wind stress onto water level. Two mathematical models are used, one giving an analytical solution and the other requiring numerical methods for solution. The first model assumes that the lake is a homogeneous rectangular body of water and the second uses the mean depth $h(x)$ and area of cross section $A(x)$, considered as functions of distance x directed along the longitudinal axis of the lake.

1. Introduction

There exist many investigations of wind effects on water levels in enclosed basins. For instance, the steady state response of a homogeneous rectangular lake to a uniform steady wind has been treated in [3], [4], [6], [9], [10], [11]. For constant surface wind stress τ_s , the surface slope of the water is given by

$$\frac{\partial\zeta}{\partial x} = \frac{\tau_s+\tau_b}{\rho g H} , \tag{1.1}$$

where

ζ is surface displacement from the undisturbed level,

Received 26 January 1973.

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

τ_b is the magnitude of the shearing stress exerted by the water on the lake floor,

H is the constant depth,

ρ is the water density,

g is the acceleration of gravity.

As a rule, it is assumed that $\tau_b = m\tau_s$ where the constant $m \ll 1$. For perfect streamline flow, Keulegan [4] has shown that the theoretical value of m is 0.5. Thus

(1.2) $$\zeta(x) = \frac{\tau_1}{\rho g H}(x - L/2) ,$$

where L is the length of the rectangular basin and

(1.3) $$\tau_1 = (1+m)\tau_s .$$

A typical analytical solution for response of the surface of an elongated lake to uniformly oscillating wind stress in the absence of frictional effects is given by Haurwitz [2]. The steady-state response of a rectangular basin to the surface wind stress

(1.4) $$\begin{cases} \tau(x,\ t) = \tau_0 \cos\omega t , & t \geq 0 , \\ \qquad\quad\ \ = \tau_0 , & t < 0 , \end{cases}$$

is obtained in the form

(1.5) $$\zeta(x,\ t) = \frac{-4(1+m)\tau_0 L}{\rho g H \pi^2} \sum_{n=1(2)\infty} \frac{1}{n^2\left[1-\beta_n^2/\omega^2\right]}\left\{\cos\beta_n t - \frac{\beta_n^2}{\omega^2}\cos\omega t\right\}\cos\frac{n\pi x}{L} ,$$

where $\beta_n = n\pi\sqrt{gH}/L$. This formula involves considerable computational effort when it is required to find the lake level at any given position as a function of time. A similar result to (1.5) was obtained by Saito [8] but his work, printed in Japanese, has been overlooked by most western workers. As an alternative, a much simpler expression for $\zeta(x,\ t)$ from which one may immediately deduce most features evident in water level data, is derived in Section 3.

A second model, considered in Section 4, assumes that the lake may have gradually varying cross-sections. This model uses the mean depth

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

$h(x)$ and area of cross-section $A(x)$, considered as functions of distance x directed from one end of the lake along its longitudinal axis, and is solved by a special finite difference scheme. Criteria for the numerical stability of this scheme are also derived.

2. Hydrodynamic equations for an elongated lake

The linearised equations of motion and continuity for flow in a channel of gradually varying cross-section are, in the absence of frictional damping,

$$\frac{\partial Q}{\partial t} = -gbh\,\frac{\partial \zeta}{\partial x} + \frac{b}{\rho}(1+m)\tau \ , \tag{2.1}$$

$$\frac{\partial \zeta}{\partial t} = -\,\frac{1}{b}\,\frac{\partial Q}{\partial x} \ , \tag{2.2}$$

where

x	is the coordinate along the lake,
$Q(x, t)$	the volume transport through a vertical section,
$\zeta(x, t)$	the mean level across the section relative to the undisturbed level,
$b(x)$	the width of the surface of the section,
$h(x)$	the mean depth across the section, and
$\tau(x, t)$	is the longitudinal wind stress.

(See, for example, [2].)

Let

$$\begin{cases} \tau(x,\ t) = \tau_0 Re\{e^{i\omega t}\} \ , \\ \zeta(x,\ t) = \zeta(x)\,Re\{e^{i\omega t}\} \ , \\ Q(x,\ t) = Q(x)\,Im\{e^{i\omega t}\} \ . \end{cases} \tag{2.3}$$

Then (2.1) and (2.2) yield

$$\frac{dQ}{dx} = \frac{\omega A \zeta}{h} \ , \tag{2.4}$$

$$\frac{d\zeta}{dx} = \frac{-\omega}{gA}\,Q + \frac{(1+m)}{\rho g h}\,\tau_0 \ , \tag{2.5}$$

where $A(x) = b(x)h(x)$. Obviously, at the ends of the lake $x = 0$ and

$x = L$, one has $Q = 0$.

3. Response function for a rectangular lake of constant depth H

Differentiating (2.5) with respect to x and substituting $\frac{dQ}{dx}$ from (2.4), one obtains the equation

$$(3.1) \qquad \frac{d^2\zeta}{dx^2} + \left\{\frac{\omega}{c}\right\}^2 \zeta = 0$$

with $c^2 = gH$ and boundary conditions

$$(3.2) \qquad \left.\frac{d\zeta}{dx}\right|_{x=0,L} = \frac{(1+m)}{\rho c^2}\,\tau_0 \ .$$

The general solution of (3.1),

$$(3.3) \qquad \zeta(x) = A\cos\left\{\frac{\omega x}{c}\right\} + B\sin\left\{\frac{\omega x}{c}\right\} \ ,$$

gives with (3.2),

$$(3.4) \qquad \begin{cases} A = \dfrac{(1+m)\tau_0}{\rho c\omega}\left\{\dfrac{\cos(\omega L/c)-1}{\sin(\omega L/c)}\right\} \ , \\[2ex] B = \dfrac{(1+m)\tau_0}{\rho c\omega} \ . \end{cases}$$

Thus, finally, one arrives at the solution of the problem

$$(3.5) \qquad \zeta(x,\ t) = \left\{\frac{(1+m)\tau_0}{\rho c\omega\cos(\omega L/2c)}\sin\frac{\omega}{c}\left\{x - \frac{L}{2}\right\}\right\} Re\{e^{i\omega t}\} \ ,$$

which displays several interesting features.

(1) Resonance occurs for

$$(3.6) \qquad \omega_R = (2n+1)\pi L^{-1}\sqrt{gH} \ , \quad n = 0,\ 1,\ 2,\ \ldots \ ,$$

that is, at the odd harmonics of the fundamental seiching frequency, namely

$$\omega_s = p\pi L^{-1}\sqrt{gH} \ , \quad p = 1,\ 2,\ 3,\ \ldots \ ,$$

(see [7]) with nodes in the middle of the lake and out-of-phase antinodes

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

at the ends.

This explains the nature of the results obtained during experiments involving water level measurements in lakes; levels at opposite ends move out of phase with practically no movement of water levels at field stations near the middle.

(2) At any instant, the surface is sinusoidal in shape. For instance, at $t = 0$, we obtain

$$(3.7) \qquad \zeta(x, 0) = \frac{(1+m)\tau_0}{\rho c\omega\cos(\omega L/2c)} \sin\left\{\frac{\omega}{c}(x-L/2)\right\} .$$

(3) The wind effect increases with decreasing depth. For example, at $x = L$,

$$(3.8) \qquad \zeta(L, t) = \frac{(1+m)\tau_0}{\rho c\omega} \tan\left\{\frac{\omega L}{2c}\right\}\cos\omega t ,$$

which increases as $c = \sqrt{gH}$ decreases. One would therefore expect Australia's lakes, all of which are very shallow, to show marked wind effects.

(4) The steady-state solution for a constant uniform wind is obtained as $\omega \to 0$, in equation (3.5), which gives

$$(3.9) \qquad \zeta(x) = \frac{(1+m)\tau_0}{\rho c^2}\left\{x - \frac{L}{2}\right\} ,$$

that is, equation (1.2).

(5) The response function $T_x(\omega)$ at x is given by

$$(3.10) \qquad T_x(\omega) = \zeta(x, t)/\tau(t)$$

$$= \frac{(1+m)}{\rho c\omega\cos(\omega L/2c)} \sin\left\{\frac{\omega}{c}(x-L/2)\right\} ,$$

the gain by

$$(3.11) \qquad G_x(\omega) = |T_x(\omega)| ,$$

and the phase by

$$(3.12) \qquad \theta_x(\omega) = \mathrm{Arg}\{T_x(\omega)\} .$$

However, the response function $T_x(\omega)$ is real, so the gain is equal to the

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

numerical value $T_x(\omega)$ and the phase is zero, when $T_x(\omega)$ is positive, and π when it is negative.

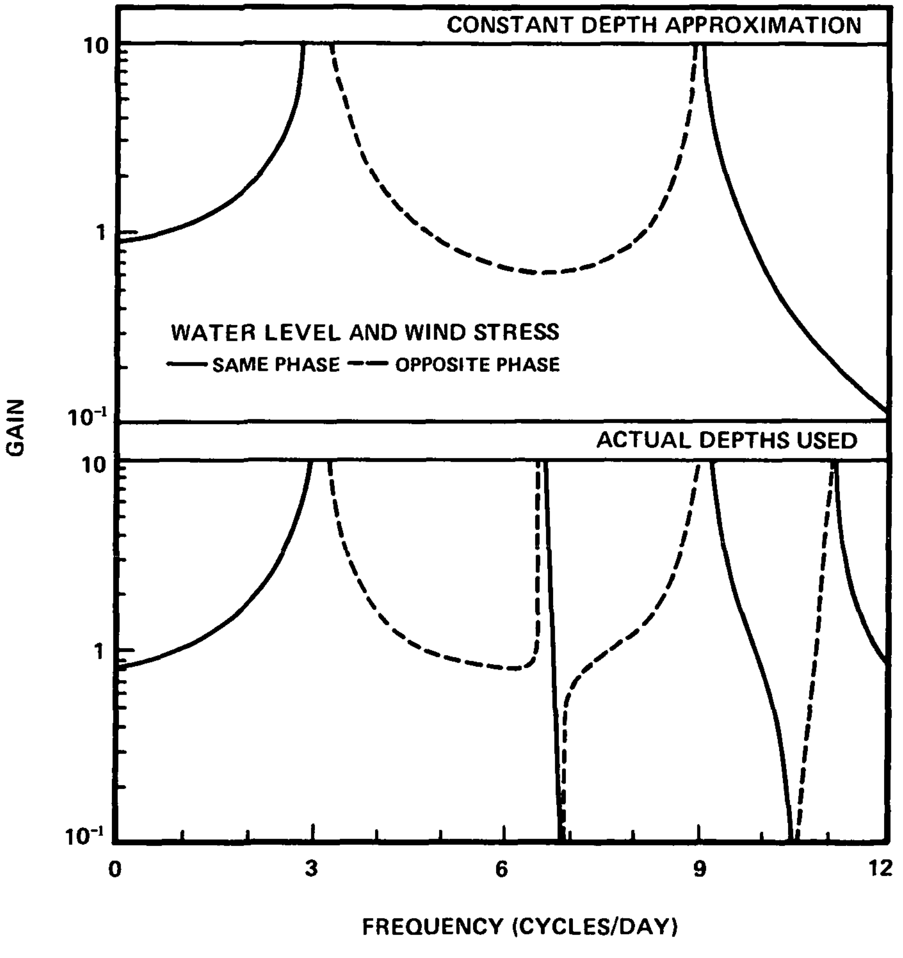

Figure 1. Theoretical response function of wind stress to water level, using constant depth rectangular basin approximation for a lake with $H = 1.3m$, $L = 50km$, $x = 0.75L$, and using actual areas and mean depths of cross sections for a real lake with the same length and mean depth.

Figure 1 shows the theoretical response function for a rectangular lake of constant depth $H = 1.3m$ and length $L = 50km$, the water-level being considered three-quarters the way along the lake, namely at

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

$x = 0.75L$. The first resonance frequency of this mathematical model is approximately 3 cycles/day, corresponding to a period of 8 hours. The next resonance period is 2.7 hours. In Figure 1 the magnitude of the gain is indicated by a solid line when the water-level oscillation is in phase with the wind-stress oscillation, for example, in the range 0-3 cycles/day, and by a dashed line when these quantities are out of phase, for example between 3 and 9 cycles/day.

4. Numerical solution for the case of gradually varying cross-sections

The derivatives $\frac{dQ}{dx}$ and $\frac{d\zeta}{dx}$ in (2.4) and (2.5) may be replaced by centred finite differences and ζ and Q evaluated at alternate sections

(4.1) $$\frac{Q_{j+1}-Q_{j-1}}{2\Delta x} = \omega A_j \zeta_j / h_j \ , \quad j = 1(2)2k-1 \ ,$$

(4.2) $$\frac{\zeta_{j+1}-\zeta_{j-1}}{2\Delta x} = -\frac{\omega}{g} \cdot \frac{Q_j}{A_j} + \frac{(1+m)\tau_0}{\rho g h_j} \ , \quad j = 2(2)2k-2$$

where $L = 2k\Delta x$,

(4.3) $$x_j = j\Delta x \ , \quad j = 0(1)2k \ .$$

The boundary conditions then become

(4.4) $$Q_0 = Q_{2k} = 0 \ .$$

Hence

(4.5) $$Q_{j+1} = Q_{j-1} + 2\Delta x \omega A_j \zeta_j / h_j \ , \quad j = 1(2)2k-1 \ ,$$

(4.6) $$\zeta_{j+1} = \zeta_{j-1} + 2\Delta x \left\{ \frac{(1+m)\tau_0}{\rho g h_j} - \frac{\omega Q_j}{g A_j} \right\} \ , \quad j = 2(2)2k-2 \ .$$

Given τ_0 and ω , the evaluation of Q and ζ commences with $j = 1$ in equation (4.5), which requires that values of Q_0 and ζ_1 be known. $Q_0 = 0$, but ζ_1 is unknown and is found by a process similar to the method of combination of solutions used to solve second-order linear ordinary differential equations with given boundary conditions (see p. 105, [1]). Assigning a value to ζ_1 , say $\zeta_1^{(0)} = 0$, $Q_2^{(0)}$ is computed, then

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

$\zeta_3^{(0)}$ and so on until $Q_{2k}^{(0)}$ is reached. If $Q_{2k}^{(0)} \neq 0$ then the true value of ζ_1 must be estimated. To do this, it is noted that equations (4.5) and (4.6) are linear, so the dependence of Q_{2k} on ζ_1 is of the form

(4.7) $$Q_{2k}(\zeta_1) = C + D\zeta_1 .$$

Clearly

(4.8) $$C = Q_{2k}^{(0)} .$$

The value of D is found by passing through the grid a second time to find $Q_{2k}^{(1)}$ for $\zeta_1^{(1)} = 1$, when

(4.9) $$D = Q_{2k}^{(1)} - Q_{2k}^{(0)} .$$

A third pass through the grid with

(4.10) $$\zeta_1 = -C/D$$

yields the proper distribution of values of Q and ζ at alternate grid-points. It is clear from (4.5) and (4.6) that the depths are only required for $j = 2(2)2k - 2$, that is, at the even grid-points.

Stability criteria for this particular numerical scheme are found by introducing a small error at one step of the computation of Q (or ζ) and choosing Δx so that this error diminishes in magnitude as values of Q (or ζ) are computed at further steps. For instance, we may assume that an error of ΔQ_{j-1} has occurred in the computation of Q_{j-1} , yielding instead

(4.11) $$Q_{j-1}^* = Q_{j-1} + \Delta Q_{j-1} .$$

Substitution of Q_{j-1}^* for Q_{j-1} in equation (4.6) yields

(4.12) $$\zeta_j^* = \zeta_j - \left(2\Delta x \omega \Delta Q_{j-1}/gA_j\right) ,$$

instead of ζ_j . Putting ζ_j^* for ζ_j and Q_{j-1}^* for Q_{j-1} in equation (4.5) then gives

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press

(4.13) $$Q^{*}_{j+1} = Q_{j+1} + \left\{1 - \frac{(2\omega\Delta x)^2}{gh_j}\right\}\Delta Q_{j-1} .$$

The error introduced at this step diminishes if, for all succeeding values of j ,

(4.14) $$\left|1 - \frac{(2\omega\Delta x)^2}{gh_j}\right| < 1 ,$$

that is, if

(4.15) $$\Delta x < (2\omega)^{-1}\sqrt{2gh_m} \text{ , for } m \geq j .$$

Choosing Δx smaller than $(2\omega)^{-1}\sqrt{2gh}$, where $h = \min(h_j)$, therefore implies numerical stability for a given value of ω . The same criterion is found when an error of $\Delta\zeta_j$ is introduced in the computation of ζ_j .

Using the values of ζ at the gridpoint corresponding to $x = 0.75L$ the response function for wind stress to water level was calculated for a real lake of the same average dimensions considered in Section 3 (Figure 1). This more complicated model increased the number of resonance frequencies in the range 0-12 cycles/day from two to four. The first resonance frequency occurs now at 3 cycles/day (8 hours period), the second near 7 cycles/day (about 3.5 hours period), with further harmonics at 9 cycles/day (2.7 hours period) and 11 cycles/day (2.2 hours period), and so on, the phase either being zero or π .

5. Conclusion

The response to unsteady wind stresses of the constant depth lake in the absence of frictional dissipation has been found in an analytic form involving only trigonometrical functions. This result is clearly better than the ones previously found in terms of infinite series; many of the observed properties of movements of water levels in lakes are immediately evident from this analytic solution. Extending the model to include a slowly varying cross-section indicates, to some degree, the manner in which variations in depth and breadth may alter this transfer function.

A common feature of the response functions computed using these two models is the general fall in gain as the frequency increases, except at the resonance frequencies. These resonance frequencies correspond to those of the odd harmonics of the fundamental natural frequency of the lake. The response function for each model gives a low frequency gain of approximate approximately 0.8 , with negligible phase lag. This corresponds to the steady-state response of the rectangular lake to a constant wind of unit stress, described by equation (1.2).

The results for the lake response indicate that the hydrodynamic model chosen may be deficient in some respects. At resonance frequencies, the gain is unbounded and the phase restricted to 0 or π , features which are typical of mathematical models of dynamical systems in which frictional damping is ignored (see [5], p. 134). Inclusion of a frictional damping term in equation (2.1) should lead to more realistic theoretical results; one would expect the discontinuity in the gain to become a peak and the phase to vary over a whole cycle.

References

[1] Ivo Babuška, Milan Práger and Emil Vitásek (in cooperation with R. Radok), *Numerical processes in differential equations* (SNTL, Prague; Interscience [John Wiley & Sons], London, New York, Sydney; 1966).

[2] B. Haurwitz, *The slope of lake surfaces under variable wind stresses* (Technical Memorandum No. 25; Beach Erosion Board, 1951).

[3] B. Hellström, *Wind effect on lakes and rivers* (IngvetenskAkad. Handl. 158, Stockholm, 1941).

[4] Garbis H. Keulegan, "Wind tides in small closed channels", *J. Res. Nat. Bur. Standards* 46 (1951), 358-381.

[5] Erwin Kreyszig, *Advanced engineering mathematics*, 2nd ed. (John Wiley & Sons, New York, London, Sydney, 1967).

[6] H.L. Langhaar, "Wind tides in inland waters", *Proc. First Midwest. Conf. Fluid Dynam.*, May 1950, 278-296 (J.W. Edwards, Ann Arbor, Michigan, 1951).

[7] J. Proudman, *Dynamical oceanography* (Methuen, London; John Wiley & Sons, New York; 1953).

[8] Y. Saito, "A solution to the oscillation of lake water generated by wind" (Japanese), *J. Meteor. Soc. Japan* (2) 27 (1949), 20-25.

[9] T. Saville, Jr, *Wind set-up and waves in shallow water* (Technical Memorandum No. 27; Beach Erosion Board, 1952).

[10] B.A. Tareyev, "Stationary wind set up and circulation in a shallow rectangular basin", *Bull. (Izv.) Acad. Sci. USSR, Geophys. Ser.* 1958, 661-663.

[11] William G. Van Dorn, "Wind stress on an artificial pond", *J. Mar. Res.* 12 (1953), 249-276.

Department of Applied Mathematics,
University of Adelaide,
Adelaide,
South Australia.

https://doi.org/10.1017/S0004972700042714 Published online by Cambridge University Press