INVERSES OF BOOLEAN MATRICES

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1. This note is concerned with square matrices, denoted by capital letters, whose elements belong to a Boolean algebra with null element 0 and all element 1. Such matrices, which have important applications in the theory of electric circuits, can be compounded in the three following ways.

$$A \wedge B = C \quad \Leftrightarrow \quad a_{ij} \cap b_{ij} = c_{ij},$$
$$A \vee B = C \quad \Leftrightarrow \quad a_{ij} \cup b_{ij} = c_{ij},$$
$$AB = C \Leftrightarrow \bigcup_j (a_{ij} \cap b_{jk}) = c_{ik}.$$

More than a quarter of a century ago, J. H. M. Wedderburn [1] showed that the equations (1), (2), (3), (4) of the present paper were necessary and sufficient conditions for the existence of a Boolean matrix X satisfying the matrix equations

$$AX = XA = I,$$

in which I is the Boolean matrix $[\delta_{ij}]$. Such a matrix X, if it exists, is therefore in the Boolean sense a two-sided inverse of A. More recently R. D. Luce [2] has shown that a Boolean matrix A possesses a two-sided inverse if and only if A is an orthogonal matrix in the sense that $AA^T = I$, where A^T denotes the transpose of A, and that, in this case, A^T is a two-sided inverse. The question examined in the following pages is whether a matrix A might possess one-sided inverses or whether $AX = I$ implies $XA = I$.

It will be shown that the latter alternative is the correct one and that consequently one half of Wedderburn's conditions are superfluous. Thus (1) and (2), or alternatively (3) and (4), are necessary and sufficient conditions that the matrix A should possess an inverse.

We define the determinant of a Boolean matrix A to be

$$|A| = \bigcup_\sigma a_{1i_1} \cap \dots \cap a_{ni_n},$$

in which σ ranges over all permutations $i_1, \dots, i_n$ of $1 \dots n$. At first sight this function more closely resembles the permanent $\overset{+}{|} A \overset{+}{|} = \sum_\sigma a_{1i_1} \dots a_{ni_n}$ than the determinant $|A| = \sum_\sigma (\text{sign } \sigma)\, a_{1i_1} \dots a_{ni_n}$ of elementary algebra, but, since the concept of sign is entirely absent in Boolean algebra, such a distinction would be illusory. The Boolean determinant as here defined has many properties reminiscent of elementary algebra. For instance expansion by a row or by a column is permissible. We shall not enlarge on this here but we mention that such determinants have important practical applications in the theory of switching circuits.

Quite a different definition is given by Wedderburn. If we denote the complement of a_{ik} by a'_{ik} and write

$$\bar{a}_{ij} = a_{ij} \cap \left(\bigcap_{k \neq j} a'_{ik} \right),$$

then Wedderburn calls $|\bar{A}|$ the determinant of A, where $\bar{A}$ is the matrix composed of the elements $\bar{a}_{ij}$. This definition has the desirable properties that $|\overline{AB}| = |\bar{A}| \cap |\bar{B}|$ and that $|\bar{A}| = 0$ if two columns of A are identical, but it does not permit expansion by a row or column of A according to the familiar formula. It is stated by Wedderburn that a necessary and sufficient condition that A should possess an inverse is that $|\bar{A}| = 1$.

2. Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

be a square Boolean matrix such that

$$\bigcup_j a_{ij} = 1 \quad (i = 1, \dots, n), \tag{1}$$

$$a_{ij} \cap a_{kj} = 0 \quad (i \neq k). \tag{2}$$

From (1) it follows that

$$\bigcap_i \left(\bigcup_j a_{ij} \right) = 1.$$

We may express the left member of this equation as a union of intersections, a typical intersection being of the form

$$a_{1j_1} \cap a_{2j_2} \cap \dots \cap a_{nj_n}.$$

However, any such term will vanish in virtue of (2) unless the $j_1, \dots, j_n$ are distinct. The remaining terms comprise all the terms and no others of the Boolean determinant of A. Consequently

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = 1.$$

Expanding this determinant by its jth column, we obtain

$$\bigcup_i a_{ij} \geqq \bigcup_i (a_{ij} \cap A_{ij}) = 1.$$

Consequently

$$\bigcup_i a_{ij} = 1 \quad (j = 1, \dots, n). \tag{3}$$

Again, we readily obtain

$$a_{ij} \cap \left(\bigcup_{k \neq i} a_{kj} \right) = 0$$

from (2). This equation taken in conjunction with (3) demonstrates that the complement of a_{ij} is

$$a'_{ij} = \bigcup_{k \neq i} a_{kj}.$$

By taking complements of both sides of (1) we deduce that

$$\bigcap_j \left(\bigcup_{k \neq i} a_{kj} \right) = \bigcap_j a'_{ij} = 0 \quad (i = 1, \ldots, n).$$

We can now write the expression on the left as a union of intersections each of which must vanish and each of which is of the form

$$a_{k_1 1} \cap a_{k_2 2} \cap \ldots \cap a_{k_n n} = 0,$$

in which the $k_1, \ldots, k_n$ cannot all be distinct since none of them can take some particular value i. Any selection of $k_1, \ldots, k_n$ is indeed possible provided we stipulate that at least two of them have the same value. Then the union of all such terms which have $k_1 = k_2 = k$ must vanish and the repeated use of the distributive law yields

$$a_{k1} \cap a_{k2} \cap \left(\bigcup_i a_{i3} \right) \cdots \left(\bigcup_i a_{in} \right) = 0,$$

which, in view of (3), yields

$$a_{k1} \cap a_{k2} = 0.$$

In the same manner we can prove that

$$a_{ki} \cap a_{kj} = 0 \quad (i \neq j). \tag{4}$$

It follows that (3) and (4) are consequences of (1) and (2). Conversely, (1) and (2) are consequences of (3) and (4), as may be seen by repeating the foregoing argument using the transposed matrix A^T in place of A. We have therefore established the following result:

THEOREM. *The relations* (1) *and* (2) *imply and are implied by the relations* (3) *and* (4).

3. We next consider the Boolean matrix equation

$$BC^T = I, \tag{5}$$

where B, C are square Boolean matrices each of order n. For notational convenience it is desirable to write the second factor on the left as C^T rather than C.

The matrix equation (5) implies the following relations between the matrix elements:

$$\bigcup_j (b_{ij} \cap c_{ij}) = 1 \quad (i = 1, \ldots, n); \tag{6}$$

$$\bigcup_j (b_{ij} \cap c_{kj}) = 0 \quad (i \neq k).$$

The second of these yields immediately

$$b_{ij} \cap c_{kj} = 0 \quad (i \neq k). \tag{7}$$

If we now write

$$a_{ij} = b_{ij} \cap c_{ij},$$

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then evidently

$$a_{ij} \cap a_{kj} = b_{ij} \cap c_{ij} \cap b_{kj} \cap c_{kj} = 0 \quad (i \neq k). \tag{8}$$

It is now clear from (6) and (8) that the matrix A, which can be defined as $B \wedge C$, satisfies (1) and (2) and must also satisfy (3) and (4). Furthermore, since $a_{ij} \leqq b_{ij}$ and $a_{ij} \leqq c_{ij}$, we can obtain from (3) the relations

$$\bigcup_i b_{ij} = 1, \quad \bigcup_i c_{ij} = 1 \quad (j = 1, \ldots, n).$$

Then, using (7),

$$\begin{aligned} b_{kj} = b_{kj} \cap \Big(\bigcup_i c_{ij}\Big) &= \bigcup_i (b_{kj} \cap c_{ij}) = b_{kj} \cap c_{kj} \\ &= \bigcup_i (b_{ij} \cap c_{kj}) = \Big(\bigcup_i b_{ij}\Big) \cap c_{kj} = c_{kj}. \end{aligned}$$

It has therefore been established that B and C are the same matrix, and indeed that

$$A = B = C.$$

In other words the matrix equation $BC^T = I$ can only be satisfied if $C = B$ and consequently, if B has a right inverse, this must be B^T. This is only possible if B is an orthogonal matrix in the sense that $BB^T = I$. Furthermore, in this case, (6) and (7) reduce to (1) and (2) which, by the theorem already proved, imply (3) and (4). These in turn imply that $B^T B = I$. Moreover, B^T is the only left inverse of B, for if $KB = I$, then $B = K^T$ and so $K = B^T$. This establishes our main result, which incorporates the theorem of Luce.

THEOREM. *If a Boolean matrix B possesses a one-sided inverse, that inverse is also a two-sided inverse. Furthermore such an inverse, if it exists, is unique and is B^T.*

4. We conclude with a few remarks showing the relationship of the foregoing to Wedderburn's remark concerning $|\bar{A}|$. According to our definition, $\bar{a}_{ij} \leqq a_{ij}$. From the isotone property it is evident that

$$|\bar{A}| \leqq |A| = \bigcup_i a_{ij} \cap A_{ij} \leqq \bigcup_i a_{ij},$$

in which A_{ij} is the cofactor of a_{ij} in $|A|$. Thus $|\bar{A}| = 1$ implies (3). A similar argument shows that $|\bar{A}| = 1$ implies that $\bigcup_j \bar{a}_{ij} = 1$. Now $\bar{a}_{ij} \leqq a'_{ik}$ for all values of j except k, whereas $\bar{a}_{ik} \leqq a'_{il}$ for any l not equal to k. Hence

$$0' = 1 = \bigcup_j \bar{a}_{ij} \leqq a'_{ik} \cup a'_{il} = (a_{ik} \cap a_{il})'.$$

Thus $0 = a_{ik} \cap a_{il}$ if $k \neq l$, which is (4). We have now proved that (3) and (4), and therefore (1) and (2), are consequences of $|\bar{A}| = 1$.

Conversely, (1) and (4) show that $a'_{ij} = \bigcup_{k \neq j} a_{ik}$, whence $a_{ij} = \bigcap_{k \neq j} a'_{ik}$, showing that $\bar{A} = A$.

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On the other hand we proved earlier that (1) and (2) imply that $| A | = 1$. It follows that, if (1), (2), (4) are valid, then

$$| \bar{A} | = | A | = 1.$$

It follows that $| \bar{A} | = 1$ is a necessary and sufficient condition that A should possess an inverse.

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