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Values of the Dedekind Eta Function at Quadratic Irrationalities

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Abstract. Let d be the discriminant of an imaginary quadratic field. Let a, b, c be integers such that

$$b^2 - 4ac = d, \quad a > 0, \quad \gcd(a, b, c) = 1.$$

The value of $|\eta\big((b + \sqrt{d})/2a\big)|$ is determined explicitly, where $\eta(z)$ is Dedekind's eta function

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} (1 - e^{2\pi imz}) \qquad \big(\mathrm{im}(z) > 0\big).$$

1 Introduction

The Dedekind eta function $\eta(z)$ is defined for all complex numbers $z = x + iy$ with $y > 0$ by

$$\eta(z) := e^{\pi iz/12} \prod_{m=1}^{\infty} (1 - e^{2\pi imz}). \tag{1.1}$$

The basic properties of $\eta(z)$ are given for example in [9, pp. 14–22].

Let F be an imaginary quadratic field. Let d denote the discriminant of F. Thus d is a negative integer with $d \equiv 0$ or $1 \pmod 4$, $F = Q(\sqrt{d})$, and the largest positive integer m such that $m^2 \mid d$ and $d/m^2 \equiv 0$ or $1 \pmod 4$ is $m = 1$. Let a, b and c be integers satisfying

$$b^2 - 4ac = d, \quad a > 0, \quad \gcd(a, b, c) = 1. \tag{1.2}$$

Such integers exist as we may take

$$a = 1, \quad b = 0, \quad c = -d/4, \quad \text{if } d \equiv 0 \pmod 4,$$
$$a = 1, \quad b = 1, \quad c = (1 - d)/4, \quad \text{if } d \equiv 1 \pmod 4.$$

The main result of this paper is the explicit determination of $|\eta\big((b + \sqrt{d})/2a\big)|$ (Theorem 9.3). We describe briefly how Theorem 9.3 is proved. Section 2 is devoted to giving

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the basic definitions, notation and results necessary for the rest of the paper. In Section 3 we prove an explicit formula for the number $P_K(n)$ of proper representations of a positive integer n by the class K of the form class group $H(d)$ of discriminant d (Proposition 3.1). In Section 4 it is shown (Proposition 4.1) that the following linear combination of the $P_L(n)$ $(L \in H(d))$

$$W_K(n) := \frac{1}{w(d)} \sum_{L \in H(d)} f(K,L) P_L(n) \quad \text{(see (2.23))} \tag{1.3}$$

is a multiplicative function of n for each $K \in H(d)$. The quantities $w(d)$ $(= 2$, 4 or 6) and $f(K,L)$ (= a certain root of unity) are defined in (2.9) and (2.21) respectively. In Sections 5 and 6 certain infinite products are examined which are needed in later sections. In Sections 7 and 8 the behaviour of $\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s}$ as $s \to 1^+$ is determined (Propositions 7.1, 7.2 and 8.1) in terms of the infinite products discussed in Sections 5 and 6. Inverting (1.3) we obtain

$$P_K(n) = \frac{w(d)}{h(d)} \sum_{L \in H(d)} f(L,K)^{-1} W_L(n), \tag{1.4}$$

see proof of Proposition 9.1. The total number of representations of n by the class K of $H(d)$ is denoted by $R_K(n)$. Clearly

$$R_K(n) = \sum_{e^2 \mid n} P_K(n/e^2) \quad \text{(see (2.5))} \tag{1.5}$$

so that by (1.4) and (1.5)

$$\sum_{n=1}^{\infty} \frac{R_K(n)}{n^s} = \zeta(2s) \sum_{n=1}^{\infty} \frac{P_K(n)}{n^s} = \frac{\zeta(2s) w(d)}{h(d)} \sum_{L \in H(d)} f(L,K)^{-1} \sum_{n=1}^{\infty} \frac{W_L(n)}{n^s}, \tag{1.6}$$

see proof of Proposition 9.1. From (1.6) knowing the behaviour of $\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s}$ as $s \to 1^+$ for each $K \in H(d)$, we can determine the behaviour of $\sum_{n=1}^{\infty} \frac{R_K(n)}{n^s}$ as $s \to 1^+$ in the form

$$\sum_{n=1}^{\infty} \frac{R_K(n)}{n^s} = \frac{2\pi/\sqrt{|d|}}{s-1} + A(K,d) + o(s-1)$$

for a certain constant $A(K,d)$ (Proposition 9.1). On the other hand we can obtain a second representation

$$\sum_{n=1}^{\infty} \frac{R_K(n)}{n^s} = \frac{2\pi/\sqrt{|d|}}{s-1} + B(a,b,c) + o(s-1)$$

for a certain constant $B(a,b,c)$, where $K = [a,b,c]$, by making use of Kronecker's limit formula for $\sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0)}}^{\infty} \frac{1}{(ax^2+bxy+cy^2)^s}$, see Proposition 9.2. The equality $A(K,d) = B(a,b,c)$ gives Theorem 9.3.

Two numerical examples are discussed in Section 9. In Section 10 it is shown that the famous Chowla-Selberg formula for $\prod_{[a,b,c]\in H(d)} a^{-1/4}|\eta\big((b+\sqrt{d})/2a\big)|$ (Theorem 10.1) is a simple consequence of Theorem 9.3. In Section 11 it is proved that the Chowla-Selberg formula for genera (Theorem 11.1) due to Williams and Zhang [10] in 1993 is also a consequence of Theorem 9.3. In Section 12 an expression appearing in the evaluation of $|\eta\big((b+\sqrt{d})/2a\big)|$ given in Theorem 9.3 is discussed.

2 Notation

Let d be the discriminant of an imaginary quadratic field F. Let a, b, c be integers satisfying (1.2). Thus

$$f = f(x,y) = ax^2 + bxy + cy^2$$

is a positive-definite, primitive, integral, binary quadratic form of discriminant d. We call f a form for short and write $f = (a,b,c)$. Let Γ denote the classical modular group, that is,

$$\Gamma := \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} \;\middle|\; r,s,t,u \in Z, \quad ru - st = 1 \right\}.$$

The class of the form $f = (a,b,c)$ is the set of forms

$$[f] = \left\{ f(rx+sy, tx+uy) \;\middle|\; \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \Gamma \right\}.$$

We write $[a,b,c]$ for $[f] = [(a,b,c)]$. With respect to Gaussian composition the set $H(d)$ of classes of forms of discriminant d is a finite abelian group called the form class group. The number $h(d)$ of classes in $H(d)$ is called the form class number. The identity I of the form class group $H(d)$ is the class

$$I = \begin{cases} [1,0,-d/4], & \text{if } d \equiv 0 \pmod 4, \\ \big[1,1,(1-d)/4\big], & \text{if } d \equiv 1 \pmod 4, \end{cases}$$

and the inverse of the class $[a,b,c] \in H(d)$ is the class $[a,b,c]^{-1} = [a,-b,c] \in H(d)$.

A positive integer n is said to be represented by the form $f = (a,b,c)$ if there exist integers x and y such that

$$n = ax^2 + bxy + cy^2. \tag{2.1}$$

A pair (x,y) of integers satisfying (2.1) is called a representation of n by the form (a,b,c). If $\gcd(x,y) = 1$ the representation is said to be proper. As $d < 0$ there are only finitely many representations of n by (a,b,c). The number of representations of n by the form f is denoted by $R_f(n)$ and the number of proper representations by $P_f(n)$. It is well known and easily proved that

$$R_f(n) = \sum_{e^2|n} P_f(n/e^2), \tag{2.2}$$

where e runs through the positive integers whose squares divide n. If the forms f_1 and f_2 belong to the same class $K \in H(d)$ then there is a one-to-one correspondence between the representations of n by f_1 and those of n by f_2, as well as a one-to-one correspondence between the proper representations of n by f_1 and those of n by f_2. Hence if $K \in H(d)$ and $f_1, f_2 \in K$ then $R_{f_1}(n) = R_{f_2}(n)$ and $P_{f_1}(n) = P_{f_2}(n)$. Thus we can define the number $R_K(n)$ of representations of n by the class of K of $H(d)$ by

$$R_K(n) := R_f(n) \quad \text{for any } f \in K \tag{2.3}$$

and the number $P_K(n)$ of proper representations of n by the class K by

$$P_K(n) := P_f(n) \quad \text{for any } f \in K. \tag{2.4}$$

From (2.2), (2.3) and (2.4), we see that

$$R_K(n) = \sum_{e^2 \mid n} P_K(n/e^2) \tag{2.5}$$

for any positive integer n and any $K \in H(d)$. We also set

$$R(n) := \sum_{K \in H(d)} R_K(n), \quad P(n) := \sum_{K \in H(d)} P_K(n), \tag{2.6}$$

so that by (2.5) we have

$$R(n) = \sum_{e^2 \mid n} P(n/e^2). \tag{2.7}$$

It is known that

$$R_K(1) = P_K(1) = \begin{cases} w(d), & \text{if } K = I, \\ 0, & \text{if } K \neq I, \end{cases} \tag{2.8}$$

where

$$w(d) := \begin{cases} 6, & \text{if } d = -3, \\ 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4. \end{cases} \tag{2.9}$$

The definition of the Legendre-Jacobi-Kronecker symbol $(\frac{d}{k})$ for a discriminant d and a positive integer k is recalled in [5, p. 278]. If p is a prime for which $(\frac{d}{p}) = -1$ then $R_K(p) = P_K(p) = 0$ for any class K of $H(d)$. If $(\frac{d}{p}) = +1$ the congruence $t^2 \equiv d \pmod{4p}$ has exactly two solutions satisfying $0 \leq t < 2p$. We let t denote the smaller of these and define the class $K_p \in H(d)$ by

$$K_p = \left[p, t, (t^2 - d)/4p\right]. \tag{2.10}$$

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The prime p is represented by the classes K_p and K_p^{-1} of $H(d)$ and by no others. If $(\frac{d}{p}) = 0$ (equivalently $p \mid d$) then p is represented only by the class

$$K_p = K_p^{-1} = \big[p, \lambda p, (\lambda^2 p^2 - d)/4p\big], \tag{2.11}$$

where

$$\lambda := \begin{cases} 0, & \text{if } p > 2, d \equiv 0 \pmod 4, \\ & \text{or } p = 2, d \equiv 8 \pmod{16}, \\ 1, & \text{if } p > 2, d \equiv 1 \pmod 4, \\ & \text{or } p = 2, d \equiv 12 \pmod{16}. \end{cases} \tag{2.12}$$

Moreover

$$R_K(p) = P_K(p) = \begin{cases} 0, & \text{if } (\frac{d}{p}) = -1, \\ & \text{or } (\frac{d}{p}) = 0 \text{ or } 1 \text{ and } K \neq K_p, K_p^{-1}, \\ w(d), & \text{if } (\frac{d}{p}) = 1 \text{ and } K = K_p \neq K_p^{-1}, \\ & \text{or } (\frac{d}{p}) = 1 \text{ and } K = K_p^{-1} \neq K_p, \\ & \text{or } (\frac{d}{p}) = 0 \text{ and } K = K_p (= K_p^{-1}), \\ 2w(d), & \text{if } (\frac{d}{p}) = 1 \text{ and } K = K_p = K_p^{-1}. \end{cases} \tag{2.13}$$

More generally in Section 3 we give an explicit formula for $P_K(n)$ for any positive integer n and any class K of $H(d)$, see Proposition 3.1. This formula for $P_K(n)$ is given in terms of the quantity $N_K(n)$ defined in Definition 2.1.

Definition 2.1 Let n be a positive integer. Let $K \in H(d)$. Suppose first that $n > 1$. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the prime power decomposition of n, that is, $p_1, \dots, p_r$ are $r(\geq 1)$ distinct primes and $a_1, \dots, a_r$ are positive integers. If $(\frac{d}{p_i}) = -1$ for some i $(1 \leq i \leq r)$ we set $N_K(n) = 0$. If $(\frac{d}{p_i}) = 0$ or 1 for every i $(1 \leq i \leq r)$ we set

$$\begin{aligned} N_K(n) := {} & \text{number of } (\varepsilon_1, \dots, \varepsilon_r) \in \{-1, 1\}^r \text{ such that} \\ & (K_{p_1})^{a_1\varepsilon_1} \cdots (K_{p_r})^{a_r\varepsilon_r} = K. \end{aligned} \tag{2.14}$$

Now suppose that $n = 1$. In this case we set

$$N_K(1) := \begin{cases} 1, & \text{if } K = I, \\ 0, & \text{if } K \neq I. \end{cases} \tag{2.15}$$

We note that

$$\sum_{K \in H(d)} N_K(n) = 2^r. \tag{2.16}$$

We let the type of the finite abelian group $H(d)$ be $(h_1,\dots,h_\ell)$. The positive integers $h_1,\dots,h_\ell$ are called the *invariants of* $H(d)$ and are such that

$$h_1\cdots h_\ell = h(d),\quad 1 < h_1 \mid h_2 \mid \cdots \mid h_\ell. \tag{2.17}$$

Moreover there exist $A_1,\dots,A_\ell \in H(d)$ such that $\operatorname{ord}(A_i) = h_i$ $(i = 1,\dots,\ell)$ and for each $K \in H(d)$ there exist unique integers $k_1,\dots,k_\ell$ such that

$$K = A_1^{k_1}\cdots A_\ell^{k_\ell},\quad 0 \le k_j < h_j \quad (j = 1,\dots,\ell). \tag{2.18}$$

We fix the choice of the generators $A_1,\dots,A_\ell$ of $H(d)$ once and for all. The nonnegative integer k_j is called the index of K with respect to the j-th element of the ordered set of generators $\mathcal{A} = \{A_1,\dots,A_\ell\}$ and is written

$$\operatorname{ind}_{A_j}(K) = k_j \quad (j = 1,\dots,\ell). \tag{2.19}$$

For $K, L, M \in H(d)$ we set

$$[K,L] := \sum_{j=1}^{\ell} \frac{\operatorname{ind}_{A_j}(K)\operatorname{ind}_{A_j}(L)}{h_j}. \tag{2.20}$$

We note that $[K,L] = [L,K]$, $[K,I] = 0$, $[KL,M] \equiv [K,M] + [L,M] \pmod 1$, and $[K^r,L^s] \equiv rs[K,L] \pmod 1$ for integers r and s. Then we define

$$f(K,L) := e^{2\pi i[K,L]}. \tag{2.21}$$

If it is important to indicate the basis $\mathcal{A}$ we write $[K,L]_{\mathcal{A}}$ for $[K,L]$ and $f_{\mathcal{A}}(K,L)$ for $f(K,L)$. Clearly $f(K,L) = f(L,K)$, $f(K,I) = 1$, $f(KL,M) = f(K,M)f(L,M)$ and $f(K^r,L^s) = f(K,L)^{rs}$ for any integers r and s.

Simple calculations show that

$$\begin{cases} \displaystyle\sum_{M\in H(d)} f(K,M) = \begin{cases} h(d), & \text{if } K = I,\\ 0, & \text{if } K \ne I,\end{cases} \\ \displaystyle\sum_{M\in H(d)} f(K,M)f(M,L)^{-1} = \begin{cases} h(d), & \text{if } K = L,\\ 0, & \text{if } K \ne L.\end{cases}\end{cases} \tag{2.22}$$

Next, for a positive integer n and $K \in H(d)$, we define

$$W_K(n) := \frac{1}{w(d)} \sum_{L\in H(d)} f(K,L)P_L(n). \tag{2.23}$$

We note that

$$W_K(1) = \frac{1}{w(d)} \sum_{L\in H(d)} f(K,L)P_L(1) = f(K,I) = 1 \tag{2.24}$$

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and that

$$W_I(n) = \frac{1}{w(d)} \sum_{L\in H(d)} f(I,L)P_L(n) = \frac{1}{w(d)} \sum_{L\in H(d)} P_L(n) = \frac{P(n)}{w(d)}. \tag{2.25}$$

We will also use the following notation:

$$\tau(n) := \text{ number of distinct prime divisors of the positive integer } n \text{ (so that } \tau(1) = 0), \tag{2.26}$$

$$n^* := \text{radical of } n = \prod_{p\mid n} p \quad (\text{so that } 1^* = 1), \tag{2.27}$$

$$\begin{aligned}\gamma := \text{ Euler's constant } &= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)\\ &= 0.57721566 \text{ (approx)},\end{aligned} \tag{2.28}$$

(2.29)

$$\begin{aligned}\zeta(s) := \text{ Riemann zeta function } &= \sum_{n=1}^{\infty} \frac{1}{n^s}\\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} (s > 1) \quad (\text{so that } \zeta(2) = \pi^2/6),\end{aligned}$$

$$\begin{aligned}L(s,d) := &\text{ Dirichlet's } L\text{-series for discriminant } d\\ &= \sum_{n=1}^{\infty} \frac{\left(\frac{d}{n}\right)}{n^s} (s > 0),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\Gamma(x) := \text{ gamma function } &= \int_0^{\infty} t^{x-1}e^{-t}\,dt \quad (x > 0)\\ &(\text{so that } \Gamma(n) = (n-1)!),\end{aligned} \tag{2.31}$$

$$\begin{aligned}t_j(d) &:= \prod_{\substack{p\\ (\frac{d}{p})=j}} \left(1 - \frac{1}{p^2}\right) \quad (j = -1, 0, 1)\\ &(\text{so that } t_{-1}(d)t_0(d)t_1(d) = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}),\end{aligned} \tag{2.32}$$

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$$\ell(K,d) := \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f(K,K_p)}{p}\right) \quad \bigl(K \in H(d)\bigr). \tag{2.33}$$

With regard to (2.33), we note that writing (m,n) for $\gcd(m,n)$ from now on

$$\begin{aligned}\left(\frac{d}{p}\right) = 0 \Rightarrow K_p = K_p^{-1} \\ &\Rightarrow K_p = \prod_{j=1}^{\ell} A_j^{\frac{h_j c_j}{(2,h_j)}}, \quad c_j = 0,\dots,(2,h_j)-1, \quad j = 1,\dots,\ell, \\ &\Rightarrow f(K,K_p) = e^{2\pi i \sum_{j=1}^{\ell} \frac{h_j c_j}{(2,h_j)} k_j/h_j} = (-1)^{\sum_{j=1}^{\ell} \frac{2}{(2,h_j)} k_j c_j} = \pm 1,\end{aligned}$$

so that $\ell(K,d)$ is real. Also

$$\begin{aligned}\ell(K^{-1},d) &= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f(K^{-1},K_p)}{p}\right) \\ &= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f(K,K_p)^{-1}}{p}\right) \\ &= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f(K,K_p)}{p}\right),\end{aligned}$$

as $f(K,K_p) = \pm 1$, so that

$$\ell(K^{-1},d) = \ell(K,d). \tag{2.34}$$

3 Formula for $P_K(n)$

We prove

Proposition 3.1 *Let n be a positive integer and let $K \in H(d)$. Then*

$$P_K(n) = \frac{w(d)}{2^{\tau(n)}} \left(\frac{d}{n/n^*}\right) \left(\sum_{g|n}{}' \left(\frac{d}{g}\right)\right) N_K(n),$$

where the prime ($'$) indicates that the (positive) divisors g of n are restricted to be squarefree.

Proof When $n = 1$ the right hand side of the asserted formula is $w(d)N_K(1)$, which is equal to $P_K(1)$ by (2.8) and (2.15). Thus we may assume that $n > 1$.

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If n has a prime factor q such that $q^2 \mid n$, $q \mid d$ then it is easy to show that n cannot be represented properly by any form of discriminant d so that $P_K(n) = 0$. Set $n = q^\beta n_1$, where $q \nmid n_1$ and $\beta \geq 2$. Then

$$\left(\frac{d}{n/n^*}\right) = \left(\frac{d}{q^{\beta-1} n_1/n_1^*}\right) = \left(\frac{d}{q}\right)^{\beta-1} \left(\frac{d}{n_1/n_1^*}\right) = 0,$$

showing that the asserted formula holds in this case.

Further, if n has a prime factor r such that $(\frac{d}{r}) = -1$ then again it is easy to show that n is not properly represented by any form of discriminant d so that $P_K(n) = 0$. Set $n = r^\gamma n_1$, where $r \nmid n_1$ and $\gamma \geq 1$. Then

$$\sum_{g \mid n}{}' \left(\frac{d}{g}\right) = \left(1 + \left(\frac{d}{r}\right)\right) \sum_{g_1 \mid n_1}{}' \left(\frac{d}{g_1}\right) = 0,$$

so the asserted formula works in this case too.

Hence we may suppose that n has a prime factorization of the form

$$n = p_1^{a_1} \cdots p_r^{a_r} q_1 \cdots q_s,$$

where the p_i $(i = 1, \ldots, r)$ are distinct primes with $(\frac{d}{p_i}) = 1$, the q_i $(i = 1, \ldots, s)$ are distinct primes with $(\frac{d}{q_i}) = 0$ (equivalently $q_i | d$), and the a_i are positive integers. In the ring of integers O_F of $F = Q(\sqrt{d})$ we have (see for example [1, p. 142])

$$p_i O_F = P_i P_i' \quad (i = 1, \ldots, r),$$

where P_i and P_i' are distinct conjugate prime ideals with $N(P_i) = N(P_i') = p_i$, and

$$q_i O_F = Q_i^2 \quad (i = 1, \ldots, s),$$

where Q_i is a self-conjugate prime ideal with $N(Q_i) = q_i$. Thus the prime ideal decomposition of the principal ideal nO_F is

$$nO_F = P_1^{a_1} P_1'^{a_1} \cdots P_r^{a_r} P_r'^{a_r} Q_1^2 \cdots Q_s^2.$$

A nonzero ideal A of O_F is said to be integerfree if $kO_F \mid A$, where k is an integer, implies that $k = \pm 1$. From the prime ideal decomposition of nO_F, we see that all integerfree ideals A of O_F with norm n are given by

$$A = P_1^{a_1} P_1'^{a_1 - u_1} \cdots P_r^{u_r} P_r'^{a_r - u_r} Q_1 \cdots Q_s, \tag{3.1}$$

where each $u_i = 0$ or a_i. For $K = [a, b, c] \in H(d)$ we set

$$I_K(n) := \text{number of integerfree ideals } A \text{ of } O_F \text{ with}$$

$$N(A) = n \text{ and } \overline{A} = \overline{\left[a, \frac{-b + \sqrt{d}}{2}\right]}. \tag{3.2}$$

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Here $\overline{A}$ denotes the class of the ideal A in the ideal class group. We note that $I_K(n)$ is well-defined for if $K = [a, b, c] = [a', b', c']$ then $\overline{[a, \frac{-b+\sqrt{d}}{2}]} = \overline{[a', \frac{-b'+\sqrt{d}}{2}]}$, see for example [2, Theorem 7.7]. From (3.1) and (3.2) we deduce that

$$I_K(n) = \text{number of } r\text{-tuples } (u_1, \ldots, u_r) \text{ with}$$
$$\text{each } u_i = 0 \text{ or } a_i \text{ such that}$$
$$\overline{P}_1^{u_1}\overline{P}_1'^{a_1-u_i} \cdots \overline{P}_r^{u_r}\overline{P}_r'^{a_r-u_r}\overline{Q}_1 \cdots \overline{Q}_s = \overline{\left[a, \frac{-b+\sqrt{d}}{2}\right]}.$$

As P_iP_i' and $P_i^{h(d)}$ are both principal ideals in O_F, we have

$$\overline{P_i}\,\overline{P_i'} = \overline{P_iP_i'} = \overline{P_i^{h(d)}} = (\overline{P}_i)^{h(d)},$$

so that

$$\overline{P_i'} = (\overline{P}_i)^{h(d)-1}, \quad i = 1, \ldots, r.$$

Set

$$\varepsilon_i = \begin{cases} 1, & \text{if } u_i = a_i, \\ -1, & \text{if } u_i = 0. \end{cases}$$

Hence

$$\begin{aligned} \overline{P}_i^{u_i}\overline{P}_i'^{a_i-u_i} &= \overline{P}_i^{u_i+(h(d)-1)(a_i-u_i)} \\ &= \begin{cases} \overline{P}_i^{a_i}, & \text{if } u_i = a_i, \\ (\overline{P}_i)^{-a_i}, & \text{if } u_i = 0, \end{cases} \\ &= \overline{P}_i^{\varepsilon_i a_i}. \end{aligned}$$

Thus

$$I_K(n) = \text{number of } (\varepsilon_1, \ldots, \varepsilon_r) \in \{-1, 1\}^r \text{ such that}$$
$$\overline{P}_1^{\varepsilon_1 a_1} \cdots \overline{P}_r^{\varepsilon_r a_r}\overline{Q}_1 \cdots \overline{Q}_s = \overline{\left[a, \frac{-b+\sqrt{d}}{2}\right]}.$$

As Q_i^2 is principal, $\overline{Q}_i = \overline{Q}_i^{-1}$ so

$$I_K(n) = \frac{1}{2^s}\text{number of } (\varepsilon_1, \ldots, \varepsilon_{r+s}) \in \{-1, 1\}^{r+s} \text{ such that}$$
$$\overline{P}_1^{\varepsilon_1 a_1} \cdots \overline{P}_r^{\varepsilon_r a_r}\overline{Q}_1^{\varepsilon_{r+1}} \cdots \overline{Q}_s^{\varepsilon_{r+s}} = \overline{\left[a, \frac{-b+\sqrt{d}}{2}\right]}.$$

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Let α be the isomorphism between the form class group and the ideal class group given by

$$\alpha\big([a,b,c]\big) = \overline{\left[a, \frac{-b+\sqrt{d}}{2}\right]},$$

see for example [2, Theorem 7.7]. Then

$$\alpha(K_{p_i}) = \alpha\Big(\big[p_i, t_i, (t_i^2 - d)/4p_i\big]\Big) = \overline{\left[p_i, \frac{-t_i+\sqrt{d}}{2}\right]} = \overline{P}_i$$

and

$$\alpha(K_{q_i}) = \alpha\Big(\big[q_i, \lambda_i q_i, (\lambda_i^2 q_i^2 - d)/4q_i\big]\Big) = \overline{\left[q_i, \frac{-\lambda_i q_i+\sqrt{d}}{2}\right]} = \overline{Q}_i,$$

see for example [1, pp. 144–145], so that

$$\begin{aligned} I_K(n) &= \frac{1}{2^s} \text{ number of } (\varepsilon_1, \dots, \varepsilon_{r+s}) \in \{-1,1\}^{r+s} \text{ such that} \\ &\qquad K_{p_1}^{\varepsilon_1 a_1} \cdots K_{p_r}^{\varepsilon_r a_r} K_{q_1}^{\varepsilon_{r+1}} \cdots K_{q_s}^{\varepsilon_{r+s}} = K \\ &= \frac{1}{2^s} N_K(n). \end{aligned}$$

Now each integerfree ideal A of O_F with $N(A) = n$ and $\overline{A} = \overline{[a, \frac{-b+\sqrt{d}}{2}]}$ gives rise to exactly $w(d)$ proper representations of n by $K = [a,b,c]$, see for example [2, pp. 137–142], so that

$$P_K(n) = w(d) I_K(n).$$

Thus

$$\begin{aligned} P_K(n) &= \frac{w(d)}{2^s} N_K(n) \\ &= \frac{w(d)}{2^{r+s}} 2^r N_K(n) \\ &= \frac{w(d)}{2^{r+s}} \left(\frac{d}{p_1^{a_1-1} \cdots p_r^{a_r-1}}\right) \prod_{i=1}^{r} \left(1 + \left(\frac{d}{p_i}\right)\right) \prod_{j=1}^{s} \left(1 + \left(\frac{d}{q_i}\right)\right) N_K(n) \\ &\qquad \left(\text{as } \left(\frac{d}{p_i}\right) = 1, \left(\frac{d}{q_j}\right) = 0\right) \\ &= \frac{w(d)}{2^{\tau(n)}} \left(\frac{d}{n/n^*}\right) \sum_{g \mid n}{}' \left(\frac{d}{g}\right) N_K(n), \end{aligned}$$

as asserted. This completes the proof of Proposition 3.1. ■

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4 $W_K(n)$ is a Multiplicative Function of n

In this section we use Proposition 3.1 to prove

Proposition 4.1 *Let n be a positive integer and let $K \in H(d)$. Then $W_K(n)$ is a multiplicative function of n.*

Proof Let n_1 and n_2 be positive integers with $(n_1, n_2) = 1$. It follows immediately from Definition 2.1 that for all $K \in H(d)$

$$N_K(n_1 n_2) = \sum_{K_1 K_2 = K} N_{K_1}(n_1) N_{K_2}(n_2), \tag{4.1}$$

where the sum is over all pairs (K_1, K_2) of classes of $H(d)$ with $K_1 K_2 = K$.

From (2.23) and Proposition 3.1, we obtain

$$W_K(n) = \frac{1}{2^{\tau(n)}} \left(\frac{d}{n/n^*}\right) \left({\sum_{g|n}}' \left(\frac{d}{g}\right)\right) \sum_{L \in H(d)} f(K, L) N_L(n).$$

Since each of $\frac{1}{2^{\tau(n)}}$, $(\frac{d}{n/n^*})$, $\sum'_{g|n}(\frac{d}{g})$ is a multiplicative function of n, to prove that $W_K(n)$ is a multiplicative function of n, it suffices to show that $\sum_{L \in H(d)} f(K, L) N_L(n)$ is a multiplicative function of n. We have

$$\begin{aligned}
&\sum_{L \in H(d)} f(K, L) N_L(n_1 n_2) \\
&\quad = \sum_{L \in H(d)} f(K, L) \sum_{L_1 L_2 = L} N_{L_1}(n_1) N_{L_2}(n_2) \\
&\quad = \sum_{L \in H(d)} \sum_{L_1 L_2 = L} f(K, L_1 L_2) N_{L_1}(n_1) N_{L_2}(n_2) \\
&\quad = \sum_{L_1 \in H(d)} \sum_{L_2 \in H(d)} f(K, L_1) f(K, L_2) N_{L_1}(n_1) N_{L_2}(n_2) \\
&\quad = \sum_{L_1 \in H(d)} f(K, L_1) N_{L_1}(n_1) \sum_{L_2 \in H(d)} f(K, L_2) N_{L_2}(n_2),
\end{aligned}$$

which completes the proof of

$$W_K(n_1 n_2) = W_K(n_1) W_K(n_2), \quad (n_1, n_2) = 1. \qquad \blacksquare$$

5 Estimation of a Certain Infinite Product

Our aim in this section is to prove the following result. We make use of the ideas in [4, pp. 346–353].

Proposition 5.1 *Let $K \in H(d)$. Let ω be a complex number with $|\omega| = 1$. Then there exists a nonzero complex number $C(K, d, \omega)$ depending only on K, d and ω such that*

$$\prod_{\substack{p \\ (\frac{d}{p})=1 \\ K_p=K}} \left(1 - \frac{\omega}{p^s}\right) = (s-1)^{\omega/2h(d)} C(K, d, \omega)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+,$$

where p runs through prime numbers.

This proposition will be used in the proof of Proposition 6.1. In order to prove Proposition 5.1 we require a number of lemmas. For $x \in R$ and $K \in H(d)$ we set

$$\pi_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} 1,$$

$$\theta_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} \log p,$$

$$\kappa_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} \frac{\log p}{p},$$

$$\lambda_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} \frac{1}{p},$$

where p runs through prime numbers.

Lemma 5.2 *Let $K \in H(d)$. Then*

$$\pi_{K,d}(x) = \frac{1}{2h(d)} \frac{x}{\log x} + O_{K,d}\left(\frac{x}{\log^2 x}\right),$$

where the constant implied by the O-symbol depends on K and d, and not on x.

Proof From the prime ideal theorem with remainder for ideal classes, see for example [7, Corollary (i), p. 369], and the relationship between the ideal classes in $F = Q(\sqrt{d})$ and the form classes of $H(d)$, we have

$$\pi_{K,d}(x) = \frac{1}{2h(d)} \ell i(x) + O_{K,d}(x e^{-b(K,d)\sqrt{\log x}}),$$

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for some positive number $b(K,d)$ depending only on K and d. As $\ell i(x) = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$ and $e^{-b(K,d)\sqrt{\log x}} = O_{K,d}(\frac{1}{\log^2 x})$, the asserted result follows. ■

Lemma 5.3 *Let $K \in H(d)$. Then*

$$\theta_{K,d}(x) = \frac{1}{2h(d)}x + O_{K,d}\left(\frac{x}{\log x}\right).$$

Proof By partial summation we have

$$\theta_{K,d}(x) = \pi_{K,d}(x)\log x - \int_2^x \frac{\pi_{K,d}(t)}{t}\,dt, \quad x \geq 2,$$

see for example [4, Theorem 421, p. 346]. The result now follows on using Lemma 5.2. ■

Lemma 5.4 *Let $K \in K(d)$. Then*

$$\kappa_{K,d}(x) = \frac{1}{2h(d)}\log x + O_{K,d}(\log\log x).$$

Proof By partial summation we have

$$\kappa_{K,d}(x) = \frac{\theta_{K,d}(x)}{x} + \int_2^x \frac{\theta_{K,d}(t)}{t^2}\,dt.$$

The result follows on using Lemma 5.3. ■

Lemma 5.5 *Let $K \in H(d)$. Then there exists a constant $c(K,d)$ depending only on K and d such that*

$$\lambda_{K,d}(x) = \frac{1}{2h(d)}\log\log x + c(K,d) + O_{K,d}\left(\frac{1}{\log\log x}\right).$$

Proof Set

$$\kappa_{K,d}(x) = \frac{1}{2h(d)}\log x + \tau_{K,d}(x).$$

By Lemma 5.4 we have $\tau_{K,d}(x) = O_{K,d}(\log\log x)$. Next, by partial summation, we have

$$\lambda_{K,d}(x) = \frac{\kappa_{K,d}(x)}{\log x} + \int_2^x \frac{\kappa_{K,d}(t)}{t\log^2 t}\,dt.$$

Appealing to Lemma 5.4, we obtain

$$\lambda_{K,d}(x) = \frac{1}{2h(d)} + O_{K,d}\left(\frac{\log\log x}{\log x}\right) + \frac{1}{2h(d)}(\log\log x - \log\log 2) + \int_2^x \frac{\tau_{K,d}(t)}{t\log^2 t}\,dt.$$

As $\tau_{K,d}(t) = O_{K,d}(\log\log t)$ the integrals $\int_2^\infty \frac{\tau_{K,d}(d)}{t\log^2 t}\,dt$ and $\int_x^\infty \frac{\tau_{K,d}(t)}{t\log^2 t}\,dt$ are convergent. Moreover

$$\int_x^\infty \frac{\tau_{K,d}(t)\,dt}{t\log^2 t} = O_{K,d}\left(\frac{1}{\log\log x}\right),$$

so

$$\lambda_{K,d}(x) = \frac{1}{2h(d)} \log\log x + c(K,d) + O_{K,d}\left(\frac{1}{\log\log x}\right),$$

with

$$c(K,d) = \frac{1}{2h(d)}(1 - \log\log 2) + \int_2^\infty \frac{\left(\kappa_{K,d}(t) - \frac{1}{2h(d)}\log t\right)}{t\log^2 t}\,dt. \qquad \blacksquare$$

Lemma 5.6 *Let $K \in H(d)$. Then*

$$\sum_{\substack{p\\ K_p = K}} \frac{1}{p^s} = -\frac{1}{2h(d)}\log(s-1) + \left(c(K,d) - \frac{\gamma}{2h(d)}\right) + o(s-1),$$

as $s \to 1^+$.

Proof Let δ be a real number satisfying $0 < \delta < 1/4$. By partial summation we have

$$\sum_{\substack{p\le x\\ K_p = K}} \frac{1}{p^{1+\delta}} = \frac{\lambda_{K,d}(x)}{x^\delta} + \delta\int_2^x \frac{\lambda_{K,d}(t)}{t^{1+\delta}}\,dt, \quad x \ge 2.$$

Let $x \to +\infty$. By Lemma 5.5 we obtain

$$\sum_{\substack{p\\ K_p = K}} \frac{1}{p^{1+\delta}} = \delta\int_2^\infty \frac{\lambda_{K,d}(t)}{t^{1+\delta}}\,dt.$$

Set

$$\lambda_{K,d}(x) = \frac{1}{2h(d)}\log\log x + c(K,d) + E_{K,d}(x).$$

By Lemma 5.5 we have $E_{K,d}(x) = O_{K,d}(\frac{1}{\log\log x})$, say

$$|E_{K,d}(x)| \le \frac{e(K,d)}{\log\log x}, \quad x > e,$$

for some positive number $e(K,d)$. Then

$$\sum_{\substack{p\\ K_p = K}} \frac{1}{p^{1+\delta}} = \frac{\delta}{2h(d)}\int_2^\infty \frac{\log\log t}{t^{1+\delta}}\,dt + \frac{c(K,d)}{2^\delta} + \delta\int_2^\infty \frac{E_{K,d}(t)}{t^{1+\delta}}\,dt,$$

as $\delta\int_2^\infty \frac{dt}{t^{1+\delta}} = \frac{1}{2^\delta}$. Now

$$\left|\int_1^2 \frac{\log\log t}{t^{1+\delta}}\,dt\right| \le \int_1^2 \frac{|\log\log t|}{t}\,dt = \text{constant}$$

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so that

$$\delta \int_1^2 \frac{\log\log t}{t^{1+\delta}}\,dt = O(\delta).$$

Further, putting $t = e^{u/\delta}$, we obtain

$$\begin{aligned}\delta \int_1^\infty \frac{\log\log t}{t^{1+\delta}}\,dt &= \int_0^\infty e^{-u} \log\left(\frac{u}{\delta}\right)\,du \\ &= \int_0^\infty e^{-u} \log u\,du - \log\delta \int_0^\infty e^{-u}\,du \\ &= -\gamma - \log\delta,\end{aligned}$$

as

$$\int_0^\infty e^{-u} \log u\,du = -\gamma,$$

see for example [3, p. 602]. Hence

$$\delta \int_2^\infty \frac{\log\log t}{t^{1+\delta}}\,dt = -\gamma - \log\delta + O(\delta).$$

Now set $T = e^{1/\sqrt{\delta}}$ so that

$$\log T = 1/\sqrt{\delta}, \quad \log\log T = \frac{1}{2}|\log\delta|, \quad T > e^2.$$

We also set $g(K,d) = \int_2^{e^2} \frac{|E_{K,d}(t)|}{t}\,dt$. Then

$$\begin{aligned}&\left|\delta \int_2^\infty \frac{E_{K,d}(t)}{t^{1+\delta}}\,dt\right| \\ &\quad\le \delta \int_2^{e^2} \frac{|E_{K,d}(t)|}{t^{1+\delta}}\,dt + \delta \int_{e^2}^T \frac{|E_{K,d}(t)|}{t^{1+\delta}}\,dt + \delta \int_T^\infty \frac{|E_{K,d}(t)|}{t^{1+\delta}}\,dt \\ &\quad\le \delta \int_2^{e^2} \frac{|E_{K,d}(t)|}{t}\,dt + \delta \frac{e(K,d)}{\log\log(e^2)} \int_{e^2}^T \frac{dt}{t^{1+\delta}} + \delta \frac{e(K,d)}{\log\log T} \int_T^\infty \frac{dt}{t^{1+\delta}} \\ &\quad\le \delta g(K,d) + \delta \frac{e(K,d)}{\log 2} \int_{e^2}^T \frac{dt}{t} + \delta \frac{e(K,d)}{\log\log T} \frac{1}{\delta T^\delta} \\ &\quad\le \delta g(K,d) + \delta \frac{e(K,d)}{\log 2} \log T + \frac{e(K,d)}{\log\log T} \\ &\quad\le \delta g(K,d) + 2\delta e(K,d) \log T + \frac{2e(K,d)}{|\log\delta|} \\ &\quad= g(K,d)\delta + 2e(K,d)\sqrt{\delta} + \frac{2e(K,d)}{|\log\delta|},\end{aligned}$$

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so that

$$\delta \int_2^\infty \frac{E_{K,d}(t)}{t^{1+\delta}}\,dt = o(\delta), \quad \text{as } \delta \to 0^+.$$

Hence

$$\begin{aligned}\sum_{\substack{p\\K_p=K}} \frac{1}{p^{1+\delta}} &= \frac{1}{2h(d)}\big(-\gamma - \log\delta + O(\delta)\big) + c(K,d)\big(1 + o(\delta)\big) + o(\delta)\\ &= -\frac{1}{2h(d)}\log\delta + \left(c(K,d) - \frac{\gamma}{2h(d)}\right) + o(\delta),\end{aligned}$$

as $\delta \to 0^+$. Finally we set $s = 1 + \delta$ to obtain the asserted result. ■

Lemma 5.7 *Let $K \in H(d)$. Let ω be a complex number such that $|\omega| = 1$.*

(i) The series

$$\sum_{\substack{p\\K_p=K}} \left(\sum_{n=2}^\infty \frac{\omega^n}{np^n}\right)$$

converges.

(ii) Denoting the sum of the series in (i) by $A(K,d,\omega)$, we have

$$\sum_{\substack{p\\K_p=K}} \left(\sum_{n=2}^\infty \frac{\omega^n}{np^{ns}}\right) = A(K,d,\omega) + o(s-1), \quad \text{as } s \to 1^+.$$

Proof For $s \geq 1$ we have

$$\left|\sum_{n=2}^\infty \frac{\omega^n}{np^{ns}}\right| \leq \sum_{n=2}^\infty \frac{1}{np^{ns}} \leq \sum_{n=2}^\infty \frac{1}{np^n} \leq \frac{1}{2}\sum_{n=2}^\infty \frac{1}{p^n} = \frac{1}{2}\frac{1/p^2}{1-1/p} \leq \frac{1}{p^2},$$

so the series $\sum_{\substack{p\\K_p=K}} (\sum_{n=2}^\infty \frac{\omega^n}{np^{ns}})$ is uniformly convergent for $s \geq 1$. Thus, in particular, $\sum_{\substack{p\\K_p=K}} (\sum_{n=2}^\infty \frac{\omega^n}{np^n})$ converges, proving (i). Moreover, the uniform convergence ensures that

$$\lim_{s\to 1^+} \sum_{\substack{p\\K_p=K}} \left(\sum_{n=2}^\infty \frac{\omega^n}{np^{ns}}\right) = \sum_{\substack{p\\K_p=K}} \left(\sum_{n=2}^\infty \frac{\omega^n}{np^n}\right) = A(K,d,\omega),$$

proving (ii). We note that $\overline{A(K,d,\omega)} = A(K,d,\bar{\omega})$. ■

Lemma 5.8 *Let $K \in H(d)$. Let ω be a complex number with $|\omega| = 1$. Then there exists a nonzero complex number $B(K,d,\omega)$ depending only on K, d and ω such that*

$$\prod_{\substack{p\\K_p=K}} \left(1 - \frac{\omega}{p^s}\right) = (s-1)^{\frac{\omega}{2h(d)}} B(K,d,\omega)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+.$$

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Proof Let s be a real number with $s > 1$. We have as $|\omega/p^s| < 1$

$$\begin{aligned}
\prod_{\substack{p \\ K_p=K}} \left(1 - \frac{\omega}{p^s}\right) &= \prod_{\substack{p \\ K_p=K}} e^{\log(1-\frac{\omega}{p^s})} \\
&= e^{\sum\limits_{\substack{p \\ K_p=K}} \log(1-\frac{\omega}{p^s})} \\
&= e^{-\sum\limits_{\substack{p \\ K_p=K}} \sum\limits_{n=1}^{\infty} \frac{\omega^n}{np^{ns}}} \\
&= e^{-\omega \sum\limits_{\substack{p \\ K_p=K}} \frac{1}{p^s} - \sum\limits_{\substack{p \\ K_p=K}} \sum\limits_{n=2}^{\infty} \frac{\omega^n}{np^{ns}}} \\
&= e^{-\omega\left(-\frac{1}{2h(d)}\log(s-1)+c(K,d)-\frac{\gamma}{2h(d)}+o(s-1)\right)-(A(K,d,\omega)+o(s-1))} \\
&\qquad \text{(by Lemmas 5.6 and 5.7(ii))} \\
&= (s-1)^{\omega/2h(d)} B(K,d,\omega)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+,
\end{aligned}$$

where

$$B(K,d,\omega) := e^{\omega\left(\frac{\gamma}{2h(d)} - c(K,d)\right) - A(K,d,\omega)} \neq 0.$$

We note that $\overline{B(K,d,\omega)} = B(K,d,\overline{\omega})$. ∎

Proof of Proposition 5.1 Let $K \in H(d)$. If p is a prime with $K_p = K$ then $(\frac{d}{p}) = 0$ or 1. Hence

$$\begin{aligned}
\prod_{\substack{p \\ (\frac{d}{p})=1 \\ K_p=K}} \left(1 - \frac{\omega}{p^s}\right) &= \frac{\prod\limits_{\substack{p \\ K_p=K}} \left(1 - \frac{\omega}{p^s}\right)}{\prod\limits_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K}} \left(1 - \frac{\omega}{p^s}\right)} \\
&= \frac{(s-1)^{\omega/2h(d)} B(K,d,\omega)\big(1 + o(s-1)\big)}{\prod\limits_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K}} \left(1 - \frac{\omega}{p}\right)\big(1 + o(s-1)\big)} \quad \text{(by Lemma 5.8)} \\
&= (s-1)^{\frac{\omega}{2h(d)}} C(K,d,\omega)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+,
\end{aligned}$$

where

$$C(K,d,\omega) := \frac{B(K,d,\omega)}{\prod\limits_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K}} \left(1 - \frac{\omega}{p}\right)} \neq 0.$$

We note that $\overline{C(K,d,\omega)} = C(K,d,\overline{\omega})$. ∎

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6 The Quantity $j(K,d)$

In this section we make use of Proposition 5.1 to determine the limiting behaviour of the infinite product

$$\prod_{\substack{p\\ (\frac{d}{p})=1}}\left(1-\frac{f(K,K_p)}{p^s}\right)\left(1-\frac{f(K,K_p)^{-1}}{p^s}\right)$$

as $s\to 1^+$ for $K(\neq I)\in H(d)$. We prove

Proposition 6.1 *If $K(\neq I)\in H(d)$ then*

$$\lim_{s\to 1^+}\prod_{\substack{p\\ (\frac{d}{p})=1}}\left(1-\frac{f(K,K_p)}{p^s}\right)\left(1-\frac{f(K,K_p)^{-1}}{p^s}\right)$$

exists and is a nonzero real number which we denote by $j(K,d)$.

Proof Let s be a real number with $s>1$. Then

$$\prod_{\substack{p\\ (\frac{d}{p})=1}}\left(1-\frac{f(K,K_p)}{p^s}\right)\left(1-\frac{f(K,K_p)^{-1}}{p^s}\right)$$

$$=\prod_{\substack{p\\ (\frac{d}{p})=1}}\left(1-\frac{e^{2\pi i[K,K_p]}}{p^s}\right)\left(1-\frac{e^{-2\pi i[K,K_p]}}{p^s}\right)\quad\text{(by (2.21))}$$

$$=\prod_{\substack{p\\ (\frac{d}{p})=1}}\left(1-\frac{e^{2\pi i\sum_{j=1}^{\ell}\operatorname{ind}_{A_j}(K)\operatorname{ind}_{A_j}(K_p)/h_j}}{p^s}\right)\left(1-\frac{e^{-2\pi i\sum_{j=1}^{\ell}\operatorname{ind}_{A_j}(K)\operatorname{ind}_{A_j}(K_p)/h_j}}{p^s}\right)$$

(by (2.20))

$$=\prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1}\prod_{\substack{p\\ (\frac{d}{p})=1\\ K_p=A_1^{b_1}\cdots A_\ell^{b_\ell}}}\left(1-\frac{e^{2\pi i\sum_{j=1}^{\ell}k_jb_j/h_j}}{p^s}\right)\left(1-\frac{e^{-2\pi i\sum_{j=1}^{\ell}k_jb_j/h_j}}{p^s}\right)$$

(by (2.19))

$$=\prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1}(s-1)^{\frac{1}{2h(d)}\exp(2\pi i\sum_{j=1}^{\ell}k_jb_j/h_j)}C\left(A_1^{b_1}\cdots A_\ell^{b_\ell},d,\exp\Bigl(2\pi i\sum_{j=1}^{\ell}k_jb_j/h_j\Bigr)\right)$$

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$$\times (s-1)^{\frac{1}{2h(d)}\exp(-2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j)} C\Big(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Big(-2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Big)\Big)$$

$$\times \big(1+o(s-1)\big) \quad \text{(by Proposition 5.1)}$$

$$= (s-1)^{\frac{1}{2h(d)}\big(\prod_{j=1}^{\ell}(\sum_{b_j=0}^{h_j-1}\exp(2\pi i k_j b_j/h_j)) + \prod_{j=1}^{\ell}(\sum_{bj=0}^{h_j-1}\exp(-2\pi i k_j b_j/h_j))\big)}$$

$$\times \prod_{b_1,\cdots,b_\ell=0}^{h_1-1,\dots,h_\ell-1} C\Big(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Big(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Big)\Big)$$

$$\times C\Big(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Big(-2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Big)\Big)\big(1+o(s-1)\big).$$

As $K \neq I$ at least one of $k_1, \dots, k_\ell$ is nonzero, say k_j, in which case $0 < k_j < h_j$ and

$$\sum_{b_j=0}^{h_j-1} \exp(\pm 2\pi i k_j b_j/h_j) = 0.$$

Thus

$$\prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K,K_p)}{p^s}\right)\left(1 - \frac{f(K,K_p)^{-1}}{p^s}\right)$$

$$= \prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1} C\Big(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Big(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Big)\Big)$$

$$\times C\Big(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Big(-2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Big)\Big) \times \big(1+o(s-1)\big)$$

$$= \prod_{L\in H(d)} C\big(L, d, f(K,L)\big) C\big(L, d, f(K,L)^{-1}\big)\big(1+o(s-1)\big),$$

as $s \to 1^+$, by (2.18)–(2.21). Hence

$$\lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K,K_p)}{p^s}\right)\left(1 - \frac{f(K,K_p)^{-1}}{p^s}\right)$$

exists and is equal to

$$\text{(6.1)} \qquad j(K,d) := \prod_{L\in H(d)} C\big(L, d, f(K,L)\big) C\big(L, d, f(K,L)^{-1}\big).$$

Since each $C\big(L, d, f(K, L)^{\pm 1}\big)$ $\big(L \in H(d)\big)$ is a nonzero complex number and

$$C\big(L, d, f(K, L)^{-1}\big) = C\big(L, d, \overline{f(K, L)}\big) = \overline{C\big(L, d, f(K, L)\big)},$$

we see that $j(K, d)$ is a nonzero real number. ■

As $f(K, L)^{-1} = f(K^{-1}, L)$ we see from (6.1) that

$$j(K, d) = \prod_{L \in H(d)} C\big(L, d, f(K, L)\big) C\big(L, d, f(K^{-1}, L)\big) = j(K^{-1}, d). \tag{6.2}$$

It is convenient to set

$$m(K, d) := \frac{t_1(d)}{j(K, d)}, \quad K \in H(d), \tag{6.3}$$

where $t_1(d)$ is defined in (2.32). Thus, appealing to (2.32), Proposition 6.1, and (6.3) we obtain

$$\begin{aligned} m(K, d) &= \frac{\displaystyle\prod_{\substack{p \\ (\frac{d}{p})=1}} \Big(1 - \frac{1}{p^2}\Big)}{\displaystyle\lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \Big(1 - \frac{f(K, K_p)}{p^s}\Big)\Big(1 - \frac{f(K, K_p)^{-1}}{p^s}\Big)} \\ &= \frac{\displaystyle\lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \Big(1 - \frac{1}{p^{2s}}\Big)}{\displaystyle\lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \Big(1 - \frac{f(K, K_p)}{p^s}\Big)\Big(1 - \frac{f(K, K_p)^{-1}}{p^s}\Big)}, \end{aligned}$$

that is

$$m(K, d) = \lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \frac{\Big(1 - \frac{1}{p^s}\Big)\Big(1 + \frac{1}{p^s}\Big)}{\Big(1 - \frac{f(K, K_p)}{p^s}\Big)\Big(1 - \frac{f(K, K_p)^{-1}}{p^s}\Big)}. \tag{6.4}$$

From (6.2) and (6.3) we deduce that

$$m(K, d) = m(K^{-1}, d). \tag{6.5}$$

7 Evaluation of $\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s}$, $K \neq I$

In this section we prove the following result.

Proposition 7.1 *Let $K(\neq I) \in H(d)$. Then*

$$\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \ell(K, d) m(K, d)\big(1 + o(s - 1)\big), \quad \text{as } s \to 1^+,$$

where $\ell(K, d)$ is defined in (2.33) and $m(K, d)$ in (6.3).

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Proof Let $K(\neq I) \in H(d)$. Set $k_j = \operatorname{ord}_{A_j}(K)$ $(j = 1, \dots, \ell)$, so that $(k_1, \dots, k_\ell) \neq (0, \dots, 0)$. Let p be a prime and let α be a positive integer.

First we consider those p for which $(\frac{d}{p}) = 0$ or 1. Set $b_j = \operatorname{ord}_{A_j}(K_p)$ $(j = 1, \dots, \ell)$. Let $s_1, \dots, s_\ell$ be integers with $0 \leq s_j \leq h_j - 1$ $(j = 1, \dots, \ell)$. Appealing to Definition 2.1, we see that

$$\begin{aligned}
N_{A_1^{s_1} \cdots A_\ell^{s_\ell}}(p^\alpha) &= \text{number of } \varepsilon(=\pm 1) \text{ such that } K_p^{\alpha\varepsilon} = A_1^{s_1} \cdots A_\ell^{s_\ell} \\
&= \text{number of } \varepsilon(=\pm 1) \text{ such that } A_1^{b_1\alpha\varepsilon} \cdots A_\ell^{b_\ell\alpha\varepsilon} = A_1^{s_1} \cdots A_\ell^{s_\ell} \\
&= \text{number of } \varepsilon(=\pm 1) \text{ such that } b_j\alpha\varepsilon \equiv s_j \pmod{h_j}\ (j = 1, \dots, \ell) \\
&= \begin{cases}
2, & \text{if (A) } s_j \equiv b_j\alpha \equiv -b_j\alpha \pmod{h_j}\ (j = 1, \dots, \ell) \\
1, & \text{if (B) } s_j \equiv b_j\alpha \pmod{h_j}\ (j = 1, \dots, \ell) \text{ and} \\
& \qquad b_j\alpha \not\equiv -b_j\alpha \pmod{h_j} \text{ for some } j, \\
& \text{or (C) } s_j \equiv -b_j\alpha \pmod{h_j}\ (j = 1, \dots, \ell) \text{ and} \\
& \qquad b_j\alpha \not\equiv -b_j\alpha \pmod{h_j} \text{ for some } j, \\
0, & \text{if (D) } s_j \not\equiv b_j\alpha \pmod{h_j} \text{ for some } j \\
& \qquad \text{and } s_k \not\equiv -b_k\alpha \pmod{b_k} \text{ for some } k.
\end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{L \in H(d)} f(K, L) N_L(p^\alpha) &= \sum_{s_1, \dots, s_\ell = 0}^{h_1 - 1, \dots, h_\ell - 1} e^{2\pi i \sum_{j=1}^{\ell} k_j s_j / h_j} N_{A_1^{s_1} \cdots A_\ell^{s_\ell}}(p^\alpha) \\
&= \Big\{ 2 \sum_{(A)} + \sum_{(B)} + \sum_{(C)} \Big\} e^{2\pi i \sum_{j=1}^{\ell} k_j s_j / h_j} \\
&= \Big\{ \Big(\sum_{(A)} + \sum_{(B)} \Big) + \Big(\sum_{(A)} + \sum_{(C)} \Big) \Big\} e^{2\pi i \sum_{j=1}^{\ell} k_j s_j / h_j} \\
&= \Big\{ \sum_{(A) \cup (B)} + \sum_{(A) \cup (C)} \Big\} e^{2\pi i \sum_{j=1}^{\ell} k_j s_j / h_j} \\
&= e^{2\pi i \alpha \sum_{j=1}^{\ell} k_j s_j / h_j} + e^{-2\pi i \alpha \sum_{j=1}^{\ell} k_j s_j / h_j} \\
&= \theta^\alpha + \theta^{-\alpha},
\end{aligned}$$

where we have set for convenience

$$\theta := e^{2\pi i \sum_{j=1}^{\ell} k_j b_j / h_j} = e^{2\pi i [K, K_p]} = f(K, K_p).$$

Note that θ depends on p, d and K, and that $|\theta| = 1$. If $(\frac{d}{p}) = 1$, from Proposition 3.1, we deduce that

$$P_L(p^\alpha) = w(d) N_L(p^\alpha) \quad \big(L \in H(d)\big)$$

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so that

$$\begin{aligned} W_K(p^\alpha) &= \frac{1}{w(d)} \sum_{L\in H(d)} f(K,L)P_L(p^\alpha) \\ &= \sum_{L\in H(d)} f(K,L)N_L(p^\alpha) \\ &= \theta^\alpha + \theta^{-\alpha}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\alpha=1}^{\infty} \frac{W_K(p^\alpha)}{p^{\alpha s}} &= \sum_{\alpha=1}^{\infty} \frac{\theta^\alpha + \theta^{-\alpha}}{p^{\alpha s}} \\ &= \frac{\theta/p^s}{1-\theta/p^s} + \frac{\theta^{-1}/p^s}{1-\theta^{-1}/p^s} \\ &= \frac{\theta/p^s + \theta^{-1}/p^s - 2/p^{2s}}{(1-\theta/p^s)(1-\theta^{-1}/p^s)}, \end{aligned}$$

so that by (2.24)

$$\sum_{j=0}^{\infty} \frac{W_K(p^j)}{p^{js}} = 1 + \frac{\theta/p^s + \theta^{-1}/p^s - 2/p^{2s}}{(1-\theta/p^s)(1-\theta^{-1}/p^s)},$$

that is

$$\sum_{j=0}^{\infty} \frac{W_K(p^j)}{p^{js}} = \frac{1-1/p^{2s}}{(1-\theta/p^s)(1-\theta^{-1}/p^s)}, \quad \text{if } \left(\frac{d}{p}\right) = 1. \tag{7.1}$$

If $(\frac{d}{p}) = 0$ then, by Proposition 3.1, we have

$$P_L(p^\alpha) = \begin{cases} 0, & \text{if } \alpha \geq 2, \\ \frac{w(d)}{2} N_L(p), & \text{if } \alpha = 1. \end{cases}$$

Hence

$$\begin{aligned} W_K(p^\alpha) &= \frac{1}{w(d)} \sum_{L\in H(d)} f(K,L)P_L(p^\alpha) \\ &= \begin{cases} 0, & \text{if } \alpha \geq 2, \\ \frac{1}{2} \sum_{L\in H(d)} f(K,L)N_L(p), & \text{if } \alpha = 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } \alpha \geq 2, \\ \frac{1}{2}(\theta + \theta^{-1}), & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

Thus

$$\sum_{j=0}^{\infty} \frac{W_K(p^j)}{p^{js}} = 1 + \frac{\frac{1}{2}(\theta + \theta^{-1})}{p^s}, \quad \text{if } \left(\frac{d}{p}\right) = 0. \tag{7.2}$$

Now suppose that $(\frac{d}{p}) = -1$. By Proposition 3.1 we have $P_L(p^\alpha) = 0$. Then, by (2.23), we deduce that $W_K(p^\alpha) = 0$. Thus

$$\sum_{j=0}^{\infty} \frac{W_K(p^j)}{p^{js}} = 1, \quad \text{if } \left(\frac{d}{p}\right) = -1. \tag{7.3}$$

Next, by Proposition 4.1, $W_K(n)$ is a multiplicative function of n so that

$$\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \prod_p \left(\sum_{j=0}^{\infty} \frac{W_k(p^j)}{p^{js}} \right).$$

Appealing to (7.1), (7.2) and (7.3), we obtain

$$\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \prod_{\substack{p \\ (\frac{d}{p})=1}} \frac{(1 - \frac{1}{p^{2s}})}{(1 - \frac{\theta}{p^s})(1 - \frac{\theta^{-1}}{p^s})} \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{\frac{1}{2}(\theta + \theta^{-1})}{p^s}\right).$$

We now consider the product

$$\begin{aligned} \prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{\theta}{p^s}\right)\left(1 - \frac{\theta^{-1}}{p^s}\right) &= \prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K, K_p)}{p^s}\right)\left(1 - \frac{f(K, K_p)^{-1}}{p^s}\right) \\ &= j(K, d)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+, \end{aligned}$$

by Proposition 6.1. Here we used $K \neq I$.

Further

$$\prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{1}{p^{2s}}\right) = t_1(d)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+,$$

where $t_1(d)$ is defined in (2.32). Also

$$\prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{\frac{1}{2}(\theta + \theta^{-1})}{p^s}\right) = \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{\frac{1}{2}\big(f(K,K_p) + f(K,K_p)^{-1}\big)}{p^s}\right)$$

$$= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{\frac{1}{2}\big(f(K,K_p) + f(K,K_p)^{-1}\big)}{p}\right)\big(1 + o(s-1)\big)$$

$$= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f(K,K_p)}{p}\right)\big(1 + o(s-1)\big)$$

$$= \ell(K,d)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+,$$

since

$$\left(\frac{d}{p}\right) = 0 \Rightarrow K_p = K_p^{-1} \Rightarrow f(K,K_p)^{-1} = f(K,K_p^{-1}) = f(K,K_p).$$

We have shown that

$$\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \frac{t_1(d)\big(1 + o(s-1)\big)\ell(K,d)\big(1 + o(s-1)\big)}{j(K,d)\big(1 + o(s-1)\big)}$$

$$= \ell(K,d)m(K,d)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+. \qquad \blacksquare$$

The next result is a slight modification of Proposition 7.1, in a form convenient for use in the proof of Proposition 9.1.

Proposition 7.2 *Let $K(\neq I) \in H(d)$. Then*

$$\zeta(2s) \sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \frac{\pi^2}{6}\ell(K,d)m(K,d)\big(1 + o(s-1)\big), \quad \textit{as } s \to 1^+.$$

Proof By Proposition 7.1 we have

$$\sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \ell(K,d)m(K,d)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+.$$

Also

$$\zeta(2s) = \zeta(2)\big(1 + o(s-1)\big) = \frac{\pi^2}{6}\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+.$$

Hence

$$\zeta(2s) \sum_{n=1}^{\infty} \frac{W_K(n)}{n^s} = \frac{\pi^2}{6}\ell(K,d)m(K,d)\big(1 + o(s-1)\big), \quad \text{as } s \to 1^+. \qquad \blacksquare$$

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8 Evaluation of $\sum_{n=1}^{\infty} \frac{W_I(n)}{n^s}$

In this section we determine $\zeta(2s)\sum_{n=1}^{\infty}\frac{W_K(n)}{n^s}$ as $s \to 1^+$ in the excluded case $K = I$. This is the companion result to Proposition 7.2.

Proposition 8.1

$$\begin{aligned}\zeta(2s)\sum_{n=1}^{\infty}\frac{W_I(n)}{n^s}\\ &= \frac{2\pi h(d)}{w(d)\sqrt{|d|}}\frac{1}{s-1}\\ &\quad+\left\{\frac{4\pi\gamma h(d)}{w(d)\sqrt{|d|}}+\frac{2\pi(\log 2\pi)h(d)}{w(d)\sqrt{|d|}}-\frac{\pi}{\sqrt{|d|}}\sum_{m=1}^{|d|}\left(\frac{d}{m}\right)\log\Gamma\left(\frac{m}{|d|}\right)\right\}\\ &\quad+o(s-1),\end{aligned}$$

as $s \to 1^+$.

Proof From (2.25) we have

$$W_I(n) = \frac{P(n)}{w(d)}$$

so that

$$\sum_{n=1}^{\infty}\frac{W_I(n)}{n^s} = \frac{1}{w(d)}\sum_{n=1}^{\infty}\frac{P(n)}{n^s}, \quad s > 1.$$

Thus

$$\begin{aligned}\zeta(2s)\sum_{n=1}^{\infty}\frac{W_I(n)}{n^s} &= \frac{1}{w(d)}\sum_{m=1}^{\infty}\frac{1}{m^{2s}}\sum_{n=1}^{\infty}\frac{P(n)}{n^s}\\ &= \frac{1}{w(d)}\sum_{\ell=1}^{\infty}\frac{1}{\ell^s}\sum_{\substack{m,n\\ m^2n=\ell}}P(n)\\ &= \frac{1}{w(d)}\sum_{\ell=1}^{\infty}\frac{1}{\ell^s}\sum_{m^2|\ell}P(\ell/m^2)\\ &= \frac{1}{w(d)}\sum_{\ell=1}^{\infty}\frac{R(\ell)}{\ell^s},\end{aligned}$$

by (2.7). Now, by Dirichlet's formula (see for example [5], [6]), we have

$$R(\ell) = w(d)\sum_{e|\ell}\left(\frac{d}{e}\right),$$

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so that

$$\begin{aligned}\zeta(2s)\sum_{n=1}^{\infty}\frac{W_I(n)}{n^s} &= \sum_{\ell=1}^{\infty}\frac{1}{\ell^s}\sum_{e\mid\ell}\left(\frac{d}{e}\right)\\ &= \sum_{e,k=1}^{\infty}\frac{1}{e^sk^s}\left(\frac{d}{e}\right)\\ &= \left(\sum_{e=1}^{\infty}\frac{\left(\frac{d}{e}\right)}{e^s}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k^s}\right)\\ &= L(s,d)\zeta(s).\end{aligned}$$

As $s \to 1^+$ we have

$$\zeta(s) = \frac{1}{s-1} + \gamma + o(s-1),$$

where γ denotes Euler's constant, and

$$L(s,d) = L(1,d) + (s-1)L'(1,d) + o\big((s-1)^2\big),$$

so that

$$\zeta(2s)\sum_{n=1}^{\infty}\frac{W_I(n)}{n^s} = \frac{L(1,d)}{s-1} + \big(\gamma L(1,d) + L'(1,d)\big) + o(s-1).$$

Now, from [1, p. 171] and [8, p. 110], we have

$$L(1,d) = \frac{2\pi h(d)}{w(d)\sqrt{|d|}}$$

and

$$L'(1,d) = \frac{2h(d)\pi(\gamma + \log 2\pi)}{w(d)\sqrt{|d|}} - \frac{\pi}{\sqrt{|d|}}\sum_{m=1}^{|d|}\left(\frac{d}{m}\right)\log\Gamma\left(\frac{m}{|d|}\right),$$

so that

$$\begin{aligned}\zeta(2s)&\sum_{n=1}^{\infty}\frac{W_I(n)}{n^s}\\ &= \frac{2\pi h(d)}{w(d)\sqrt{|d|}}\frac{1}{s-1}\\ &\quad + \left\{\frac{4\pi\gamma h(d)}{w(d)\sqrt{|d|}} + \frac{2\pi(\log 2\pi)h(d)}{w(d)\sqrt{|d|}} - \frac{\pi}{\sqrt{|d|}}\sum_{m=1}^{|d|}\left(\frac{d}{m}\right)\log\Gamma\left(\frac{m}{|d|}\right)\right\}\\ &\quad + o(s-1), \quad \text{as } s \to 1^+,\end{aligned}$$

as asserted. ■

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9 Evaluation of $|\eta\big((b+\sqrt{d})/2a\big)|$

Let $K \in H(d)$. In this section we determine $\sum_{n=1}^{\infty} \frac{R_K(n)}{n^s}(s > 1)$ in two different ways (Propositions 9.1 and 9.2). First we make use of Propositions 7.2 and 8.1 to prove Proposition 9.1. Secondly we appeal to Kronecker's limit formula to prove Proposition 9.2. Equating the expressions in Propositions 9.1 and 9.2, we obtain the main result of this paper which gives a formula for $|\eta\big((b+\sqrt{d})/2a\big)|$ (Theorem 9.3).

Proposition 9.1 *Let $K \in H(d)$. Then*

$$\sum_{n=1}^{\infty} \frac{R_K(n)}{n^s} = \frac{2\pi/\sqrt{|d|}}{s-1} + A(K,d) + o(s-1), \quad \text{as } s \to 1^+,$$

with

$$A(K,d) = \frac{4\pi\gamma}{\sqrt{|d|}} + \frac{2\pi(\log 2\pi)}{\sqrt{|d|}} - \frac{\pi w(d)}{h(d)\sqrt{|d|}} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log\Gamma\left(\frac{m}{|d|}\right)$$
$$+ \frac{\pi^2 w(d)}{6h(d)} \sum_{\substack{L\in H(d)\\ L\neq I}} f(L,K)\ell(L,d)m(L,d),$$

where $f(L,K)$ is defined in (2.21), $\ell(L,d)$ in (2.33), and $m(L,d)$ in (6.3).

Proof Appealing to (2.22) and (2.23), we obtain

$$\begin{aligned}\sum_{L\in H(d)} f(L,K)^{-1} W_L(n) &= \frac{1}{w(d)} \sum_{L\in H(d)} f(L,K)^{-1} \sum_{M\in H(d)} f(L,M)P_M(n)\\ &= \frac{1}{w(d)} \sum_{M\in H(d)} P_M(n) \sum_{L\in H(d)} f(M,L)f(L,K)^{-1}\\ &= \frac{1}{w(d)} \sum_{\substack{M\in H(d)\\ M=K}} P_M(n)h(d)\\ &= \frac{h(d)}{w(d)} P_K(n)\end{aligned}$$

so that

$$P_K(n) = \frac{w(d)}{h(d)} \sum_{L\in H(d)} f(L,K)^{-1} W_L(n).$$

Hence

$$\sum_{n=1}^{\infty} \frac{P_K(n)}{n^s} = \frac{w(d)}{h(d)} \sum_{L\in H(d)} f(L,K)^{-1} \sum_{n=1}^{\infty} \frac{W_L(n)}{n^s}.$$

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By (2.5) we have

$$\sum_{n=1}^{\infty}\frac{R_K(n)}{n^s}=\zeta(2s)\sum_{n=1}^{\infty}\frac{P_K(n)}{n^s}$$

so that

$$\sum_{n=1}^{\infty}\frac{R_K(n)}{n^s}=\frac{w(d)}{h(d)}\sum_{L\in H(d)}f(L,K)^{-1}\left(\zeta(2s)\sum_{n=1}^{\infty}\frac{W_L(n)}{n^s}\right).$$

Appealing to Proposition 7.2 and 8.1, we obtain as $f(I,K)=1$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{R_K(n)}{n^s}&=\frac{w(d)}{h(d)}\Bigg\{\frac{2\pi h(d)}{w(d)\sqrt{|d|}}\frac{1}{s-1}+\frac{4\pi\gamma h(d)}{w(d)\sqrt{|d|}}\\&\qquad+\frac{2\pi(\log 2\pi)h(d)}{w(d)\sqrt{|d|}}-\frac{\pi}{\sqrt{|d|}}\sum_{m=1}^{|d|}\left(\frac{d}{m}\right)\log\Gamma\left(\frac{m}{|d|}\right)+o(s-1)\\&\qquad+\sum_{\substack{L\in H(d)\\L\neq I}}f(L,K)^{-1}\frac{\pi^2}{6}\ell(L,d)m(L,d)\big(1+o(s-1)\big)\Bigg\}\\&=\frac{2\pi/\sqrt{|d|}}{s-1}+\Bigg\{\frac{4\pi\gamma}{\sqrt{|d|}}+\frac{2\pi(\log 2\pi)}{\sqrt{|d|}}-\frac{\pi w(d)}{h(d)\sqrt{|d|}}\sum_{m=1}^{|d|}\left(\frac{d}{m}\right)\log\Gamma\left(\frac{m}{|d|}\right)\\&\qquad+\frac{\pi^2}{6}\frac{w(d)}{h(d)}\sum_{\substack{L\in H(d)\\L\neq I}}f(L,K)^{-1}\ell(L,d)m(L,d)\Bigg\}+o(s-1),\end{aligned}$$

as $s\to 1^+$. The asserted formula now follows as

$$\begin{aligned}\sum_{\substack{L\in H(d)\\L\neq I}}f(L,K)^{-1}\ell(L,d)m(L,d)&=\sum_{\substack{L\in H(d)\\L\neq I}}f(L^{-1},K)\ell(L,d)m(L,d)\\&=\sum_{\substack{L\in H(d)\\L\neq I}}f(L,K)\ell(L^{-1},d)m(L^{-1},d)\quad(L\to L^{-1})\\&=\sum_{\substack{L\in H(d)\\L\neq I}}f(L,K)\ell(L,d)m(L,d),\end{aligned}$$

by (2.34) and (6.5). ■

Proposition 9.2 *Let $K=[a,b,c]\in H(d)$. Then*

$$\sum_{n=1}^{\infty}\frac{R_K(n)}{n^s}=\frac{2\pi/\sqrt{|d|}}{s-1}+B(a,b,c)+o(s-1),\quad \textit{as } s\to 1^+,$$

where

$$B(a,b,c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi \log |d|}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}} \log \left(a^{-1/4} \left| \eta \left(\frac{b+\sqrt{d}}{2a} \right) \right| \right).$$

Proof We have for $n \geq 1$

$$R_K(n) = R_{(a,b,c)}(n) = \sum_{\substack{x,y=-\infty \\ ax^2+bxy+cy^2=n}}^{\infty} 1 = \sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0) \\ ax^2+bxy+cy^2=n}}^{\infty} 1,$$

so that, by Kronecker's limit formula (see for example [9, Theorem 1, p. 14]), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{R_K(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0) \\ ax^2+bxy+cy^2=n}}^{\infty} 1 \\ &= \sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0)}}^{\infty} \frac{1}{(ax^2+bxy+cy^2)^s} \\ &= \frac{2\pi/\sqrt{|d|}}{s-1} + B(a,b,c) + o(s-1), \end{aligned}$$

as $s \to 1^+$, where

$$B(a,b,c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi \log |d|}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}} \log \left(a^{-1/4} \left| \eta \left(\frac{b+\sqrt{d}}{2a} \right) \right| \right). \qquad \blacksquare$$

Theorem 9.3 *Let $K = [a,b,c] \in H(d)$. Then*

$$\begin{aligned} & a^{-1/4} \left| \eta \left(\frac{b+\sqrt{d}}{2a} \right) \right| \\ & = (2\pi|d|)^{-1/4} \left\{ \prod_{m=1}^{|d|} \Gamma \left(\frac{m}{|d|} \right)^{\left(\frac{d}{m}\right)} \right\}^{\frac{w(d)}{8h(d)}} e^{-\frac{\pi w(d)\sqrt{|d|}}{48h(d)} \sum_{\substack{L\in H(d) \\ L\neq I}} f(L,K)\ell(L,d)m(L,d)}, \end{aligned} \tag{9.1}$$

where $f(L,K)$ is defined in (2.21), $\ell(L,d)$ in (2.33), and $m(L,d)$ in (6.3).

Proof From Propositions 9.1 and 9.2, we deduce that

$$A(K,d) = B(a,b,c),$$

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so that

$$\frac{4\pi\gamma}{\sqrt{|d|}} + \frac{2\pi(\log 2\pi)}{\sqrt{|d|}} - \frac{\pi w(d)}{h(d)\sqrt{|d|}} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log\Gamma\left(\frac{m}{|d|}\right) + \frac{\pi^2 w(d)}{6h(d)} \sum_{\substack{L\in H(d)\\ L\neq I}} f(L,K)\ell(L,d)m(L,d)$$

$$= \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi\log|d|}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}} \log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right).$$

Multiplying both sides by $\sqrt{|d|}/8\pi$, and rearranging terms, we obtain

$$\log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right) = \log\left((2\pi|d|)^{-1/4}\left(\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right)^{\frac{w(d)}{8h(d)}}\right) - \frac{\pi w(d)\sqrt{|d|}}{48h(d)} \sum_{\substack{L\in H(d)\\ L\neq I}} f(L,K)\ell(L,d)m(L,d).$$

Exponentiating both sides, we obtain the asserted formula. ■

We close this section by illustrating Theorem 9.3 in the cases $d = -15$ and $d = -31$. First we treat $d = -15$.

Example 9.4 $d = -15$. In this case we have

$$H(-15) = \{I, A\}, \quad A^2 = I,$$

where

$$I = [1,1,4], \quad A = [2,1,2],$$

so that $h(-15) = 2$. The primes dividing $d = -15$ are 3 and 5. Appealing to formulae (2.11) and (2.12) we obtain

$$K_3 = [3,3,2] = [2,-3,3] = [2,1,2] = A$$

and

$$K_5 = [5,5,2] = [2,-5,5] = [2,-1,2] = [2,1,2] = A.$$

Next (with $A_1 = A$, $h_1 = 2$, $\ell = 1$) we have from (2.20)

$$[I,I] = [I,A] = [A,I] = 0, \quad [A,A] = 1/2,$$

so that by (2.21)

$$f(I,I) = f(I,A) = f(A,I) = 1, \quad f(A,A) = -1.$$

Then, by (2.33), we have

$$\ell(A,-15) = \left(1 + \frac{f(A,K_3)}{3}\right)\left(1 + \frac{f(A,K_5)}{5}\right) = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = \frac{8}{15}.$$

Further, by (6.4) we deduce that

$$m(A,-15) = \lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{-15}{p})=1}} \frac{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}\right)}{\left(1-\frac{f(A,K_p)}{p^s}\right)\left(1-\frac{f(A,K_p)^{-1}}{p^s}\right)}.$$

As $H(-15)$ is a cyclic group of order 2, by Gauss' theory of genera, we have for $p \neq 3, 5$

$$f(A,K_p) = 1 \longleftrightarrow K_p = I \longleftrightarrow \left(\frac{-3}{p}\right) = \left(\frac{5}{p}\right) = 1$$

and

$$f(A,K_p) = -1 \longleftrightarrow K_p = A \longleftrightarrow \left(\frac{-3}{p}\right) = \left(\frac{5}{p}\right) = -1,$$

so that

$$m(A,-15) = \lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{-15}{p})=1}} \frac{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}\right)}{\left(1-\frac{(\frac{-3}{p})}{p^3}\right)\left(1-\frac{(\frac{5}{p})}{p^s}\right)}.$$

If p is a prime with $(\frac{-15}{p}) = -1$, we have $\left(\left(\frac{-3}{p}\right), \left(\frac{5}{p}\right)\right) = (1,-1)$ or $(-1,1)$ so

$$m(A,-15) = \lim_{s\to 1^+} \prod_{p\neq 3,5} \frac{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}\right)}{\left(1-\frac{\left(\frac{-3}{p}\right)}{p^s}\right)\left(1-\frac{(\frac{5}{p})}{p^s}\right)}.$$

As $(\frac{-3}{5}) = (\frac{5}{3}) = -1$ we have

$$\begin{aligned} m(A,-15) &= \lim_{s\to 1^+} \frac{1}{\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)} \prod_{p} \frac{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}\right)}{\left(1-\frac{\left(\frac{-3}{p}\right)}{p^s}\right)\left(1-\frac{\left(\frac{5}{p}\right)}{p^s}\right)} \\ &= \frac{15}{8} \lim_{s\to 1^+} \prod_{p} \frac{\left(1-\frac{\left(\frac{-3}{p}\right)}{p^s}\right)^{-1}\left(1-\frac{\left(\frac{5}{p}\right)}{p^s}\right)^{-1}}{\left(1-\frac{1}{p^{2s}}\right)^{-1}} \\ &= \frac{15}{8} \lim_{s\to 1^+} \frac{L(s,-3)L(s,5)}{\zeta(2s)} \\ &= \frac{15}{8}\frac{L(1,-3)L(1,5)}{\zeta(2)} \end{aligned}$$

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$$= \frac{15}{8} \frac{\frac{\pi}{3\sqrt{3}} \frac{2}{\sqrt{5}} \log\left(\frac{1+\sqrt{5}}{2}\right)}{\pi^2/6}$$
$$= \frac{\sqrt{15}}{2\pi} \log\left(\frac{1+\sqrt{5}}{2}\right).$$

By Theorem 9.3, for $K = [a, b, c] \in H(-15)$, we have

$$a^{-1/4} \left| \eta\left(\frac{b+\sqrt{-15}}{2a}\right) \right| = (30\pi)^{-1/4} E^{1/8} e^{-\frac{\pi\sqrt{15}}{48} f(A,K)\ell(A,-15)m(A,-15)},$$

where

$$E = \frac{\Gamma(1/15)\Gamma(2/15)\Gamma(4/15)\Gamma(8/15)}{\Gamma(7/15)\Gamma(11/15)\Gamma(13/15)\Gamma(14/15)}.$$

Taking $(a, b, c) = (1, 1, 4)$, we obtain

$$\left| \eta\left(\frac{1+\sqrt{-15}}{2}\right) \right| = (30\pi)^{-1/4} E^{1/8} \left(\frac{1+\sqrt{5}}{2}\right)^{-1/12},$$

and with $(a, b, c) = (2, 1, 2)$

$$2^{-1/4} \left| \eta\left(\frac{1+\sqrt{-15}}{4}\right) \right| = (30\pi)^{-1/4} E^{1/8} \left(\frac{1+\sqrt{5}}{2}\right)^{1/12}.$$

Hence we have proved the following result.

Corollary 9.5

$$\eta\left(\frac{1+\sqrt{-15}}{2}\right) = e^{\pi i/24} \left| \eta\left(\frac{1+\sqrt{-15}}{2}\right) \right|$$
$$= e^{\pi i/24} (30\pi)^{-1/4} E^{1/8} \left(\frac{1+\sqrt{5}}{2}\right)^{-1/12},$$
$$\left| \eta\left(\frac{1+\sqrt{-15}}{4}\right) \right| = (15\pi)^{-1/4} E^{1/8} \left(\frac{1+\sqrt{5}}{2}\right)^{1/12}.$$

Now we treat $d = -31$.

Example 9.6 $d = -31$. In this case we have

$$H(-31) = \{I, A, A^2\}, \quad A^3 = I,$$

where

$$I = [1, 1, 8], \quad A = [2, 1, 4], \quad A^2 = [2, -1, 4],$$

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so that $h(-31) = 3$. Also

$$\begin{aligned} K_{31} &= [31, 31, 8] = [8, -31, 31] = [8, 1, 1] \\ &= [1, -1, 8] = [1, 1, 8] = I. \end{aligned}$$

Next, with $A_1 = A$, $h_1 = 3$, $\ell = 1$, we have from (2.20)

$$[I, I] = [I, A] = [I, A^2] = [A, I] = [A^2, I] = 0,$$
$$[A, A] = 1/3, \quad [A, A^2] = [A^2, A] = 2/3, \quad [A^2, A^2] = 4/3,$$

so that by (2.21)

$$f(I, I) = f(I, A) = f(I, A^2) = f(A, I) = f(A^2, I) = 1,$$
$$f(A, A) = f(A^2, A^2) = \omega, \quad f(A, A^2) = f(A^2, A) = \omega^2,$$

where $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$. From (2.33), we obtain for $r = 0, 1, 2$

$$\begin{aligned} \ell(A^r, -31) &= \prod_{\substack{p \\ (\frac{-31}{p})=0}} \left(1 + \frac{f(A^r, K_p)}{p}\right) = 1 + \frac{f(A^r, K_{31})}{31} = 1 + \frac{f(A^r, I)}{31} \\ &= 1 + \frac{1}{31} = \frac{32}{31}. \end{aligned}$$

It is convenient to set

$$\lambda(p) := \begin{cases} 1, & \text{if } (\frac{-31}{p}) = 1, K_p = I, \\ \omega, & \text{if } (\frac{-31}{p}) = 1, K_p = A, \\ \omega^2, & \text{if } (\frac{-31}{p}) = 1, K_p = A^2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, from (6.4), for $r = 1, 2$ we have

$$\begin{aligned} m(A^r, -31) &= \lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{-31}{p})=1}} \frac{\left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{p^s}\right)}{\left(1 - \frac{f(A^r, K_p)}{p^s}\right)\left(1 - \frac{f(A^r, K_p)^{-1}}{p}\right)} \\ &= \lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{-31}{p})=1}} \frac{\left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{p^s}\right)}{\left(1 - \frac{\lambda(p)}{p^s}\right)\left(1 - \frac{\lambda(p)^2}{p^s}\right)}. \end{aligned}$$

Appealing to Theorem 9.3 with $(a, b, c) = (1, 1, 8)$, $(2, 1, 4)$ and $(2, -1, 4)$, we obtain

$$\left|\eta\left(\frac{1 + \sqrt{-31}}{2}\right)\right| = (62\pi)^{-1/4} E e^{-2R}$$

and

$$\left|\eta\left(\frac{1+\sqrt{-31}}{4}\right)\right| = \left|\eta\left(\frac{-1+\sqrt{-31}}{4}\right)\right| = (31\pi)^{-1/4}Ee^{R},$$

where

$$E := \left\{\prod_{m=1}^{31}\Gamma\left(\frac{m}{31}\right)^{\left(\frac{-31}{m}\right)}\right\}^{1/12} \tag{9.2}$$

and

$$R := \frac{4\pi\sqrt{31}}{279}\lim_{s\to 1^+}\prod_{\substack{p\\ \left(\frac{-31}{p}\right)=1}}\frac{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}\right)}{\left(1-\frac{\lambda(p)}{p^s}\right)\left(1-\frac{\lambda(p)^2}{p^s}\right)}. \tag{9.3}$$

Hence we have the following result.

Corollary 9.7

$$\eta\left(\frac{1+\sqrt{-31}}{2}\right) = e^{\pi i/24}\left|\eta\left(\frac{1+\sqrt{-31}}{2}\right)\right| = e^{\pi i/24}(62\pi)^{-1/4}Ee^{-2R}$$

and

$$\left|\eta\left(\frac{1+\sqrt{-31}}{4}\right)\right| = \left|\eta\left(\frac{-1+\sqrt{-31}}{4}\right)\right| = (31\pi)^{-1/4}Ee^{R},$$

where E is defined in (9.2) and R is defined in (9.3).

10 Chowla-Selberg Formula

In this section we recover the Chowla-Selberg formula [8] from Theorem 9.3.

***Theorem 10.1* (Chowla-Selberg formula)**

$$\prod_{[a,b,c]\in H(d)} a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-h(d)/4}\left\{\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{w(d)/8}.$$

Proof Multiplying formula (9.1) together over the $h(d)$ classes $K = [a,b,c]$ of $H(d)$, we obtain

$$\prod_{[a,b,c]\in H(d)} a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-h(d)/4}\left\{\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{w(d)/8} \times e^{-\frac{\pi w(d)}{48h(d)}\sum_{K\in H(d)}\sum_{\substack{L\in H(d)\\ L\neq I}} f(L,K)\ell(L,d)m(L,d)}.$$

The result now follows as

$$\sum_{K\in H(d)} \sum_{\substack{L\in H(d)\\ L\neq I}} f(L,K)\ell(L,d)m(L,d) = \sum_{\substack{L\in H(d)\\ L\neq I}} \ell(L,d)m(L,d) \sum_{K\in H(d)} f(L,K)$$

$$= 0,$$

by (2.22). ∎

11 Chowla-Selberg Formula for Genera

In this section we deduce from Theorem 9.3 the Chowla-Selberg formula for genera in the case of a fundamental discriminant d. This formula was first discovered by Williams and Zhang [10] in 1993, see also [5, formula (1.7)]. Note that in this formula Δ should be replaced by d in 5 places and the subscript 1 in $w(d_1)$ should be deleted. For the basic properties of genera and generic characters, the reader is referred to [5, Section 2]. We will use the notation and terminology of [5] throughout this section.

Let G be a genus of classes of $H(d)$, that is, G is a coset of the subgroup $H^2(d)$ of squares in $H(d)$. The group of genera of discriminant d is denoted by $G(d)$. The order of $G(d)$ is 2^t, where $t = t(d)$ is a nonnegative integer such that $t+1$ is the number of distinct prime divisors of d. Here we have made use of the fact that d is a fundamental discriminant. Each genus contains $\frac{|H(d)|}{|G(d)|} = \frac{h(d)}{2^t}$ classes, so that

$$|G| = h(d)/2^t. \tag{11.1}$$

By Theorem 9.3 and (11.1), we have

$$\prod_{K=[a,b,c]\in G} a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = \prod_{K\in G}\left\{2\pi|d|^{-1/4}\left\{\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{w(d)/8h(d)} \times e^{-\frac{\pi w(d)\sqrt{|d|}}{48h(d)}\sum_{\substack{L\in H(d)\\ L\neq I}} f(L,K)\ell(L,d)m(L,d)}\right\}$$

$$= (2\pi|d|)^{-h(d)/2^{t+2}}\left\{\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{w(d)}{2^{t+3}}} \times e^{-\frac{\pi w(d)\sqrt{|d|}}{48h(d)}\sum_{K\in G}\sum_{L\neq I} f(L,K)\ell(L,d)m(L,d)},$$

so that

$$\prod_{[a,b,c]\in G} a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-h(d)/2^{t+2}}\left\{\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{w(d)}{2^{t+3}}} e^{-\frac{\pi w(d)\sqrt{|d|}}{48h(d)}E}, \tag{11.2}$$

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where

$$E = \sum_{L \neq I} \ell(L,d) m(L,d) \sum_{K \in G} f(L,K). \tag{11.3}$$

In order to determine $\sum_{K \in G} f(L,K)$ we recall from (2.17) and (2.18) that $\{A_1, \ldots, A_\ell\}$ is a set of generators of $H(d)$ such that the $h_i = \operatorname{ord} A_i$ $(i = 1, \ldots, \ell)$ satisfy $h_1 | h_2 | \cdots | h_\ell$. Thus we may suppose that $h_1, \ldots, h_m$ are odd and $h_{m+1}, \ldots, h_\ell$ are even, where m is an integer satisfying $0 \le m \le \ell$. The genera of $H(d)$ are given by

$$A_{m+1}^{r_{m+1}} \cdots A_\ell^{r_\ell} H^2(d)$$

with $r_{m+1}, \ldots, r_\ell = 0$ or 1. There are $2^{\ell - m}$ genera in total so that

$$t = \ell - m. \tag{11.4}$$

We suppose that the genus G is given by

$$G = A_{m+1}^{\varepsilon_{m+1}} \cdots A_\ell^{\varepsilon_\ell} H^2(d), \quad \text{where each } \varepsilon_{m+1}, \ldots, \varepsilon_\ell = 0 \text{ or } 1. \tag{11.5}$$

The classes K of the genus G are given by

$$K = A_1^{k_1} \cdots A_m^{k_m} A_{m+1}^{2k_{m+1} + \varepsilon_{m+1}} \cdots A_\ell^{2k_\ell + \varepsilon_\ell}, \tag{11.6}$$

where

$$k_j = \begin{cases} 0, 1, \ldots, h_j - 1; \; j = 1, \ldots, m, \\ 0, 1, \ldots, (h_j/2) - 1; \; j = m+1, \ldots, \ell. \end{cases} \tag{11.7}$$

Thus, for $L \in H(d)$, say $L = A_1^{t_1} \cdots A_\ell^{t_\ell}$, $0 \le t_j < h_j$, $j = 1, \ldots, \ell$, we have

$$[L,K] = \sum_{j=1}^{m} \frac{t_j k_j}{h_j} + \sum_{j=m+1}^{\ell} \frac{t_j (2k_j + \varepsilon_j)}{h_j}, \tag{11.8}$$

so that appealing to (2.17), (2.21), (11.4), (11.6), (11.7) and (11.8), we obtain

$$\begin{aligned} &\sum_{K \in G} f(L,K) \\ &\quad = \sum_{K \in G} e^{2\pi i [L,K]} \\ &\quad = \prod_{j=1}^{m} \left\{ \sum_{k_j = 0}^{h_j - 1} e^{\frac{2\pi i t_j k_j}{h_j}} \right\} \prod_{j=m+1}^{\ell} \left\{ \sum_{k_j=0}^{(h_j/2)-1} e^{\frac{2\pi i t_j (2k_j + \varepsilon_j)}{h_j}} \right\} \\ &\quad = \prod_{j=1}^{m} \begin{Bmatrix} h_j, & \text{if } t_j = 0 \\ 0, & \text{if } t_j \neq 0 \end{Bmatrix} \prod_{j=m+1}^{\ell} e^{\frac{2\pi i t_j \varepsilon_j}{h_j}} \prod_{j=m+1}^{\ell} \left\{ \sum_{k_j=0}^{(h_j/2)-1} e^{\frac{2\pi i t_j k_j}{(h_j/2)}} \right\} \end{aligned}$$

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$$= e^{2\pi i \sum_{j=m+1}^{\ell} \frac{t_j\varepsilon_j}{h_j}} \prod_{j=1}^{m} \begin{Bmatrix} h_j, & \text{if } t_j = 0 \\ 0, & \text{if } t_j \neq 0 \end{Bmatrix} \prod_{j=m+1}^{\ell} \begin{Bmatrix} h_j/2, & \text{if } t_j = 0 \text{ or } h_j/2 \\ 0, & \text{if } t_j \neq 0, h_j/2 \end{Bmatrix}$$

$$= \begin{cases} e^{2\pi i \sum_{j=m+1}^{\ell} \frac{t_j\varepsilon_j}{h_j}} h_1 \cdots h_m/2^{\ell-m}, & \text{if } t_j = 0\ (1 \le j \le m) \\ & \text{and } t_j = \frac{h_j x_j}{2} (m+1 \le j \le \ell) \\ & \text{and each } x_j = 0 \text{ or } 1, \\ 0, & \text{otherwise,} \end{cases}$$

that is

$$\sum_{K\in G} f(L,K)$$

$$(11.9)\qquad = \begin{cases} \frac{h(d)}{2^t}(-1)^{\sum_{j=m+1}^{\ell} x_j\varepsilon_j}, & \text{if } L = \prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, \text{ where each } x_j = 0 \text{ or } 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, from (11.3) and (11.9), we obtain

$$(11.10)\qquad E = \frac{h(d)}{2^t} \sum_{x_{m+1},\dots,x_\ell=0}^{1}{}' (-1)^{\sum_{j=m+1}^{\ell} x_j\varepsilon_j} \ell\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) m\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big),$$

where the prime ($'$) indicates that $(x_{m+1},\dots,x_\ell) = (0,\dots,0)$ is omitted.

Next we turn to the evaluation of the terms appearing in the sum on the right side of (11.10). We denote the distinct primes dividing d by $p_1,\dots,p_{t+1}$ and the corresponding prime discriminants by $p_1^*,\dots,p_{t+1}^*$, see Definition 2.1 of [5], so that

$$(11.11)\qquad d = p_1^* \cdots p_{t+1}^*.$$

We define the sets $F(d)$, $F_+(d)$ and $F_-(d)$ as follows:

$$(11.12)\qquad F(d) := \{d_1 \mid d_1 = p_1^{*y_1} \cdots p_{t+1}^{*y_{t+1}}, y_1,\dots,y_{t+1} = 0 \text{ or } 1\},$$

$$(11.13)\qquad F_+(d) := \{d_1 \in F(d) \mid d_1 > 0\},$$

$$(11.14)\qquad F_-(d) := \{d_1 \in F(d) \mid d_1 < 0\}.$$

Clearly, from (11.12), (11.13) and (11.14), we see that

$$(11.15)\qquad F(d) = F_+(d) \cup F_-(d), \quad F_+(d) \cap F_-(d) = \varnothing, \quad |F(d)| = 2^{t+1}.$$

Moreover the mapping $\theta\colon F_+(d) \to F_-(d)$ given by $\theta(d_1) = d/d_1$ is a bijection so that

$$(11.16)\qquad |F_+(d)| = |F_-(d)|.$$

Hence, from (11.15) and (11.16), we deduce that

(11.17) $$|F_+(d)| = \frac{1}{2}|F(d)| = 2^t.$$

Now set

$$X := \{0,1\}^{\ell-m}$$

so that by (11.4) and (11.17)

(11.18) $$|X| = 2^{\ell-m} = 2^t = |F_+(d)|,$$

and define a mapping

(11.19) $$\lambda\colon F_+(d) \to X$$

by

(11.20) $$\lambda(d_1) = (x_{m+1}, \dots, x_\ell),$$

where

(11.21) $$\gamma_{d_1}(A_j) = (-1)^{x_j}, \quad x_j = 0 \text{ or } 1, \quad j = m+1, \dots, \ell.$$

We show that λ is a bijection. By (11.18) it suffices to show that λ is one-to-one. Let $d_1, d_1' \in F_+(d)$ be such that

$$\lambda(d_1) = \lambda(d_1').$$

Then

$$\gamma_{d_1}(A_j) = \gamma_{d_1'}(A_j), \quad j = m+1, \dots, \ell.$$

Hence

$$\gamma_{\Delta(d_1 d_1')}(A_j) = \gamma_{d_1 d_1'}(A_j) = \gamma_{d_1}(A_j)\gamma_{d_1'}(A_j) = 1, \quad j = m+1, \dots, \ell.$$

This shows that

$$\gamma_{\Delta(d_1 d_1')}(G_1) = 1, \quad \text{for all genera } G_1 \text{ of } H(d).$$

Hence

$$\Delta(d_1 d_1') = 1 \text{ or } d,$$

that is

$$d_1' = d_1 \quad \text{or} \quad d_1' = d/d_1.$$

The latter possibility cannot occur as $d_1 > 0$, $d_1' > 0$, $d < 0$. Hence λ is a bijection. Thus the inverse map $\lambda^{-1}\colon X \to F_+(d)$ associates with each $\ell - m$ tuple $(x_{m+1}, \dots, x_\ell)$ (each $x_j = 0$ or 1) a unique positive fundamental discriminant d_1 such that $\gamma_{d_1}(A_j) = (-1)^{x_j}$ $(j = m+1, \dots, \ell)$. Further $\lambda^{-1}\big((0, \dots, 0)\big) = 1$.

We now consider $(-1)^{\sum_{j=m+1}^{\ell} x_j \varepsilon_j}$. We have

$$(-1)^{\sum_{j=m+1}^{\ell} x_j \varepsilon_j} = \prod_{j=m+1}^{\ell} \big((-1)^{x_j}\big)^{\varepsilon_j}$$

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$$= \prod_{j=m+1}^{\ell} \gamma_{d_1}(A_j)^{\varepsilon_j}$$

$$= \prod_{j=m+1}^{\ell} \gamma_{d_1}(A_j^{\varepsilon_j})$$

$$= \gamma_{d_1}\Big(\prod_{j=m+1}^{\ell} A_j^{\varepsilon_j}\Big),$$

that is

$$(-1)^{\sum_{j=m+1}^{\ell} x_j \varepsilon_j} = \gamma_{j_1}(G). \tag{11.22}$$

Next we consider $\ell(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d)$. For $s = 1, \ldots, t+1$ we have $p_s | d$ so that $K_{p_s} = K_{p_s}^{-1}$ and thus

$$K_{p_s} = \prod_{j=m+1}^{\ell} A_j^{h_j u_{js}/2},$$

where

$$u_{js} = 0 \text{ or } 1; \quad j = m+1, \ldots, \ell, \quad s = 1, \ldots, t+1.$$

Hence

$$f\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, K_{p_s}\Big) = f\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, \prod_{j=m+1}^{\ell} A_j^{h_j u_{js}/2}\Big)$$

$$= e^{2\pi i \left[\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, \prod_{j=m+1}^{\ell} A_j^{h_j u_{js}/2}\right]}$$

$$= e^{2\pi i \sum_{j=m+1}^{\ell} \frac{(h_j x_j/2)(h_j u_{js}/2)}{h_j}}$$

$$= (-1)^{\sum_{j=m+1}^{\ell} h_j x_j u_{js}/2}$$

$$= \prod_{j=m+1}^{\ell} \big((-1)^{x_j}\big)^{h_j u_{js}/2}$$

$$= \prod_{j=m+1}^{\ell} \big(\gamma_{d_1}(A_j)\big)^{h_j u_{js}/2}$$

$$= \gamma_{d_1}\Big(\prod_{j=m+1}^{\ell} A_j^{h_j u_{js}/2}\Big)$$

$$= \gamma_{d_1}(K_{p_s}).$$

If $p_s \nmid d_1$ then $\gamma_{d_1}(K_{p_s}) = (\frac{d_1}{p_s})$. If $p_s \mid d_1$ then $p_s \nmid d/d_1$ and $\gamma_{d_1}(K_{p_s}) = \gamma_{d/d_1}(K_{p_s}) = (\frac{d/d_1}{p_s})$. Hence

$$\begin{aligned}
\ell\Big(\prod_{j=m+1}^{\ell} A_{h_j}^{h_j x_j/2}, d\Big) &= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f\big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, K_p\big)}{p}\right) \\
&= \prod_{s=1}^{t+1}\left(1 + \frac{f\big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, K_{p_s}\big)}{p_s}\right) \\
&= \prod_{\substack{s=1 \\ p_s \nmid d_1}}^{t+1} \left(1 + \frac{(\frac{d_1}{p_s})}{p_s}\right) \prod_{\substack{s=1 \\ p_s \mid d_1}}^{t+1} \left(1 + \frac{(\frac{d/d_1}{p_s})}{p_s}\right) \\
&= \prod_{p \mid d} \left(1 + \frac{(\frac{d_1}{p})}{p}\right)\left(1 + \frac{(\frac{d/d_1}{p})}{p}\right) \\
&= \prod_{p \mid d} \frac{(1 - \frac{1}{p})(1 + \frac{1}{p})}{(1 - \frac{(\frac{d_1}{p})}{p})(1 - \frac{(\frac{d/d_1}{p})}{p})},
\end{aligned}$$

that is

$$\ell\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) = \lim_{s \to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=0}} \frac{(1 - \frac{1}{p^s})(1 + \frac{1}{p^s})}{\left(1 - \frac{(\frac{d_1}{p})}{p^s}\right)\left(1 - \frac{(\frac{d/d_1}{p})}{p^s}\right)}. \tag{11.23}$$

Now we consider $m(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d)$. Let p be a prime such that $(\frac{d}{p}) = 1$ so that $K_p \in H(d)$. Set

$$K_p = \prod_{j=1}^{\ell} A_j^{z_j}, \quad z_j = 0, 1, \ldots, h_j - 1,\ j = 1, \ldots, \ell.$$

Then

$$\begin{aligned}
f\Big(\prod_{j=m+1}^{\ell} A_j^{\frac{h_j x_j}{2}}, K_p\Big) &= f\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, \prod_{j=1}^{\ell} A_j^{z_j}\Big) \\
&= e^{2\pi i \left[\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, \prod_{j=1}^{\ell} A_j^{z_j}\right]}
\end{aligned}$$

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$$
\begin{aligned}
&= e^{2\pi i \sum_{j=m+1}^{\ell} \frac{(h_j x_j/2)(z_j)}{h_j}} \\
&= (-1)^{\sum_{j=m+1}^{\ell} x_j z_j} \\
&= \prod_{j=m+1}^{\ell} \big((-1)^{x_j}\big)^{z_j} \\
&= \prod_{j=m+1}^{\ell} \big(\gamma_{d_1}(A_j)\big)^{z_j} \\
&= \prod_{j=m+1}^{\ell} \gamma_{d_1}(A_j^{z_j}) \\
&= \gamma_{d_1}\Big(\prod_{j=m+1}^{\ell} A_j^{z_j}\Big) \\
&= \gamma_{d_1}(K_p) \\
&= \left(\frac{d_1}{p}\right) = \left(\frac{d/d_1}{p}\right).
\end{aligned}
$$

Thus by (6.4) we obtain

$$
m\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) = \lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \frac{(1-\frac{1}{p^s})(1+\frac{1}{p^s})}{\Big(1-\frac{(\frac{d_1}{p})}{p^s}\Big)\Big(1-\frac{(\frac{d/d_1}{p})}{p^s}\Big)}. \tag{11.24}
$$

Since

$$
\prod_{\substack{p \\ (\frac{d}{p})=-1}} \frac{(1-\frac{1}{p^s})(1+\frac{1}{p^s})}{\Big(1-\frac{(\frac{d_1}{p})}{p^s}\Big)\Big(1-\frac{(\frac{d/d_1}{p})}{p^s}\Big)} = 1, \tag{11.25}
$$

we deduce from (11.23), (11.24) and (11.25) that

$$
\begin{aligned}
\ell\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) m\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) &= \lim_{s\to 1^+} \prod_{p} \frac{(1-\frac{1}{p^s})(1+\frac{1}{p^s})}{\Big(1-\frac{(\frac{d_1}{p})}{p^s}\Big)\Big(1-\frac{(\frac{d/d_1}{p})}{p^s}\Big)} \\
&= \lim_{s\to 1^+} \frac{L(s,d_1), L(s,d/d_1)}{\zeta(2s)},
\end{aligned}
$$

that is

$$
\ell\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) m\Big(\prod_{j=m+1}^{\ell} A_j^{h_j x_j/2}, d\Big) = \frac{6}{\pi^2} L(1,d_1) L(1,d/d_1). \tag{11.26}
$$

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Hence, by (11.22) and (11.26), we obtain

$$\sum_{x_{m+1},\ldots,x_\ell=0}^{1}{}' (-1)^{\sum_{j=m+1}^{\ell} x_j\varepsilon_j} \ell\Big(\prod_{j=m+1}^{\ell} A_j^{h_jx_j/2}, d\Big) m\Big(\prod_{j=m+1}^{\ell} A_j^{h_jx_j/2}, d\Big)$$

$$= \frac{6}{\pi^2} \sum_{\substack{d_1\in F_+(d)\\ d_1\neq 1}} \gamma_{d_1}(G) L(1,d_1) L(1,d/d_1)$$

$$= \frac{6}{\pi^2} \sum_{1<d_1\in F(d)} \gamma_{d_1}(G) \frac{2h(d_1)\log\varepsilon_{d_1}}{\sqrt{d_1}} \frac{2\pi h(d/d_1)}{w(d/d_1)\sqrt{|d/d_1|}}$$

$$= \frac{24}{\pi\sqrt{|d|}} \sum_{1<d_1\in F(d)} \gamma_{d_1}(G) \frac{h(d_1)h(d/d_1)}{w(d/d_1)} \log\varepsilon_{d_1},$$

so that by (11.10)

$$E = \frac{24h(d)}{2^t\pi\sqrt{|d|}} \sum_{1<d_1\in F(d)} \gamma_{d_1}(G) \frac{h(d_1)h(d/d_1)}{w(d/d_1)} \log\varepsilon_{d_1}.$$

Finally

$$e^{\frac{-\pi w(d)\sqrt{|d|}}{48h(d)}E} = e^{-\frac{\pi w(d)\sqrt{|d|}}{48h(d)} \frac{24h(d)}{2^t\pi\sqrt{|d|}} \sum_{1<d_1\in F(d)} \gamma_{d_1}(G) \frac{h(d_1)h(d/d_1)}{w(d/d_1)} \log\varepsilon_{d_1}}$$

$$= e^{-\sum_{1<d_1\in F(d)} \frac{w(d)\gamma_{d_1}(G)h(d_1)h(d/d_1)}{w(d/d_1)2^{t+1}} \log\varepsilon_{d_1}}$$

$$= \prod_{1<d_1\in F(d)} \varepsilon_{d_1}^{-\frac{w(d)\gamma_{d_1}(G)h(d_1)h(d/d_1)}{w(d/d_1)2^{t+1}}}.$$

This completes the proof of the Chowla-Selberg formula for genera as given by Williams and Zhang in the case of a fundamental discriminant.

Theorem 11.1 *Let G be a genus of classes of $H(d)$. Then*

$$\prod_{[a,b,c]\in G} a^{-1/4} \left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-h(d)/2^{t+2}} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{w(d)}{2^{t+3}}}$$

$$\times \prod_{\substack{d_1\in F(d)\\ d_1>1}} \varepsilon_{d_1}^{-\frac{w(d)\gamma_{d_1}(G)h(d_1)h(d/d_1)}{w(d/d_1)2^{t+1}}},$$

where ε_{d_1} denotes the fundamental unit (>1) of the real quadratic field $Q(\sqrt{d_1})$, $\gamma_{d_1}(G)(=\pm1)$ is defined in [5, Section 2], $F(d)$ is defined in (11.12), and $t+1$ is the number of distinct prime factors of the fundamental discriminant d.

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12 Concluding remarks

Let $K \in H(d)$. It is clear from Theorem 9.3 that the sum

$$\sum_{\substack{L \in H(d) \\ L \neq I}} f(L,K)\ell(L,d)m(L,d)$$

does not depend upon the choice of basis $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ of $H(d)$. In this section we prove this assertion directly (Theorem 12.4). Let $\mathcal{B} = \{B_1, \ldots, B_\ell\}$ be another basis of $H(d)$ with $\operatorname{ord}(B_i) = h_i$ $(i = 1, \ldots, \ell)$. Hence there exist unique integers a_{ij} $(i, j = 1, \ldots, \ell)$ such that

$$(12.1) \qquad B_i = \prod_{j=1}^{\ell} A_j^{a_{ij}}, \quad 0 \leq a_{ij} < h_j,$$

and unique integers b_{ij} $(i, j = 1, \ldots, \ell)$ such that

$$(12.2) \qquad A_i = \prod_{j=1}^{\ell} B_j^{b_{ij}}, \quad 0 \leq b_{ij} < h_j.$$

Now, by (12.2), we have

$$\prod_{j=1}^{\ell} B_j^{b_{ij}h_i} = \Big(\prod_{j=1}^{\ell} B_j^{b_{ij}}\Big)^{h_i} = A_i^{h_i} = I,$$

so that

$$(12.3) \qquad b_{ij}h_i \equiv 0 \pmod{h_j}, \quad i, j = 1, \ldots, \ell.$$

Also, by (12.1), we have

$$\prod_{j=1}^{\ell} A_j^{a_{ij}h_i} = \Big(\prod_{j=1}^{\ell} A_j^{a_{ij}}\Big)^{h_i} = B_i^{h_i} = I,$$

so that

$$(12.4) \qquad a_{ij}h_i \equiv 0 \pmod{h_j}, \quad i, j = 1, \ldots, \ell.$$

Further

$$\prod_{k=1}^{\ell} A_k^{\sum_{j=1}^{\ell} a_{jk}b_{ij}} = \prod_{k=1}^{\ell}\prod_{j=1}^{\ell} A_k^{a_{jk}b_{ij}} = \prod_{j=1}^{\ell}\Big(\prod_{k=1}^{\ell} A_k^{a_{jk}}\Big)^{b_{ij}} = \prod_{j=1}^{\ell} B_j^{b_{ij}} = A_i,$$

so that

(12.5) $$\sum_{j=1}^{\ell} a_{jk} b_{ij} \equiv \delta_{ik} \pmod{h_k}, \quad i,k=1,\dots,\ell.$$

Similarly we have

$$\prod_{k=1}^{\ell} B_k^{\sum_{j=1}^{\ell} a_{ij} b_{jk}} = \prod_{k=1}^{\ell}\prod_{j=1}^{\ell} B_k^{a_{ij}b_{jk}} = \prod_{j=1}^{\ell}\Big(\prod_{k=1}^{\ell} B_k^{b_{jk}}\Big)^{a_{ij}} = \prod_{j=1}^{\ell} A_j^{a_{ij}} = B_i,$$

so that

(12.6) $$\sum_{j=1}^{\ell} a_{ij} b_{jk} \equiv \delta_{ik} \pmod{h_k}, \quad i,k=1,\dots,\ell.$$

As $K \in H(d)$ we can define integers k_i $(i=1,\dots,\ell)$ and k_i' $(i=1,\dots,\ell)$ by

(12.7) $$K = \prod_{i=1}^{\ell} A_i^{k_i} = \prod_{i=1}^{\ell} B_i^{k_i'}, \quad 0 \le k_i,\ k_i' < h_i.$$

From (12.2) and (12.7), we obtain

$$\prod_{j=1}^{\ell} B_j^{k_j'} = \prod_{i=1}^{\ell}\Big(\prod_{j=1}^{\ell} B_j^{b_{ij}}\Big)^{k_i} = \prod_{j=1}^{\ell} B_j^{\sum_{i=1}^{\ell} k_i b_{ij}},$$

so that

(12.8) $$k_j' \equiv \sum_{i=1}^{\ell} k_i b_{ij} \pmod{h_j}, \quad j=1,\dots,\ell.$$

From (12.1) and (12.7) we have

$$\prod_{j=1}^{\ell} A_j^{k_j} = \prod_{i=1}^{\ell}\Big(\prod_{j=1}^{\ell} A_j^{a_{ij}}\Big)^{k_i'} = \prod_{j=1}^{\ell} A_j^{\sum_{i=1}^{\ell} k_i' a_{ij}}$$

so that

(12.9) $$k_j \equiv \sum_{i=1}^{\ell} k_i' a_{ij} \pmod{h_j}.$$

Now let $L \in H(d)$ and define integers ℓ_i and ℓ_i' $(i = 1, \dots, \ell)$ by

$$L = \prod_{i=1}^{\ell} A_i^{\ell_i} = \prod_{i=1}^{\ell} B_i^{\ell_i'}, \quad 0 \le \ell_i,\ \ell_i' < h_i. \tag{12.10}$$

Set

$$x_i = \sum_{j=1}^{\ell} b_{ij} \ell_j' h_i / h_j, \quad i = 1, \dots, \ell. \tag{12.11}$$

By (12.3) x_i $(i = 1, \dots, \ell)$ is a nonnegative integer. Set

$$L^* = \prod_{i=1}^{\ell} A_i^{x_i} \in H(d). \tag{12.12}$$

Lemma 12.1 $[L^*, K]_{\mathcal{A}} \equiv [L, K]_{\mathcal{B}} \pmod 1$.

Proof We have

$$\begin{aligned}
[L^*, K]_{\mathcal{A}} &= \sum_{i=1}^{\ell} \frac{x_i k_i}{h_i} \\
&= \sum_{i=1}^{\ell} \frac{k_i}{h_i} \sum_{j=1}^{\ell} b_{ij} \ell_j' \frac{h_i}{h_j} \quad \text{(by (12.11))} \\
&= \sum_{j=1}^{\ell} \frac{\ell_j'}{h_j} \sum_{i=1}^{\ell} k_i b_{ij} \\
&\equiv \sum_{j=1}^{\ell} \frac{\ell_j'}{h_j} k_j' \pmod 1 \quad \text{(by (12.8))} \\
&= [L, K]_{\mathcal{B}}.
\end{aligned}$$

■

Lemma 12.2 *The mapping $\theta \colon H(d) \to H(d)$ given by*

$$\theta(L) = L^*$$

is a bijection with $\theta(I) = I$.

Proof Suppose $L_1, L_2 \in H(d)$ are such that

$$\theta(L_1) = \theta(L_2).$$

For $r = 1, 2$ we set $\ell_j(r) = \operatorname{ind}_{A_j}(L_r)$ and $\ell'_j(r) = \operatorname{ind}_{B_j}(L_r)$, $j = 1, \ldots, \ell$. Then, for $i = 1, \ldots, \ell$, we have

$$\sum_{j=1}^{\ell} b_{ij}\ell'_j(1)h_i/h_j \equiv \sum_{j=1}^{\ell} b_{ij}\ell'_j(2)h_i/h_j \pmod{h_i},$$

so that

$$\sum_{j=1}^{\ell} b_{ij}\big(\ell'_j(1) - \ell'_j(2)\big)/h_j \equiv 0 \pmod{1}. \tag{12.13}$$

Hence

$$\sum_{i=1}^{\ell} a_{ki} \sum_{j=1}^{\ell} b_{ij}\big(\ell'_j(1) - \ell'_j(2)\big)/h_j \equiv 0 \pmod{1}, \quad k = 1, \ldots, \ell.$$

Thus

$$\sum_{j=1}^{\ell} \frac{\big(\ell'_j(1) - \ell'_j(2)\big)}{h_j} \sum_{i=1}^{\ell} a_{ki}b_{ij} \equiv 0 \pmod{1}, \quad k = 1, \ldots, \ell.$$

Appealing to (12.6) we obtain

$$\sum_{j=1}^{\ell} \frac{\big(\ell'_j(1) - \ell'_j(2)\big)}{h_j}\delta_{jk} \equiv 0 \pmod{1},$$

that is

$$\frac{\ell'_k(1) - \ell'_k(2)}{h_k} \equiv 0 \pmod{1}, \quad k = 1, \ldots, \ell.$$

Hence $\ell'_k(1) \equiv \ell'_k(2) \pmod{h_k}$ $(k = 1, \ldots, \ell)$ so that $L_1 = L_2$. Thus θ is injective and so θ is a bijection. It is clear from (12.10), (12.11) and (12.12) that $\theta(I) = I$. ■

Lemma 12.3

(i) $f_{\mathcal{A}}(L^*, K) = f_{\mathcal{B}}(L, K)$,
(ii) $\ell_{\mathcal{A}}(L^*, d) = \ell_{\mathcal{B}}(L, d)$,
(iii) $m_{\mathcal{A}}(L^*, d) = m_{\mathcal{B}}(L, d)$.

Proof

(i)

$$\begin{aligned} f_{\mathcal{A}}(L^*, K) &= e^{2\pi i[L^*,K]_{\mathcal{A}}} \quad \text{(by (2.21))} \\ &= e^{2\pi i[L,K]_{\mathcal{B}}} \quad \text{(by Lemma 12.1)} \\ &= f_{\mathcal{B}}(L, K) \quad \text{(by (2.21).)} \end{aligned}$$

(ii)

$$\ell_{\mathcal{A}}(L^*, d) = \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f_a(L^*, K_p)}{p}\right) \quad \text{(by (2.33))}$$

$$= \prod_{\substack{p \\ (\frac{d}{p})=0}} \left(1 + \frac{f_{\mathcal{B}}(L, K_p)}{p}\right) \quad \text{(by (i))}$$

$$= \ell_{\mathcal{B}}(L, d) \quad \text{(by (2.33))}.$$

(iii)

$$m_a(L^*, d) = \lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \frac{(1 - \frac{1}{p^s})(1 + \frac{1}{p^s})}{\left(1 - \frac{f_{\mathcal{A}}(L^*, K_p)}{p^s}\right)\left(1 - \frac{f_{\mathcal{A}}(L^*, K_p)^{-1}}{p^s}\right)}$$

$$= \lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \frac{(1 - \frac{1}{p^s})(1 + \frac{1}{p^s})}{\left(1 - \frac{f_{\mathcal{B}}(L, K_p)}{p^s}\right)\left(1 - \frac{f_{\mathcal{B}}(L, K_p)^{-1}}{p^s}\right)} \quad \text{(by (i))}$$

$$= m_{\mathcal{B}}(L, d).$$

■

Theorem 12.4 *The sum*

$$\sum_{\substack{L\in H(d) \\ L\neq I}} f(L, K)\ell(L, d)m(L, d)$$

does not depend upon the choice of basis $\mathcal{A}$.

Proof We have

$$\sum_{\substack{L\in H(d) \\ L\neq I}} f_{\mathcal{A}}(L, K)\ell_{\mathcal{A}}(L, d)m_{\mathcal{A}}(L, d)$$

$$= \sum_{\substack{L\in H(d) \\ L\neq I}} f_{\mathcal{A}}(L^*, K)\ell_{\mathcal{A}}(L^*, d)m_{\mathcal{A}}(L^*, d) \quad \text{(by Lemma 12.2)}$$

$$= \sum_{\substack{L\in H(d) \\ L\neq I}} f_{\mathcal{B}}(L, K)\ell_{\mathcal{B}}(L, d)m_{\mathcal{B}}(L, d) \quad \text{(by Lemma 12.3)}.$$

■

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